MAS 4203 — Homework 3 — Spring 2017 (p.1)

Section 3.1

~~(3)~~ #4 (p.80)

Let $n \in \mathbb{Z}$ with $n > 0$. Define the arithmetic function ρ by $\rho(1) = 1$ and $\rho(n) = 2^m$ where m is the number of distinct prime numbers in the prime factoriztion of n.

(a) ρ is multiplicative but not completely multiplicative.

Proof. Let $m, n \in \mathbb{Z}$, $m, n \geq 1$ and suppose $(m, n) = 1$. We will show that

$$\rho(mn) = \rho(m)\rho(n).$$

The result is clearly true if m or $n = 1$. We assume $m, n > 1$. Suppose there are r distinct primes in the prime factorization of n and s distinct primes in the prime factorization of m. Then since m and n are relatively prime there are $r + s$ primes in the factorization of mn. Hence

$$\rho(mn) = 2^{r+s} = 2^r 2^s = \rho(m)\rho(n).$$

Finally we show that ρ is not completely multiplicative. Now $\rho(2) = 2$ and $\rho(2^2) = 2$ so that $\rho(2 \cdot 2) \neq \rho(2)\rho(2)$. □

(b) Let

$$n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$$

be the prime factorization of n. Then

$$f(n) := \sum_{d|n} \rho(d) = \prod_{i=1}^{m} (1 + 2a_i).$$

Proof. First let p be prime and suppose a is a positive integer. Then

$$\begin{aligned} f(p^a) &= \sum_{d|p^a} \rho(d) \\ &= \rho(1) + \rho(p) + \rho(p^2) + \cdots \rho(p^a) \\ &= 1 + 2 + 2 + \cdots 2 = 1 + 2a. \end{aligned}$$

Therefore

$$\begin{aligned} f(n) &= f(p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}) \\ &= f(p_1^{a_1}) f(p_2^{a_2}) \cdots f(p_m^{a_m}) \\ &\qquad \text{(since } \rho \text{ and hence } f \text{ are multiplicative by Theorem 3.1} \\ &= (1 + 2a_1)(1 + 2a_2) \cdots (1 + 2a_m) \\ &= \prod_{i=1}^{m} (1 + 2a_i). \end{aligned}$$

#6 Suppose $f: \mathbb{Z}^+ \to \mathbb{C}$ is completely multiplicative.
Let $n \in \mathbb{Z}$, with $n > 1$.
Let
$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$
be a prime factorization. Then

$$f(n) = f(p_1^{e_1}) f(p_2^{e_2}) \cdots f(p_r^{e_r})$$ (since f is multiplicative)

$$= (f(p_1))^{e_1} (f(p_2))^{e_2} \cdots (f(p_r))^{e_r}$$ (since f is completely multiplicative)

Therefore value of f at any integer > 1 is completely determined by its value at prime numbers.

NOTE: $f(1) = 0$ or 1 by ~~Ex3~~ result proved in class.
If $f(p) \neq 0$ for some prime p then $f(1) = 1$ by Ex 3.
If $f(p) = 0$ for all primes p then the value of f at 1 is not determined; $f(1) = 0$ or 1.

(p.3)

~~(4)~~ #7 (p.80)

Define $\lambda(n)$ by

$$\lambda(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = p_1 p_2 \cdots p_k \text{ where } p_1, p_2, \ldots p_k \\ & \text{are not necessarily distinct prime numbers} \end{cases}$$

Then

(a) $\lambda(n)$ is completely multiplicative.

Proof. Suppose m and n are positive integers. We show

$$\lambda(mn) = \lambda(m)\lambda(n).$$

If $m = 1$ then

$$\lambda(mn) = \lambda(n) = 1 \cdot \lambda(n) = \lambda(m)\lambda(n).$$

If $n = 1$ then the result holds similarly.

