MAS 4203 – QUIZ 3 – SUMMER 2015

Tuesday, July 21

NAME:

Instructions: All work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs make sure your reasoning is clear. Only scientific or basic calculators are allowed.

TOTAL:

1. [2+2+3+3 = 10 pts]

(a) Complete the Definition: An arithmetic function f is multiplicative if

$$f(mn) = f(m)f(n).$$

whenever m, n are positive relatively prime integers.

(b) Complete Theorem 3.1: Let f be an arithmetic function and suppose

$$F(n) = \sum_{d \mid n} f(d)$$

for $n \geq 1$. Then if f is multiplicative, then F is multipli

(c) PROVE that if f is a multiplicative arithmetic function then $f(1) = 0$ or 1.

Suppose is multiplicative. Then $(1,1)=1$

so $f(1) = f(1 \cdot 1) = f(1)f(1)$ (since f is multiplicative).

Therefore $f(1)(f(1)-1) = 0$.
So either $f(1)=0$ or $f(1)=1$. $\square$

(d) Suppose f is an arithmetic multiplicative function, and $f(1)=0$. PROVE that
$f(n) = 0$ for all $n \geq 1$.

Suppose f is multiplicative & $f(1)=0$.
Let $n \geq 1$ ($n \in \mathbb{Z}$). Then $(n,1)=1$.
So $f(n) = f(n\cdot 1) = f(n)f(1)$ (since f is multiplicative)
$= f(n)\cdot 0 = 0$.
Therefore $f(n)=0$ for all $n \geq 1$. $\square$

2.
(a) Complete

Theorem. Suppose $n>1$. Then

$$\phi(n) = n \prod_{\substack{p \mid n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)$$

(b) Complete

Theorem. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be a prime factorization where each $a_i \geq 0$. Then

$$\nu(n) = (a_1+1)(a_2+1)\cdots(a_r+1).$$

(P.3)

(c) Let $n = 1575 = 3^2 \cdot 5^2 \cdot 7$

$$\begin{array}{r|l} 5 & 1575 \\ \hline 5 & 315 \\ \hline 9 & 63 \\ \hline & 7 \end{array}$$

(i) Compute $\phi(n)$.

$$\phi(n) = \phi(3^2 \cdot 5^2 \cdot 7)$$
$$= (3^2-3)(5^2-5)(7-1)$$
$$= 6 \cdot 20 \cdot 6$$
$$= 720$$

(ii) Compute $\tau(n)$.

$$\tau(n) = \tau(3^2 \cdot 5^2 \cdot 7)$$
$$= 3 \cdot 3 \cdot 2 \quad \text{(by (b))}$$
$$= 18.$$

3.

Do TWO parts of this question.

(i) PROVE τ is multiplicative and τ is not completely multiplicative.

(ii) PROVE that $\tau(n)$ is odd if and only if n is a perfect square.

(iii) PROVE that there are no positive integers n such that $\phi(n) = \frac{n}{5}$.

(i) $f(n) = 1$ is multiplicative so
$$\nu(n) = \sum_{d|n} 1 = \sum_{d|n} f(d)$$
is multiplicative by Theorem 3.1.
$$\nu(4) = \nu(2^2) = 3$$
$$\nu(2) = \nu(2^1) = 2.$$
$$\nu(2)\nu(2) = 4 \neq \nu(2^2) = 3$$
So ν is not completely multiplicative.

(ii) Let $n \geq 1$, & $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$
be a prime factorization with each $a_i \geq 0$.
$\nu(n) = (a_1+1)(a_2+1) \cdots (a_r+1)$ is odd
iff each $a_i + 1$ is odd
iff each a_i is even
ie $a_i = 2b_i$ for some $b_i \geq 0$ & each $1 \leq i \leq r$
$(b_i \in \mathbb{Z})$

This is true if & only if n is a perfect square.
Note $p_1^{2b_1} p_2^{2b_2} \cdots p_r^{2b_r} = (p_1^{b_1} \cdots p_r^{b_r})^2$ is a perfect square. Conversely if n is a perfect square then $n = m^2$ where $m = p_1^{b_1} \cdots p_r^{b_r}$ is a prime fact & $n = m^2 = p_1^{2b_1} \cdots p_r^{2b_r}$.

(iii) ~~Suppose~~ $\phi(1) = 1$.
Suppose by way of contradiction that
$$\phi(n) = \frac{n}{5}$$ for some positive integer n.

Then $n > 1$ since $\phi(1) = 1$.

By 2(a) $\prod_{p|n}\left(1-\frac{1}{p}\right) = \prod_{p|n}\frac{(p-1)}{p} = \frac{1}{5}$ & (here p are prime). (p.5)

$5\prod_{p|n}(p-1) = \prod_{p|n} p.$

This implies that $p = 5 \,|\, n$

But then $p-1 = 4$ &

$$2^2 = 4 \,\Big|\, \prod_{p|n} p.$$

But $\prod_{p|n} p$ is a product of distinct primes & we have a contradiction. Hence there is no positive integer n for which $\varphi(n) = \frac{n}{5}$.

4. [2 Bonus pts]

(a) This is

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(b) J________

B_____________

persuaded _______'s father to let his son study mathematics

(c) He ~~studied~~ served as a

m__________

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in the Russian navy from 1727 to 1730.