

## Chapter 3 - Arithmetic Functions

(1)

### 3.1 Arithmetic Functions; Multiplicativity

Defn: An arithmetic function is a function

Some Arithmetic Functions we will study in this Chapter:

(i)  $\phi(n)$  ( )

(ii)  $\tau(n) = \#$

$n$	$\tau(n)$
1	
2	
3	
4	
5	
6	

(iii)  $\sigma(n) =$

$n$	$\sigma(n)$
1	
2	
3	
4	
5	
6	

Def<sup>n</sup> An arithmetic function  $f$  is completely multiplicative if (2)

Proposition Let  $k$  be any fixed real number. The  $f(n) = k^n$  is completely multiplicative.

Proof Let  $k \in \mathbb{R}$ , let  $f(n) = k^n$  for

Definition An arithmetic function  $f$  is multiplicative if

Proposition Let  $f$  be multiplicative.

(1)  $f(1) =$

(2) If  $f(p) = 0$  for all  $p \geq 1$ , then  $f(n) = 0$  for all  $n \geq 1$ .

(3) Suppose  $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$  is a prime factorization where each  $a_i \geq 0$ .  
Then

$$f(n) =$$

PROOF. Suppose  $f$  is multiplicative.

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(1)  $(1, 1) = 1$  so  $f(1 \cdot 1) =$

(2) Suppose  $\dots$ . Let  $n \geq 1, n \in \mathbb{Z}$ .  
Then  $(1, n) = \dots$

(3) Let  $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$   
be a prime factorization with each  $a_i \geq 0$ .  
We show  
 $f(n) =$   
by induction on  $\dots$

(4)  
Given an arithmetic function  $f(n)$  we can construct  
a new arithmetic function  
$$F(n) :=$$

NOTE: Quite often we write

Example Let  $F(n) = \sum_{d|n} \omega(d)$ .

Find  $F(6)$ .

Example Suppose  $f(n)$  is multiplicative &  
$$F(n) = \sum_{d|n} f(d)$$
.

Show that  $F(12) = F(3 \cdot 4) = F(3)F(4)$ .

NOTE Every positive divisor  $d$  of 12 can be written uniquely as

where

$d$

1

2

3

4

5

6

Ex 8. Let  $m, n$  be positive integers with  $(m, n) = 1$ .  
Every  $\neq$  divisor  $d > 0$  of  $mn$  can be written uniquely  
as

$$d = d_1 d_2$$

where  $d_1 | m$  &  $d_2 | n$ . Conversely if  $d_1 | m$  &  $d_2 | n$   
then  $d = d_1 d_2 | mn$ .

PROOF:

The result is clearly true if  $m = 1$  or  $n = 1$ .

Suppose  $m > 1$  &  $n > 1$ .

(6)

Let  $m =$

$n =$

be prime factorizations with

Note If  $d_1 | m$  &  $d_2 | n$  ~~and~~  $(m, n) = 1$  then  $(d_1, d_2) = 1$ .  
(Since)

Theorem Suppose  $f(n)$  is multiplicative. Then

$$F(n) := \sum_{d|n} f(d)$$

is

Proof: Suppose  $m, n \in \mathbb{Z}$ ,  $m, n \geq 1$  &  $(m, n) = 1$ .  
We show that  $F(mn) = F(m)F(n)$ .

$$F(mn) =$$

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Corollary

PROOF:

(See Section 3.3).



### 3.2 The Euler Phi-Function

Example Show  $\phi(4 \cdot 9) = \phi(4) \phi(9)$

1	5	9	13	17	21	25	29	33
2	6	10	14	18	22	26	30	34
3	7	11	15	19	23	27	31	35
4	8	12	16	20	24	28	32	36

$$\phi(36) = \quad \times \quad =$$

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix}$$

Theorem The Euler phi-function  $\phi(n)$  is multiplicative.

Proof: Let  $m, n$  be positive relatively prime integers. We display the integers from 1 to  $mn$  in matrix form:

1	$m+1$	$2m+1$	...	...	...
2	$m+2$	$2m+2$			
⋮	⋮	⋮			
⋮	⋮	⋮			
$i$	$m+i$	$2m+i$			
⋮	⋮	⋮			
⋮	⋮	⋮			
$m$	$2m$	$3m$			$nm$

(10)

Consider the  $i$ -th row of this matrix above

If  $(m, i) = d > 1$  then

so that

and  $d$  divides

- There are \_\_\_\_\_ remaining rows, these are the rows with  $1 \leq i \leq m$  & \_\_\_\_\_.

In each of these rows we show that there are exactly \_\_\_\_\_ numbers relatively prime to  $mn$ .

So suppose  $(m, i) = 1$ . By a previous result

$$\binom{\phantom{0}}{\phantom{0}, m} = \binom{\phantom{0}}{\phantom{0}, i} = 1.$$

So each entry is relatively prime to \_\_\_\_\_.

The entries in row  $i$  are

(\*)

$$\text{Suppose } km + i \equiv lm + i \pmod{n}$$

where  $0 \leq k, l \leq n-1$ . Then

Hence the numbers in (\*) are \_\_\_\_\_ incongruent numbers mod  $n$  and hence must be congruent to

(\*\*)

is some order. Hence exactly \_\_\_\_\_ of these numbers are relatively prime to \_\_\_\_\_, and hence to \_\_\_\_\_ (since \_\_\_\_\_).

(11)

It follows that

□

Theorem Let  $p$  be prime & suppose  $a \in \mathbb{Z}$  &  $a > 0$ .