Now suppose m and n are great than 1. Suppose further that m is a product of k (not necessarily distinct) primes and that n is a product of ℓ (not necessarily distinct) primes. Then it is clear that mn is a product of $k + \ell$ (not necessarily distinct) primes and

$$\lambda(mn) = (-1)^{k+\ell} = (-1)^k(-1)^\ell = \lambda(m)\lambda(n).$$

Hence $\lambda(n)$ is completely multiplicative. $\square$

(b)

$$\sum_{d|n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a perfect square} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We define

$$F(n) = \sum_{d|n} \lambda(d)$$

for postive integers n. Since λ is multiplicative $F(n)$ is multiplicative by Theorem 3.1 (p.79).

Now let p be prime and a a positive integer. Then

$$\begin{aligned} F(p^a) &= \lambda(1) + \lambda(p) + \lambda(p^2) + \cdots + \lambda(p^a) \\ &= 1 - 1 + 1 - 1 + \cdots + (-1)^a. \end{aligned}$$

So when a is even we have

$$F(p^a) = (1-1) + (1-1) + \cdots + (1-1) + 1 = 1,$$

(p. 4)

and when a is odd we have

$$F(p^a) = (1-1) + (1-1) + \cdots + (1-1) = 0.$$

The result is clearly true when $n = 1$. It is also clear that n is a perfect square iff n has a prime factorization of the form

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

where each a_j is even. In this case we have

$$\begin{aligned} F(n) &= F(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) \\ &= F(p_1^{a_1}) F(p_2^{a_2}) \cdots F(p_k^{a_k}) \\ &\qquad \text{since } F \text{ is multiplicative} \\ &= 1 \cdot 1 \cdots 1 \\ &= 1. \end{aligned}$$

If n is not a perfect square then at least one of the a_j is odd, say a_{j0}. In this case we have

$$\begin{aligned} F(n) &= F(p_1^{a_1} p_2^{a_2} \cdots p_{j0}^{a_{j0}} \cdots p_k^{a_k}) \\ &= F(p_1^{a_1}) F(p_2^{a_2}) \cdots F(p_{j0}^{a_{j0}}) \cdots F(p_k^{a_k}) \\ &= F(p_1^{a_1}) F(p_2^{a_2}) \cdots 0 \cdots F(p_k^{a_k}) \\ &= 0. \end{aligned}$$

This completes the proof. $\square$

Section 3.2

(~~5~~) # 10(e),(g) (p.84)

(e) Let $n = 4851$. Then

$$\begin{aligned} n &= 3^2 7^2 11, \\ \phi(n) &= 4851 \prod_{p|n} (1 - \frac{1}{p}) \\ &= 3^2 7^2 11 (2/3)(6/7)(10/11) \\ &= (3)(7)(2)(6)(10) \\ &= 2520. \end{aligned}$$

(g)

$$\begin{aligned} 15! &= (15)(14)(13)(12)(11)(10)(9)(8)(7)(6)(5)(4)(3)(2)(1) \\ &= 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13. \end{aligned}$$

$$\begin{aligned} \phi(15!) &= (2^{11} - 2^{10})(3^6 - 3^5)(5^3 - 5^2)(7^2 - 7) \cdot 10 \cdot 12 \\ &= 2^{10} 3^5 (2) 5^2 (4) 7 (6) \cdot 10 \cdot 12 \\ &= 2^{17} \cdot 3^7 \cdot 5^3 \cdot 7 = 250822656000. \end{aligned}$$

(p. 5)

~~(6)~~ #14 (p.85)

(a) There are infinitely many integers n for which $\phi(n) = \frac{n}{3}$.

Proof. Let $n = 2^a 3^b$ where a and b are any positive integers. Then by Theorem 3.4,

$$\phi(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) = n(1 - \frac{1}{2})(1 - \frac{1}{3}$$
$$= n \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{n}{3}.$$

Therefore there are infinitely many integers n for which $\phi(n) = \frac{n}{3}$. □

(b) There are no integers n for which $\phi(n) = \frac{n}{4}$.

Proof. Suppose (by way of contradiction) that $\phi(n) = \frac{n}{4}$ for some positive integer n. Then by Theorem 3.4, we have

$$\prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) = \prod_{\substack{p|n \\ p \text{ prime}}} \frac{(p-1)}{p} = \frac{1}{4},$$

and

$$4 \prod_{p|n} (p-1) = \prod_{p|n} p.$$

This implies $2^2 \mid \prod_{p|n} p$ which is impossible since the highest power of 2 that could divide $\prod_{p|n} p$ is 2^1 by unique prime factorization. We have a contradiction and so there are no integers n for which $\phi(n) = \frac{n}{4}$. □

~~Homework 6~~

(1) #15, p. 85.

