Chapter 3 – Arithmetic Functions

3.1 Arithmetic Functions; Multiplicativity

Defn: An arithmetic function is a function

Some arithmetic functions we will study in this Chapter:

(i) $\phi(n)$ ( )

(ii) $\nu(n) = \#$

n	$\nu(n)$
1	
2	
3	
4	
5	
6	

(iii) $\sigma(n) =$

n	$\sigma(n)$
1	
2	
3	
4	
5	
6	

<u>Def^{n}</u> an arithmetic function f is <u>completely multiplicative</u>
if

<u>Proposition</u> Let k be any fixed real number. Then
$f(n) =$ _ _ _ is completely multiplicative.
<u>PROOF</u> Let $k \in \mathbb{R}$, let $f(n) =$ _ _ for

<u>Definition</u> An arithmetic function f is <u>multiplicative</u>
if

<u>Proposition</u> Let f be multiplicative.
(1) $f(1) =$

(2) If _ _ _ _ then $f(n) = 0$ for all $n \geq 1$.

(3) Suppose $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$
is a prime factorization where each $a_i \geq 0$.
Then
$f(n) =$

<u>PROOF.</u> Suppose f is multiplicative.

(1) $(1,1)=1$ so $f(1\cdot 1)=$

(2) Suppose _ _ _ _ . Let $n\geqslant 1$, $n\in\mathbb{Z}$.
Then $(1,n)=$ _ _ .

(3) Let $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$
be a prime factorization with each $a_i\geqslant 0$.
We show
$f(n)=$
by induction on _ _ .

Given an arithmetic function $f(n)$ we can construct a new arithmetic function

$$F(n) :=$$

NOTE: Quite often we write

Example Let $F(n) = \sum_{d|n} \nu(d)$.

Find $F(6)$.

Example Suppose $f(n)$ is multiplicative &

$$F(n) = \sum_{d|n} f(d).$$

Show that $F(12) = F(3 \cdot 4) = F(3)F(4)$.

NOTE Every positive divisor d of 12 can be written uniquely as

where

d

1

2

3

4

5

6

Ex 8. Let m, n be positive integers with $(m,n)=1$. Every ~~p~~ divisor $d>0$ of mn can be written uniquely as

$$d = d_1 d_2$$

where $d_1 \mid m$ & $d_2 \mid n$. Conversely if $d_1 \mid m$ & $d_2 \mid n$ then $d = d_1 d_2 \mid mn$.

PROOF:

The result is clearly true if $m=1$ or $n=1$.

Suppose $m>1$ & $n>1$.

Let $m =$

$n =$

be prime factorizations with

Note If $d_1 | m$ & $d_2 | n$ ~~then~~ & (m,n) then $(d_1, d_2) =$.
(Since

Theorem Suppose $f(n)$ is multiplicative. Then

$$F(n) := \sum_{d|n} f(\;)$$

is ____ ________.

Proof: Suppose $m, n \in \mathbb{Z}$, $m, n \geq 1$ & $(m,n)=1$.
We show that $F(mn) = F(m)F(n)$.

$$F(mn) =$$

Corollary

PROOF:

(See Section 3.3).

3.2 The Euler Phi-Function

Example Show $\phi(4\cdot 9) = \phi(4)\,\phi(9)$

1	5	9	13	17	21	25	29	33
2	6	10	14	18	22	26	30	34
3	7	11	15	19	23	27	31	35
4	8	12	16	20	24	28	32	36

$\phi(36) = \quad \times \quad =$

1 2 3 4

1 2 3 4 5 6 7 8 9

Theorem The Euler phi-function $\phi(n)$ is multiplicative.

PROOF: Let m, n be positive relatively prime integers. We display the integers from 1 to mn in matrix form:

$$\begin{array}{cccccc}
1 & m+1 & 2m+1 & \cdots & \cdots & \\
2 & m+2 & 2m+2 & & & \\
\vdots & \vdots & \vdots & & & \\
i & m+i & 2m+i & & & \\
\vdots & \vdots & \vdots & & & \\
m & 2m & 3m & & & nm
\end{array}$$

Consider the i-th row of this matrix above

If $(m, i) = d > 1$ then

so that   and d divides

- There are ___ remaining rows, these are the rows with $1 \leq i \leq m$ & ____.

In each of these rows we show that there are exactly ____ numbers relatively prime to mn.

So suppose $(m, i) = 1$. By a previous result

$(\quad , m) = (m, i) = 1$.

So each entry is relatively prime to ___.

The entries in row i are

(*)

Suppose $km + i \equiv lm + i \pmod{n}$

where $0 \leq k, l \leq n-1$. Then

Hence the numbers in (*) are __ incongruent numbers mod n and hence must be congruent to

(**)

in some order. Hence exactly ___ of these numbers are relatively prime to ___, and hence to ___ (since

).

It follows that

$\square$

<u>Theorem</u> Let p be prime & suppose $a \in \mathbb{Z}$ & $a > 0$.
Then

$$\phi(p^a) =$$

<u>PROOF</u>. We list the numbers

(*) $1, 2, \ldots, p^a$.

