MAS 4203 - SPRING 2010 TEST 2

2 BONUS PTS: This math professor from Penn. State University gave the ERDOS COLLOQUIUM on February 15. What was his name? What country was Paul Erdos from?

WEDNESDAY, MARCH 17 Name: ______________________________

Instructions: Answer all questions. Show all necessary working and reasoning. Your work should be written in a proper and coherent fashion, and in a way that any student in the class can follow your work. When giving proofs your reasoning should be clear. Only scientific calculators are allowed. 50 points total.

1. [6 + 6 = 12 pts]

(i) Find the smallest positive integer x that satisfies the following linear congruences:

$$x \equiv 1 \pmod 3$$
$$x \equiv 2 \pmod 5$$
$$x \equiv 3 \pmod 7$$

Check your answer.

(ii) Find the multiplicative inverse of 47 mod 863.

Check your answer.

2. [4 + 2 + 6 + 8 = 20 points]

(i) State Fermat's Little Theorem.

(ii) Define what it means for a positive integer n to be a pseudoprime.

(iii) Prove that $645 = (3)(5)(43)$ is a pseudoprime.

(iv) Let p and q be distinct primes, and suppose that $a \in \mathbb{Z}$. Prove that

$$a^{pq} + a \equiv a^p + a^q \pmod{pq}.$$

HINT: First prove that $a^{pq} + a \equiv a^p + a^q \pmod p$.

3. $[2+2+3+(2+3+3+3)=18$ points$]$

(i) Define what it means for an arithmetic function $f(n)$ to be *multiplicative.*

(ii) Define what it means for an arithmetic function $f(n)$ to be *completely multiplicative.*

(iii) Let k be a fixed positive integer. Prove that $f(n) = n^k$ is completely multiplicative.

(iv) Let k be a fixed positive integer. Define

$$\sigma_k(n) = \sum_{d|n} d^k.$$

(As usual, in the summation it assumed that $d > 0$).

(a) Find $\sigma_2(6)$.

(b) Prove that $\sigma_k(n)$ is multiplicative, stating any theorems used in the proof.

(c) Let p be prime and suppose a is a positive integer. Find a formula for $\sigma_k(p^a)$. *HINT:* $1 + x + x^2 + \cdots + x^a = \frac{(x^{a+1}-1)}{(x-1)}$, if $x \neq 1$.

(d) Suppose $n > 1$ and let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be a prime factorization. Find a formula for $\sigma_k(n)$, giving reasons.