MAS 4203 – Final Exam – 08/06/15 – 2PM

Regular Time Allowed: 2 hours

Instructor F. GARVAN
Email fgarvan@ufl.edu

Phone 870–3886

MAS 4203 - FINAL EXAM - Summer 2015

Thursday, August 6

NAME:

Instructions: all work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear. Only scientific or basic calculators are allowed.

There are two parts A & B. Part A has 7 questions. Do six complete questions. If you do 7 then the best 6 will be taken. Part B is a bonus question.

PART A

In this part there are 7 questions. Do 6 complete questions, otherwise the best 6 will be taken.

1.

(a) Complete the Definition: An arithmetic function f is multiplicative if

(b) Prove that if f is multiplicative then $f(1) = 0$ or 1.

(c) Complete the Theorem: Let f be an arithmetic function and for $n \in \mathbb{Z}, n > 0$ let

$$F(n) = \sum_{---} \text{-----}.$$

If f is ____________ then F is ____________.

(d) Prove that if f and g are multiplicative functions then $h(n) = f(n)\, g(n)$ is also multiplicative.

(e) Define ρ to be $\rho(1) = 1$ and $\rho(n) = 2^m$ where m is the number of distinct prime numbers in the prime factorization of n. Prove or disprove that ρ is completely multiplicative.

2.

Definition: Let $n \in \mathbb{Z}$ with $n > 0$. If k is a positive integer then

$$\sigma_k(n) = \sum_{d|n} d^k.$$

(a) Find $\sigma_3(12)$ and $\sigma_4(8)$.

(b) Prove that $\sigma_k(n)$ is multiplicative.

(c) Find a formula for $\sigma_k(p^a)$ if p is prime and a is a positive integer.

(d) Let $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be a prime factorization where each $a_i > 0$. Find a formula for $\sigma_k(n)$ using (b) & (c).

3.

(i) Complete Theorem : Let $n \in \mathbb{Z}$, $n > 1$. Then

$$\phi(n) = n \prod_{\cdots\cdots\cdots} \left(\cdots\cdots\cdots \right)$$

(ii) Prove or disprove that there are infinitely many integers n such that

$$\phi(n) = \frac{n}{3}.$$

(iii) Complete Theorem: Let $n \in \mathbb{Z}$, $n > 0$. Then n is an ---- perfect number if and only if

$n =$ ....

(iv) Definition a positive integer n is superperfect if $\sigma(\sigma(n)) = 2n$.
Prove that if $2^p - 1$ is a Mersenne prime then 2^{p-1} is superperfect.

4.
(a) Complete the Definition: Let $n \in \mathbb{Z}$, with $n > 0$. The Möbius function denoted ____ is

$$\mu(n) = \begin{cases} \_\_\_ & \text{if } \_\_\_\_ \\ \_\_\_ & \text{if } \_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_ & \text{if } \_\_\_\_\_\_\_\_ \\ & \quad \_\_\_\_\_\_\_\_\_\_\_\_\_ \end{cases}$$

(b) Complete the Theorem: Let $n \in \mathbb{Z}$ with $n > 0$. Then

$$\sum_{d \mid n} \mu(d) = \begin{cases} \_\_\_ & \text{if } n = 1 \\ \_\_\_ & \text{if } n > 1. \end{cases}$$

(c) Complete the Möbius Inversion Formula: Let f, g be arithmetic functions. Then

$$f(n) = \sum_{d \mid n} \_\_\_ \quad \text{for } \_\_\_\_,$$

if and only if

$$g(n) = \sum_{d \mid n} \_\_\_\_\_\_ = \sum_{d \mid n} \_\_\_\_\_\_$$

for ____.

(d) Let $\nu(n)$ = number of positive divisors of n. Prove that

$$1 = \sum_{d \mid n} \mu(d)\, \nu\left(\frac{n}{d}\right).$$

5.

(a) Complete <u>Euler's Criterion</u>: Let p be an odd prime and $a \in \mathbb{Z}$ with $p \nmid a$. Then

$$\left(\frac{a}{p}\right) \_\_\_\_\_\_\_\_\_\_\_\_.$$

(b) Complete <u>Theorem</u> Let p be an odd prime. Then

$$\left(\frac{2}{p}\right) = \_\_\_\_\_ = \begin{cases} 1 & \text{if } \_\_\_\_\_\_\_\_\_\_ \\ -1 & \text{if } \_\_\_\_\_\_\_\_\_\_. \end{cases}$$

(c) Do ONE part.

