MAS 4203 - Quiz 4 - Summer 2015

Tuesday, July 28

NAME:

Instructions: all work should be written in a proper and coherent manner, and in a way that any student in the class can follow your work. Show all necessary working and reasoning. When giving proofs your reasoning should be clear. Any scientific or basic calculators are allowed.

TOTAL: 30 pts + 2 bonus pts

1. [$2 \times 5 = 10$ pts]

(a) Complete the Definition: Let $n \in \mathbb{Z}$ with $n > 0$. The sum of positive divisors function denoted $\sigma(n)$ is the function defined by

$$\sigma(n) = \sum_{d \mid n} d$$

(b) Prove that $\sigma(n)$ is multiplicative.

$f(n) = n$ is multiplicative so

$$\sigma(n) = \sum_{d \mid n} n = \sum_{d \mid n} f(d)$$

is multiplicative by Theorem 3.1.

(c) Complete the Theorem. Let p be prime and suppose a is a positive integer. Then

$$\sigma(p^a) = \frac{(p^{a+1} - 1)}{(p-1)}$$

(d) Let n be any positive integer. Prove that

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}.$$

$$\sigma(n) = \sum_{d|n} d = \sum_{d|n} \frac{n}{d}$$ since as d runs thru the positive divisors of n so does $\frac{n}{d}$

$$\sigma(n) = n \sum_{d|n} \frac{1}{d} \quad \&$$

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}.$$

(e) Prove that if $2^p - 1$ is a Mersenne prime then

$$n = 2^{p-1}(2^p - 1)$$

is perfect. Suppose 2^p-1 is a Mersenne prime

Then $\sigma(n) = \sigma\left(2^{p-1}(2^p-1)\right)$

$= \sigma(2^{p-1})\,\sigma(2^p-1)$ (since σ is multiplicative & $(2^{p-1}, 2^p-1) = 1$)

$= \left(\frac{2^p-1}{2-1}\right)(1+2^p-1)$ (since 2^p-1 is prime)

$= (2^p-1)2^p = 2 \cdot 2^{p-1}(2^p-1) = 2n.$

$\therefore$ n is perfect.

2. [2 + 2 + 2 + 4 = 10 pts]

(a) Complete Definition The Möbius function $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^2 | n \text{ where } p \text{ is prime} \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r \\ & \text{with } p_1, p_2, \cdots, p_r \text{ distinct primes} \end{cases}$$

(b) Complete the Theorem. Let n be a positive integer. Then

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1. \end{cases}$$

(c) Complete the Möbius Inversion Formula. Let f, g be arithmetic functions. Then

$$f(n) = \sum_{d|n} g(d) \quad \text{for } n \geq 1,$$

if and only if

$$g(n) = \sum_{d|n} f(d)\,\mu\!\left(\tfrac{n}{d}\right) = \sum_{d|n} \mu(d)\, f\!\left(\tfrac{n}{d}\right)$$

for $n \geq 1$.

(d) By a Theorem of Gauss we know that

$$F(n) = n = \sum_{d|n} \phi(d), \quad \text{for } n \geq 1.$$

Using Möbius Inversion find a formula for $\frac{\phi(n)}{n}$.

By Möbius Inversion,

$$\phi(n) = \sum_{d|n} \mu(d)\, F\!\left(\tfrac{n}{d}\right)$$

$$= \sum_{d|n} \mu(d) \cdot \frac{n}{d}$$

$$\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}, \quad \&$$

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.$$

3. [2 + 4 + 4 = 10 pts]

(a) Complete the Definition: Let $a, m \in \mathbb{Z}$ with $m > 0$ and $(a, m) = 1$. Then a is said to be a quadratic residue modulo m if $x^2 \equiv a \pmod{m}$ for some $x \in \mathbb{Z}$.

(b) Find the quadratic residues mod 17.

x	$x^2 \pmod{17}$
± 1	1
± 2	4
± 3	9
± 4	16
± 5	$25 \equiv 8$
± 6	$36 \equiv 2$
± 7	$49 \equiv 15$
± 8	$64 \equiv 13$

The q.r mod 17 are $a = 1, 2, 4, 8, 9, 13, 15, 16$

(c) ~~Prove~~ PROVE one part.

(i) $$\sum_{d \mid n} |\mu(d)| = 2^{\omega(n)}$$ for $n \geq 1$ where $\omega(n)$ is the number of distinct prime divisors of n.

(ii) $$\Lambda(n) = -\sum_{d \mid n} \mu(d) \ln d$$ for $n \geq 1$

where $\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^a \text{ where } p \text{ is prime \& } a \geq 1 \\ 0 & \text{otherwise.} \end{cases}$

[Hint: First prove $\ln n = \sum_{d \mid n} \Lambda(d)$].