Suppose $\phi(n) = k$ where $n \geq 2$.

Suppose p is prime & $p \mid n$ then by #12 $(p-1) \mid k$ & $p-1 \leq k$, $p \leq k+1$.

Suppose $p^a \mid n$ then by #12

$$p^{a-1} \mid k,\ \&$$

$$(a-1)\ln 2 \leq (a-1)\ln p = \ln(p^{a-1}) \leq \ln k,\ \&$$

$$a \leq \frac{\ln k}{\ln 2} + 1.$$

Let $n = p_1^{a_1} \cdots p_m^{a_m}$ be prime fact.

Then $\prod_{i=1}^{m}(p_i - 1) \mid \phi(n) = k$ by #12,

$$2^{m-1} \leq \prod_{i=1}^{m}(p_i - 1) \leq k \ \&$$

$$(m-1)\ln 2 \leq \ln k \ \&\ m \leq \frac{\ln k}{\ln 2} + 1.$$

Therefore

$$n = p_1^{a_1} \cdots p_m^{a_m} \leq \left((k+1)^{\frac{\ln k}{\ln 2}+1}\right)^m$$

$$\leq (k+1)^{\left(1+\frac{\ln k}{\ln 2}\right)^2},$$

n is bounded & ~~thef~~ therefore there are only finitely many possible n.

16(a) (Ch3, p.85) Let n be a positive integer. Then

$$\frac{\sqrt{n}}{2} \leq \phi(n) \leq n.$$

Proof:Let n be a positive integer. Since

$$\mathbb{Z}_n^\times = \{m : 1 \leq m \leq n, (m,n) = 1\} \subset \{1, 2, \ldots n\}$$

we have

$$\phi(n) = \left|\mathbb{Z}_n^\times\right| \leq n.$$

Next we show that

$$\frac{\sqrt{n}}{2} \leq \phi(n).$$

First, we prove that for $a \geq 1$

$$\phi(p^a) \geq \begin{cases} p^{a/2}, & \text{if } p \text{ is an odd prime,} \\ \frac{1}{2}p^{a/2}, & \text{if } p = 2. \end{cases} \quad (*)$$

Let p be an odd prime and suppose $a \geq 1$. Then

$$p \geq \frac{\sqrt{3}}{\sqrt{3}-1} \approx 2.4,$$

so that $\sqrt{3} \leq \sqrt{3}p - p$, $p \leq \sqrt{3}(p-1)$ and

$$\frac{p}{p-1} \leq \sqrt{3}.$$

Since p is odd

$$p^{a/2} \geq 3^{1/2} \geq \frac{p}{p-1}$$

$$\left(\frac{p-1}{p}\right) p^{a/2} \geq 1$$

$$\left(1 - \frac{1}{p}\right) p^a \geq p^{a/2}$$

$$p^a - p^{a-1} \geq p^{a/2}$$

and by Theorem 3.3 we have

$$\phi(p^a) = p^a - p^{a-1} \geq p^{a/2}.$$

Let $p = 2$ and suppose $a \geq 1$. Then $a \geq a/2$, $a - 1 \geq (a/2) - 1$ and

$$\phi(2^a) = 2^a - 2^{a-1} = 2^{a-1} \geq 2^{a/2-1} = \frac{1}{2}2^{a/2}.$$

Hence (*) holds.

