INTRODUCTION TO NUMBER THEORY

MAS 4203 — FINAL EXAM - PART 1 — April 18, 2016

NAME:

When giving proofs make sure your reasoning is clearly. Show all necessary work. State clearly any results you used.
There are 5 questions of EQUAL value. Do FOUR complete questions. If you do more than four questions the best four complete questions will be taken. At the end there is also an optional 2 point bonus problem. Only basic and scientific calculators are allowed. A table of primes is supplied.

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1. $[1 + 2 + 4 + 3 = 10 \text{ pts}]$

(i) Let a, $b \in \mathbb{Z}$. DEFINE $a \mid b$.

(ii) COMPLETE the statement of the **Proposition:** Let a, $b \in \mathbb{Z}$ with a and b not __________________. Then

$(a, b) :=$ _____ $\{$____$\}$.

(iii) Let a, b, $c \in \mathbb{Z}$ with $(a, b) = 1$. Using the Proposition in (ii), PROVE that if $a \mid bc$ then $a \mid c$.

(iv) Use (iii) to PROVE **Euclid's Lemma:** Let a, b, $p \in \mathbb{Z}$ with p prime. If $p \mid (ab)$ then $p \mid a$ or $p \mid b$.

2. $[2 + 4 + 4 = 10 \text{ pts}]$

(i) DEFINE the term *prime*.

(ii) PROVE that every integer greater than one has a least one prime divisor.

(iii) PROVE that there are infinitely many primes.

3. $[1+4+1+4 = 10 \text{ pts}]$

(i) Let m, a, $b \in \mathbb{Z}$ with $m > 0$. DEFINE what it means to say that a *is congruent to* b *modulo* m.

(ii) PROVE the **Proposition:** Let m, a, b, c, $d \in \mathbb{Z}$ with $m \geq 1$. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $ac \equiv bd \pmod{m}$.

(iii) DEFINE the term *pseudoprime.*

(iv) PROVE that $561 = (3)(11)(17)$ is pseudoprime.

4. $[5+5 = 10 \text{ pts}]$

(i) Let p be a positive integer. PROVE that if $2^p - 1$ is prime then p is prime.

(ii) Let n be a positive integer. PROVE that if $2^n + 1$ is prime then n is a power of 2.

5. $[1+5+4 = 10 \text{ pts}]$

(i) DEFINE what it means to say that an arithmetic function $f(n)$ is *multiplicative.*

(ii) PROVE that the function

$$f(n) = \begin{cases} -1 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

is multiplicative.

(iii) Define $g(n)$ to be the number odd positive divisors of n minus the number of even positive divisors of n. EXPLAIN why $g(n)$ is multiplicative.

2 BONUS pts

(a) This is ___ ___ ___ ___ ___ ___ C ___ ___ ___
F ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

(b) D ___ ___ ___ ___ ___ ___ ___ ___ ___ ___e s
A ___ ___ ___ ___ ___ ___ ___ ___ ___a e is the title of his famous book on Number Theory.

(c) Let left G ___ ___ ___ ___ ___ ___e n in 1798 without a diploma, but by this time he had made one of his most important discoveries — the construction of the regular ___ ___-gon by ruler and compass.