(i) Prove or disprove that there is an integer n such that n^2+2 is divisible by 2019.

(ii) Let p, q be primes with $p \equiv 3 \pmod 4$, and $q = 2p+1$. Prove that q divides $2^p - 1$.

(d) Find a prime p for which $2^p - 1$ is composite
Show all reasoning.

6.
(a) Complete the <u>Law of Quadratic Reciprocity</u>:
Let p, q be __________________ primes
Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\_\_\_\_} = \begin{cases} 1 & \text{if } \_\_\_\_\_\_\_\_\_\_\_\_\_\_. \\ -1 & \text{if } \_\_\_\_\_\_\_\_\_\_\_\_\_\_. \end{cases}$$

If __________________ then $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$.

If __________________ then $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$.

(b) Find the Legendre symbol $\left(\frac{501}{773}\right)$.
Show all reasoning.

7. [10 pts]

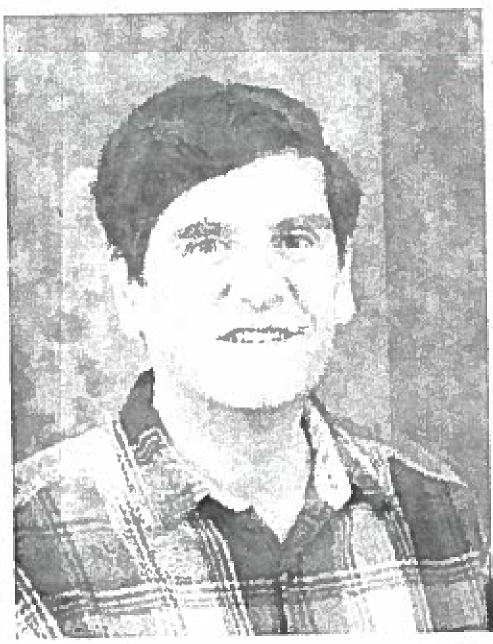
Do ONE part.

(a) Explain and describe the RSA encryption scheme and give a proof of the decoding procedure.

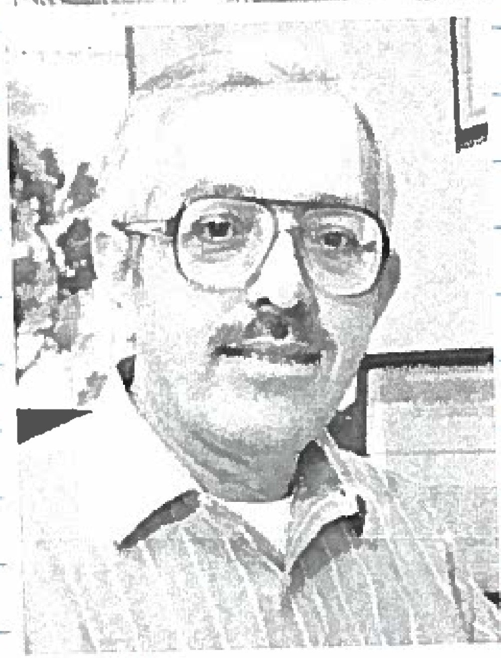
(b) State the a-test for primality and prove that p passes the a-test of all odd primes $p > a$ where a is any fixed integer $a > 1$.

PART B [3 bonus pts]

(a) This is ______________.
He was born in 1927 in __________
In 1962 he proved that 78557 is
a Sierpinski number. A Sierpinski
number is an number k such that
$k \cdot 2^n + 1$ is ______________.
for ________.



(b) This is ______________
He received his Ph.D. from ________
__________ in 1972. He is the
inventor of one of the most important
integer ______________
methods, the quadratic sieve
algorithm.



(c) This is ______________
He is a professor at __________
______________. In a talk
at the Bloomington Illinois Number Theory
Conference John Selfridge came to his rescue
helping out with saying "We use the
principle of computer ______________
______________."