The result is clearly true for $n = 1$. Now suppose $n > 1$ with prime factorization

$$n = 2^a p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

(p.8)

where $a \geq 0$. We note that (*) also holds when $a = 0$. So using (*) and the fact that ϕ is multiplicative we have

$$\begin{aligned}\phi(n) &= \phi(2^a)\phi(p_1^{a_1})\phi(p_2^{a_2})\cdots\phi(p_r^{a_r})\\ &\geq \frac{1}{2}2^{a/2}p_1^{a_1/2}p_2^{a_2/2}\cdots p_r^{a_r/2}\\ &= \frac{1}{2}\sqrt{n}. \quad \square\end{aligned}$$

16(b) (Ch 3, p.85) If n is composite, then

$$\phi(n) \leq n - \sqrt{n}.$$

<u>Proof</u>:Suppose n is composite. Then n has some prime divisor $p' \leq \sqrt{n}$, by Prop 1.7 (p.11). We have

$$\begin{aligned}\frac{1}{p'} &\geq \frac{1}{\sqrt{n}},\\ -\frac{1}{p'} &\leq -\frac{1}{\sqrt{n}},\\ 1-\frac{1}{p'} &\leq 1-\frac{1}{\sqrt{n}}.\end{aligned}$$

By Theorem 3.4 we have

$$\begin{aligned}\phi(n) &= n\prod_{p|n}(1-\frac{1}{p})\\ &\leq n(1-\frac{1}{p'}) \quad \text{(since each factor } (1-1/p) \leq 1)\\ &\leq n(1-\frac{1}{\sqrt{n}}) = n - \sqrt{n}.\end{aligned}$$

So

$$\phi(n) \leq n - \sqrt{n}. \quad \square$$

Section 3.3

30 (e)

$n = 4851 = 3^2 \cdot 7^2 \cdot 11^1.$

So

$\nu(n) = 3 \cdot 3 \cdot 2 = 18$ by Thm 3.?.

(g)

$n = 15! = 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1$

$\nu(n) = 12 \cdot 7 \cdot 4 \cdot 3 \cdot 2 \cdot 2$

$= 4032.$

32 (Ch 3, p.88) $\nu(n)$ is odd if and only if n is a perfect square.

Proof:Let

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

be the prime factorization of n where the $a_i \geq 0$. Then by Theorem 3.8 we have

$$\nu(n) = \prod_{i=1}^{r} (a_i + 1).$$

Now $\nu(n)$ is odd iff each factor $(a_i + 1)$ is odd;i.e. a_i is even for all i. Hence $\nu(n)$ is odd if and only if n is a perfect square. □

34 (Ch 3, p.88) Let $k \in \mathbb{Z}$ with $k > 1$. The equation

$$\nu(n) = k \qquad \text{(A)}$$

has infinitely many solutions.

Proof:Let $k \in \mathbb{Z}$ with $k > 1$. Let $n = p^{k-1}$, where p is any prime. Then by Theorem 3.7

$$\nu(n) = \nu(p^{k-1}) = (k-1) + 1 = k,$$

and $n = p^{k-1}$ is a solution to (A). Since there are infinitely many primes p the equation (A) has infinitely many solutions. □

#37, p 89.

Let $n \in \mathbb{Z}$, $n > 0$.

Case 1 $n = 1$. $\nu(1) = 1 \leqslant 2\sqrt{1} = 2$.

Case 2 $n = p$ where p is prime. Then

$$\nu(p) = 2 \leqslant 2\sqrt{p} \quad \text{since } p \geqslant 2 > 1 \text{ \& } \sqrt{p} > 1.$$

Case 3 n is composite.

Let d be any positive divisor of n.

Then $n = de$ some $e \in \mathbb{Z}$.

$e = \frac{n}{d}$ is also a positive divisor of n.

We may regroup the positive divisors of n in pairs d & $\frac{n}{d}$ unless $d = n/d$ & $d^2 = n$ & $d = \sqrt{n} \in \mathbb{Z}$.

If $d \leq e$ then $d^2 \leq ed = n$ & $d \leqslant \sqrt{n}$.