(c) (i) Let $g(n) = |\mu(n)|$ for $n \geq 1$.

S'pose $(m,n)=1$, $m,n \geq 1$. Then

$$g(mn) = |\mu(mn)| = |\mu(m)\mu(n)| \quad \text{(Since } \mu \text{ is multiplicative)}$$
$$= |\mu(m)|\,|\mu(n)| \quad \text{(property of absolute value)}$$
$$= g(m)\,g(n),$$

& g is multiplicative.

Hence

$$F(n) = \sum_{d|n} g(d) = \sum_{d|n} |\mu(d)|$$

is multiplicative by Theorem 3.1

$$F(1) = |\mu(1)| = 1 = 2^{\omega(1)} \quad \text{since } \omega(1)=0,$$

& result holds for $n=1$. Let p be prime & $a \geq 1$.

Then
$$F(p^a) = \sum_{d|p^a} |\mu(d)|$$
$$= |\mu(1)| + |\mu(p)| + |\mu(p^2)| + \dots + |\mu(p^a)|$$
$$= 1 + 1 + 0 + \dots + 0 \quad (\text{since } \mu(p) = -1).$$
$$= 2.$$

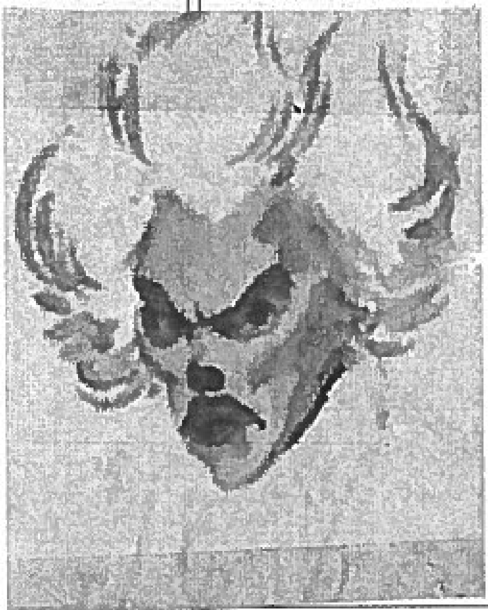
Now let $n > 1$ & $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be a prime factorization. Then

$$F(n) = F(p_1^{a_1} \cdots p_m^{a_m})$$
$$= F(p_1^{a_1}) F(p_2^{a_2}) \cdots F(p_m^{a_m}) \quad \text{(since } F \text{ is multiplicative)}$$
$$= \underbrace{2 \cdot 2 \cdots 2}_{m}$$
$$= 2^m = 2^{\omega(n)}$$

Hence

$$\sum_{d|n} |\mu(d)| = 2^{\omega(n)} \quad \text{for all } n \geq 1.$$

4. [Bonus 2 pts]

(a) Who are these guys?
Which one was the Mathematician?

(b) His proof of the Law of Quadratic Reciprocity in 1798 was ________. In 1801 Gauss did not state he was ________ L________'s result but rather claimed the result for ______ since he was ____________________.

(c) In three volumes in 1811, 1817, 1819 he published work on __________ functions.

3. (c) (ii)

First we show $\ln n = \sum_{d|n} \Lambda(d)$ for $n \geq 1$.

$\ln 1 = 0 = \Lambda(1)$ &

so it's true for $n = 1$. Let $n > 1$ &

$$n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$$

be a prime factorization.

$$\sum_{d|n} \Lambda(d) = \sum_{j=1}^{m} \sum_{e_j=1}^{a_j} \Lambda(p_j^{e_j})$$

(since the only nonzero terms come for $d = p_j^{e_j}$, $1 \leq j \leq m$, & $1 \leq e_j \leq a_j$).

$$= \sum_{j=1}^{m} \sum_{e_j=1}^{a_j} \ln p_j = \sum_{j=1}^{m} a_j \ln p_j$$

$$= a_1 \ln p_1 + a_2 \ln p_2 + \cdots + a_m \ln p_m$$

$$= \ln(p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}) = \ln n.$$

So

$$\ln n = \sum_{d|n} \Lambda(d) \text{ for all } n \geq 1.$$

By Möbius Inversion,

$$\Lambda(n) = \sum_{d|n} \mu(d) \ln\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu(d)(\ln n - \ln d)$$

$$= \sum_{d|n} \mu(d) \ln n - \sum_{d|n} \mu(d) \ln d$$

$$= \ln n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \ln d$$

$$= \cancel{(\ln 1) \cdot 1}\, 0 - \sum_{d|n} \mu(d) \ln d$$

$$\left(\text{since } \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases} \right.$$

$$\left. \& \ln 1 = 0 \right)$$

$$= - \sum_{d|n} \mu(d) \ln d.$$