<u>Example</u> Find $\phi(260000)$

Theorem Suppose $n > 1$. Then

$$\phi(n) =$$

PROOF. Suppose $n \in \mathbb{Z}$, $n > 1$ &

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

is a prime factorization where each $a_i > 0$. Then

$$\phi(n) =$$

Theorem (Gauss) Let n be a positive integer. Then

$$\sum_{d \mid n} \phi(d) = \_\_\_.$$

Proof: For $n \geq 1$, let $F(n) = \sum_{d \mid n} \phi(d)$.

Another Proof (Sketch)

Suppose $n>0$, $n \in \mathbb{Z}$. Let $1 \leq m \leq n$.

If $d=(m,n)$ then $d \mid n$.

For each $d \mid n$ let

$$S_d = \{ m \in \mathbb{Z} : \qquad \}.$$

It can be shown $|S_d| =$

Clearly

$$\bigcup_{d \mid n} S_d =$$

Since the S_d are disjoint

$$\left| \bigcup_{d \mid n} S_d \right| =$$

So that

$$\sum_{d|n} \qquad =$$

as d runs thro the positive divisors of n so does ____.

Hence $\qquad = \sum_{d|n} \qquad = \sum$

and

$$\sum_{d|n} \qquad = \qquad . \ \square$$

Ex 12 Let $n \in \mathbb{Z}$ with $n > 1$.
If $p_1^{a_1} \ldots p_m^{a_m}$ is the prime factorization of n prove that

$$\phi(n) = p_1^{a_1-1} p_2^{a_2-1} \ldots p_m^{a_m-1} \prod_{i=1}^{m} (p_i - 1).$$

3.3 The Number of Positive Divisors Function

Recall

$$\nu(n) = \#$$

$$= \sum$$

Theorem Let p be prime, $a \in \mathbb{Z}$ & $a \geq 0$.
Then

$$\nu(p^a) =$$

PROOF:

Theorem Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be a prime factorization where each $a_i \geq 0$. Then

$$\nu(n) =$$

PROOF:

<u>Example</u> Find $\tau(26\,00000)$

<u>Second Proof of Theorem</u>:

Suppose $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$.

Then d is a positive divisor of n if and only if

3.4 Sum of Positive Divisors Function

Recall that $\sigma(n) =$

Theorem

$\sigma(n)$ is

PROOF

Recall,

$\cancel{(1-x^n) =}$

$x^n - 1 =$,

and

$1 + x + \cdots + x^{n-1} =$

Theorem Let p be prime, $a \in \mathbb{Z}$ & $a \geq 0$.

Then

$\sigma(p^a) =$

PROOF:

Theorem Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be a prime factorization where each $a_i \geq 0$. Then

$$\partial(n) =$$

PROOF:

Example Find $\partial(2600000)$.

3.5 Perfect Numbers

Definition a positive integer n is perfect if

Examples

(a)

(b)

Theorem Let n be a positive integer. Then n is a perfect even number if and only if

$$n =$$

where

PROOF

($\Leftarrow$) [ ] Suppose $n =$

where

($\Longrightarrow$) [ ] Suppose n is an even perfect number.

Let $n = 2^a \cdot b$

where b is , $b \geqslant$, and $a \geqslant$.

Then

$\sigma(n) =$

3.6 The Möbius Inversion Formula

We want a multiplicative function $\mu(n)$ with the following property:

$$F(n) = \sum_{d|n} \mu(d) = \Bigg\{$$

So we need $\mu(1) =$

Let p be prime. We want

$$F(p) =$$
$$F(p^2) =$$
$$F(p^3) =$$

Hence we want $\mu(p^a) =$

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be the prime factorization of n where each $a_i > 0$. Then we want

$$\mu(n) =$$

If

If

We want

$$\mu(n) =$$

Definition of The Möbius Function

Thus we are led to define $\mu(n)$ as follows

$$\mu(n) := \begin{cases} - \\ - \\ - \\ - \end{cases}$$

Examples

n	$\mu(n)$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

Theorem $\mu(n)$ is multiplicative.

PROOF: Suppose m, n are positive relatively prime integers.

Case 1 $m = 1$

Case 2 $n = 1$

Case 3 $m > 1$ and $n > 1$.

(a)

(b)

(c) m and n are the products of distinct primes:

$m =$, $n =$.

Proposition Let $n \in \mathbb{Z}$ & $n \geq 1$. Then

$$\sum_{d|n} \mu(d) = \left\{ \right.$$

PROOF:

The Möbius Inversion Formula

Let f, g be arithmetic functions. Then

$$f(n) = \sum_{d \mid n} g(d) \quad \text{for } \_\_\_,$$

if and only if

<u>Example</u> Suppose $f(n) = \sum_{d|n} g(d)$ for all n.

~~Show that~~ $g(20) = \sum_{d|20} \mu(d) f\left(\frac{20}{d}\right)$.

$$\sum_{d|20} \mu(d) f\left(\frac{20}{d}\right)$$

$=$

Proof of the Möbius Inversion Formula

($\Longrightarrow$) Suppose $f(n) = \sum_{d|n} g(d)$ for all n.

Then

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

NOTE: ①

②

3

It follows that

($\Leftarrow$) Suppose

$$g(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) f(d) \quad \text{for all } n.$$

Then

$$\sum_{d \mid n} g(d) =$$

①

②

Hence,

<u>Example</u> By Gauss we have

$$n = \sum_{d|n} \phi(d).$$