Hence # of pairs $< \sqrt{n}$. and

$$\nu(n) \leqslant 2\sqrt{n}.$$

~~(5) #45, p 91.~~

~~Let $k \in \mathbb{Z}$, $k > 0$; and suppose~~

~~(*) $\sigma(n) = k$~~

~~where n is an integer > 1.~~

~~Then $1, n$ are positive divisors of n so that~~

~~$k = \sigma(n) \geqslant 1 + n$, &~~

~~$n \leqslant k - 1$.~~

~~Hence (*) has finitely many solutions.~~

Section 3.4

#46. Let $n>0$, $n \in \mathbb{Z}$

Since n is a positive divisor of n

$$n \leqslant \sum_{d|n} d = \sigma(n).$$

Any positive divisor d ~~such~~ of n satisfies

$$1 \leqslant d \leqslant n$$

Hence

$$\sigma(n) = \sum_{d|n} d \leqslant \sum_{d=1}^{n} d$$

$$\leqslant \sum_{d=1}^{n} n \qquad (\text{since each } d \leq n)$$

$$= n \sum_{d=1}^{n} 1 = n \cdot n = n^2.$$

Therefore

$$n \leqslant \sigma(n) \leqslant n^2.$$

47 (Ch3, p.91) Let $n \in \mathbb{Z}$ with $n > 0$. Then

$$\sum_{\substack{d|n \\ d>0}} \frac{1}{d} = \frac{\sigma(n)}{n}.$$

.

Proof:Let $n \in \mathbb{Z}$ with $n > 0$. As d runs through the positive divisors of n so does n/d. Hence

$$\sigma(n) = \sum_{\substack{d|n \\ d>0}} d = \sum_{\substack{d|n \\ d>0}} \frac{n}{d} = n \sum_{\substack{d|n \\ d>0}} \frac{1}{d}.$$

Dividing by n we obtain the result. $\square$

(6) #51 (p. 91)

(a) $\sigma_3(12) = \sum_{d|12} d^3$

$= 1^3 + 2^3 + 3^3 + 4^3 + 6^3 + 12^3$

$= 2044$

$\sigma_3(8) = \sum_{d|8} d^3$

$= 1^3 + 2^3 + 4^3 + 8^3$

$= 585$

Let k be a positive integer

(b) The function

$f(n) = n^k$ is completely multiplicative

since $f(mn) = (mn)^k = m^k n^k = f(m)f(n)$

if m, n are positive integers.

Hence

$\sigma_k(n) = \sum_{d|n} d^k = \sum_{d|n} f(d)$

is multiplicative by Theorem 3.1.

(c) Let p be prime & a be a positive integer.

Then

$\sigma_k(p^a) = \sum_{d|p^a} d^k$

$= 1 + p^k + p^{2k} + \cdots + p^{ak}$

(p.15)

$$x^{a+1} - 1 = (x-1)(x^a + x^{a-1} + \cdots + x + 1) \ \&$$

$$1 + x + x^2 + \cdots + x^a = \frac{x^{a+1} - 1}{x - 1} \quad \text{for } x \neq 1.$$

Hence by letting $x = p^k$ we see that

$$\sigma_k(p^a) = \frac{p^{k(a+1)} - 1}{p^k - 1}.$$

(d) Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be a prime factorization. Then

$$\begin{aligned} \sigma_k(n) &= \sigma_k(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}) \\ &= \sigma_k(p_1^{a_1})\, \sigma_k(p_2^{a_2}) \cdots \sigma_k(p_r^{a_r}) \quad \text{(since } \sigma_k(n) \text{ is multiplicative)} \\ &= \prod_{j=1}^{r} \frac{(p_j^{k(a_j+1)} - 1)}{(p_j^k - 1)}. \end{aligned}$$

Section 3.5

~~n/d. Hence~~

$$\cancel{\sigma(n) = \sum_{\substack{d|n\\ d>0}} d = \sum_{\substack{d|n\\ d>0}} \frac{n}{d} = n\sum_{\substack{d|n\\ d>0}} \frac{1}{d}.}$$

~~Dividing by n we obtain the result.~~ $\square$

\# 53 p.95 Let n be a perfect number. Then

$$\sum_{\substack{d|n\\ d>0}} \frac{1}{d} = 2.$$

<u>Proof</u>:Let n be a perfect number. Then

$$\sigma(n) = \sum_{\substack{d|n\\ d>0}} d = 2n.$$

From the previous problem we have

$$\sum_{\substack{d|n\\ d>0}} \frac{1}{d} = \frac{\sigma(n)}{n} = \frac{2n}{n} = 2. \quad \square$$