Then

$$\phi(p^a) =$$

PROOF. We list the numbers

$$(*) \quad 1, 2, \dots, p^a.$$

Example Find  $\phi(260000)$

Theorem Suppose  $n > 1$ . Then

$$\phi(n) =$$

Proof. Suppose  $n \in \mathbb{Z}$ ,  $n > 1$  &

$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$   
 is a prime factorization where each  $a_i > 0$ . Then

$$\phi(n) =$$

Theorem (Gauss) Let  $n$  be a positive integer. Then

$$\sum_{d|n} \phi(d) = \dots$$

Proof: For  $n > 1$ , let  $F(n) = \sum_{d|n} \phi(d)$ .

Another Proof (Sketch)

Suppose  $n > 0$ ,  $n \in \mathbb{Z}$ . Let  $1 \leq m \leq n$ .

If  $d = (m, n)$  then  $d \mid n$ .

For each  $d \mid n$  let

$$S_d = \{m \in \mathbb{Z} : \dots\}$$

It can be shown  $|S_d| =$

Clearly

$$\bigcup_{d \mid n} S_d =$$

Since the  $S_d$  are disjoint

$$\left| \bigcup_{d \mid n} S_d \right| =$$

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So that

$$\sum_{d|n} =$$

As  $d$  runs thro the positive divisors of  $n$   
so does  $\frac{n}{d}$ .

Hence  $\sum_{d|n} = \sum_{d|n} = \sum$

and

$$\sum_{d|n} = \quad \quad \quad \square$$

Ex 12 Let  $n \in \mathbb{Z}$  with  $n > 1$ .

If  $p_1^{a_1} \dots p_m^{a_m}$  is the prime factorization of  $n$   
prove that

$$\phi(n) = p_1^{a_1-1} p_2^{a_2-1} \dots p_m^{a_m-1} \prod_{i=1}^m (p_i - 1).$$

### 3.3 The Number of Positive Divisors Function

Recall

$$\tau(n) = \#$$

$$= \sum$$

Theorem Let  $p$  be prime,  $a \in \mathbb{Z}$  &  $a \geq 0$ .

Then

$$\tau(p^a) =$$

PROOF:

Theorem Let  $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$  be a prime factorization where each  $a_i \geq 0$ . Then

$$\tau(n) =$$

PROOF:

Example Find  $\tau(2600000)$

Second Proof of Theorem:

Suppose  $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$

Then  $d$  is a positive divisor of  $n$   
if and only if



### 3.4 Sum of Positive Divisors Function

Recall that  $\sigma(n) =$

Theorem

$\sigma(n)$  is

PROOF

Recall,

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \dots + x + 1)$$

$$x^n - 1 =$$

and

$$1 + x + \dots + x^{n-1} =$$

Theorem Let  $p$  be prime,  $a \in \mathbb{Z}$  &  $a \geq 0$ .

Then

$$\sigma(p^a) =$$

PROOF:

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Theorem Let  $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$  be a prime factorization where each  $a_i \geq 0$ . Then

$$\sigma(n) =$$

PROOF:

Example Find  $\sigma(2600000)$ .

### 3.5 Perfect Numbers

Definition A positive integer  $n$  is perfect if

#### Examples

(a)

(b)

Theorem Let  $n$  be a positive integer.  
Then  $n$  is a perfect even number if and only if

$$n =$$

where

PROOF

( $\Leftarrow$ ) [ ] Suppose  $n =$

where

( $\Rightarrow$ ) [ ] Suppose  $n$  is an even perfect number.

$$\text{Let } n = 2^a \cdot b$$

where  $b$  is odd,  $b \geq 3$ , and  $a \geq 1$ .

$$\sigma(n) =$$

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### 3.6 The Möbius Inversion Formula

We want a multiplicative function  $\nu(n)$  with the following property:

$$F(n) = \sum_{d|n} \nu(d) = \begin{cases} \end{cases}$$

So we need  $\nu(1) =$

Let  $p$  be prime. We want

$$F(p) =$$

$$F(p^2) =$$

$$F(p^3) =$$

Hence we want  $\nu(p^a) =$

Let  $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$  be the prime factorization of  $n$  where each  $a_i > 0$ . Then we want

$$\nu(n) =$$

If

If

we want

$$\nu(n) =$$

## Definition of the Möbius Function

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Thus we are led to define  $\mu(n)$  as follows

$$\mu(n) := \left\{ \begin{array}{l} - \\ - \\ - \\ - \end{array} \right.$$

### Examples

$n$	$\mu(n)$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

Theorem  $\mu(n)$  is multiplicative.

PROOF: Suppose  $m, n$  are positive relatively prime integers.

Case 1  $m=1$

Case 2  $n = 1$

Case 3  $m > 1$  and  $n > 1$ .

(a)

(b)

(c)  $m$  and  $n$  are the products of distinct primes:

$m =$

,  $n =$

Proposition Let  $n \in \mathbb{Z}$  &  $n > 1$ . Then

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is square-free} \\ 0 & \text{otherwise} \end{cases}$$

PROOF:



### The Möbius Inversion Formula

Let  $f, g$  be arithmetic functions. Then  
 $f(n) = \sum_{d|n} g(d)$  for  $n \geq 1$ ,  
if and only if

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Example Suppose  $f(n) = \sum_{d|n} g(d)$

for all  $n$ .

Show that  $g(20) = \sum_{d|20} \mu(d) f\left(\frac{20}{d}\right)$ .

$$\sum_{d|20} \mu(d) f\left(\frac{20}{d}\right)$$

=

Proof of the Möbius Inversion Formula

( $\Rightarrow$ ) Suppose  $f(n) = \sum_{d|n} g(d)$  for all  $n$ .

Then  $\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$

NOTE : (1)

(2)

③

*It follows that*

( $\Leftarrow$ ) Suppose  $g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$  for all  $n$ .

$$\sum_{d|n} g(d) =$$

①

②

Hence,

Example By Gauss we have

$$n = \sum_{d|n} \phi(d).$$