MAS 4203 — QUIZ 1 — SUMMER 2014

Tuesday, July 7

NAME:

Instructions: all work should be written in a proper and coherent manner, and in a way that any student can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear. Only scientific or basic calculators are allowed.

TOTAL: 30 pts = 100% (also 2 bonus pts)

1. [2 + 4×2 = 10 pts]

(a) Complete the Definition: Let $a, b \in \mathbb{Z}$. Then a divides b denoted $a|b$, if $b = ac$ for some $c \in \mathbb{Z}$.

(b) Prove or disprove the following statements:

(i) If $a, b, c \in \mathbb{Z}$ and $a \mid b + c$ then $a \mid b$ and $a \mid c$.

FALSE For example let $b = c = 1$, $a = 2$. $b+c = 2$. So $a \mid b+c$ but $a \nmid b$ & $a \nmid c$.

(ii) If $a, b, c \in \mathbb{Z}$ and $a \mid bc$ then $a \mid b$ or $a \mid c$.

FALSE For example let $a = 4$, $b = 2$, $c = 6$. Then $bc = 12$. So $a \mid bc$ since $4 \mid 12$ but $4 \nmid 2$ & $4 \nmid 6$ i.e. $a \nmid b$ & $a \nmid c$.

(iii) If $a, b, c \in \mathbb{Z}$ and

$$ax + by = c$$

for some $x, y \in \mathbb{Z}$ then $(a,b) = c$.

FALSE Let $a = 2, b=3, c=5$.

The $ax+by = c$ for $x=y=1$

But $(a,b) = (2,3) = 1 \neq 5 = c$

(iv) If $a, b, c, d \in \mathbb{Z}$, $a \mid b$ and $c \mid d$ then $ac \mid bd$.

PROOF: Suppose $a,b,c,d \in \mathbb{Z}$, $a \mid b$ & $c \mid d$.

Then $b = ae$, $d = cf$ for some $e, f \in \mathbb{Z}$

So $bd = (ae)(cf) = (ac)(ef)$,

& $ac \mid bd$ since $ef \in \mathbb{Z}$ because $e, f \in \mathbb{Z}$.

$\square$

2. [2+6+2 = 10 pts]

(a) Complete the Definition: Let $p \in \mathbb{Z}$ and $p > 1$ then p is said to be prime if the only positive divisors of p are 1 and p.

(b) Prove that there are infinitely many primes.

Suppose by way of contradiction that there are only finitely many primes say $p_1, p_2, \ldots, p_n$. Let $N = (p_1 p_2 \cdots p_n) + 1$. Then N is an integer > 1. So by a Lemma N must have a prime divisor p. So $p = p_j$ for some $1 \leq j \leq n$.
$p \mid N$ but $p = p_j \mid (p_1 p_2 \cdots p_n)$.
Therefore $p \mid N + (-1)(p_1 p_2 \cdots p_n) = 1$ which is impossible since $p > 1 > 0$. Hence there must be infinitely many primes.

(c) Complete Goldbach's Conjecture: Every even integer greater than > 2 is the sum of two not necessarily distinct primes.

3. [2+2+6 = 10 pts]

(a) Complete the definition: Let $a, b \in \mathbb{Z}$ with a, b not both zero. The greatest common divisor of a and b, denoted (a,b), is the greatest integer d such that $d \mid a$ & $d \mid b$.

(b) Complete the Proposition: Let $a, b \in \mathbb{Z}$ with a, b not both zero. Then

$$(a,b) = \text{MIN}\ \{am + bn : m, n \in \mathbb{Z} \text{ \& } am + bn > 0\}$$

(c) Use the Proposition in (b) (and not the Fundamental Theorem of Arithmetic) to prove the following

Theorem. Let $a, b, c \in \mathbb{Z}$ with $(a,b) = 1$. If $a \mid bc$ then $a \mid c$.

<u>PROOF:</u> Let $a, b, c \in \mathbb{Z}$ with $(a,b) = 1$.
Suppose $a \mid bc$. Then $bc = ad$ for some $d \in \mathbb{Z}$.
Since $(a,b)=1$, $am + bn = 1$
for some $m, n \in \mathbb{Z}$ by Proof in (b).
Then

$$c(am + bn) = c \cdot 1,$$
$$acm + bcn = c$$
$$acm + (ad)n = c \quad (\text{since } bc = ad),$$
$$a(cm + dn) = c$$

Therefore $a \mid c$ since c, m, d, n & hence $cm + dn \in \mathbb{Z}$.

$\square$

4. [Bonus 2 pts]

(a) Who is this guy?

(b) State his famous <u>Theorem</u>:

(c) Where did he do his Ph.D.?

(d) Who found a more elementary proof of this theorem?