\# 55 p.95 Let $n_1, n_2, \ldots, n_m$ be distinct even perfect numbers. Then

$$\phi(n_1 n_2 \cdots n_m) = 2^{m_1}\phi(n_1)\phi(n_2)\cdots\phi(n_m).$$

<u>Proof</u>:Let $n_1, n_2, \ldots, n_m$ be distinct even perfect numbers. Then

$$n_i = 2^{p_i-1}(2^{p_i}-1) \qquad \text{(by Theorem 3.12)},$$

for each i, where each p_i is prime and $(2^{p_i}-1)$ is prime. It is clear that the p_i are distinct and hence the $2^{p_i}-1$ are distinct primes. Now, for each i we have

$$\begin{aligned}
\phi(n_i) &= \phi(2^{p_i-1}(2^{p_i}-1)) \\
&= \phi(2^{p_i-1})\phi(2^{p_i}-1) \qquad \text{(since } \phi \text{ is multiplicative and the two factors are relatively prime} \\
&= (2^{p_i-1}-2^{p_i-2})\phi(2^{p_i}-1) \\
&= 2^{p_i-2}\phi(2^{p_i}-1).
\end{aligned}$$

We have

$$\begin{aligned}
\phi(n_1 n_2 \cdots n_m) &= \phi(2^{p_1-1}(2^{p_1}-1)2^{p_2-1}(2^{p_2}-1)\cdots 2^{p_m-1}(2^{p_m}-1)) \\
&= \phi(2^{p_1+p_2+\cdots+p_m-m})(2^{p_1}-1)(2^{p_2}-1)\cdots(2^{p_m}-1)) \\
&= \phi(2^{p_1+p_2+\cdots+p_m-m})\phi(2^{p_1}-1)\phi(2^{p_2}-1)\cdots\phi(2^{p_m}-1) \quad \text{(since } \phi \text{ is multiplicativ} \\
&= 2^{p_1+p_2+\cdots+p_m-m-1}\phi(2^{p_1}-1)\phi(2^{p_2}-1)\cdots\phi(2^{p_m}-1) \\
&= 2^{(p_1-2)+(p_2-2)+\cdots+(p_m-2)+m-1}\phi(2^{p_1}-1)\phi(2^{p_2}-1)\cdots\phi(2^{p_m}-1) \\
&= 2^{m-1}2^{p_1-2}\phi(2^{p_1}-1)2^{p_2-2}\phi(2^{p_2}-1)\cdots 2^{p_m-2}\phi(2^{p_m}-1) \\
&= 2^{m-1}\phi(n_1)\phi(n_2)\cdots\phi(n_m),
\end{aligned}$$

by (B), and this gives the desired result. $\square$

(8) #59, p. 95

(a) $\sigma(16) = \dfrac{2^5 - 1}{2 - 1} = 31.$

$\sigma(31) = 1 + 31 = 32 = 2 \cdot 16.$

$\therefore\ \sigma(\sigma(16)) = 2 \cdot 16$ & 16 is superperfect.

(b) Suppose $2^p - 1$ is Mersenne prime.

$\sigma(2^{p-1}) = \dfrac{2^p - 1}{2 - 1} = 2^p - 1.$

$\sigma(2^p - 1) = 1 + (2^p - 1)$ (since $2^p - 1$ is prime)

$= 2^p = 2 \cdot 2^{p-1}.$

Therefore

$\sigma(\sigma(2^{p-1})) = 2 \cdot 2^{p-1}$ & 2^{p-1} is superperfect.

(c) Suppose 2^a is superperfect. Then

$\sigma(\sigma(2^a)) = 2^{a+1}.$

But $\sigma(2^a) = \dfrac{2^{a+1} - 1}{1} = 2^{a+1} - 1.$

Therefore

$\sigma(2^{a+1} - 1) = 2^{a+1}.$

Then $2^{a+1} - 1$ is prime & hence a Mersenne prime

since if $2^{a+1} - 1$ is not prime

$\sigma(2^{a+1} - 1) > 1 + 2^{a+1} - 1 = 2^{a+1}$.