If we consider a θ-function in the transformed Eulerian form e.g.

(A) $1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}$,

(B) $1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)}$

and determine the nature of the singularities at the points $q = 1$, $q^2=1$, $q^3=1$, $q^4=1$, $q^5=1$, ... we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance when $q = e^{-t}$ and $t \to 0$

$$(A) = \sqrt{\frac{t}{2\pi}}\, e^{\frac{\pi^2}{6t} - \frac{t}{24}}\,{}^{*} + o(1)^{\dagger}$$

$$(B) = \frac{e^{\frac{\pi^2}{15t} - \frac{t}{60}}\,{}^{*}}{\sqrt{\frac{5-\sqrt{5}}{2}}} + o(1)^{\dagger}$$

and similar results at other singularities. * It is not necessary that there should be only one term like this. There may be many terms but the number of terms must be finite. † Also $o(1)$ may turn out to be $O(1)$. That is all. For instance when $q \to 1$ the function

$$\left\{\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}\right\}^{120}$$

is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

If we take a number of functions like (A) and (B) it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance when $q = e^{-t}$ and $t \to 0$

(C) $1 + \frac{q}{(1-q)^2} + \frac{q^3}{(1-q)^2(1-q^2)^2} + \frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2}$

$$= \sqrt{\left(\frac{t}{2\sqrt{5}}\right)}\, e^{\frac{\pi^2}{5t} + a_1 t + a_2 t^2 + \cdots + O(a_k t^k)}$$

where $a_1 = \frac{1}{8\sqrt{5}}$, and so on.

(2)

The function (C) is a simple example of a function behaving in an unclosed form at the singularities.

* The coeff^t. $\frac{1}{t}$ in the index of e happens to be $\frac{\pi^2}{5}$ in this particular case. It may be some other transcendental numbers in other cases.

† The coeff^{ts} of t, t^2, ... happen to be $\frac{1}{8\sqrt{5}}$, ... in this case. In other cases they may turn out to be some other algebraic numbers.

Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{\frac{2i\pi m}{n}}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is:— is the function taken the sum of two functions one of which is an ordinary ϑ function and the other a (trivial) function which is $O(1)$ at all the points $e^{\frac{2i\pi m}{n}}$? The answer is it is not necessarily so. When it is not so I call the function Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is not inconceivable to construct a ϑ function to cut out the singularities.

 (3)

of the original function. Also I have shown if it is necessarily so then it leads to the following assertion;— viz. it is possible to construct two power series in x namely $\sum_0^\infty a_n x^n$ and $\sum b_n x^n$ both of which have essential singularities on the unit circle, ~~while $\sum_0^\infty a_n x^n$ exists inside the circle and is regular there and $\sum_0^\infty b_n x^n$ ... one~~ are convergent when $|x|<1$, and tend to finite limits at every point $x = e^{2i\pi r/s}$ and that at the same time the limit of $\sum_0^\infty a_n x^n$ at the point $x = e^{2i\pi r/s}$ is equal to the limit of $\sum_0^\infty b_n x^n$ at the point $x = e^{-2i\pi r/s}$.

This assertion seems to be untrue. Any how we shall go to the examples and see how far our assertions are true.

I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

then

$$f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)$$

~~at all the~~ $= O(1)$

at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1$, ...;

and at the same time

$$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots) = O(1)$$

at all the points $q^2=-1$, $q^4=-1$, $q^6=-1$, ...

Also obviously $f(q) = O(1)$ at all the points $q=1$, $q^3=1$, $q^5=1$

(5)

And so $f(q)$ is a Mock ϑ function.

When $q = -e^{-t}$ and $t \to 0$

$$f(q) + \sqrt{\frac{\pi}{t}}\, e^{\frac{\pi^2}{24t} - \frac{t}{24}} \to 4.$$

The coefft of q^n in $f(q)$ is

$$(-1)^{n-1} \frac{e^{\pi\sqrt{\frac{n}{6} - \frac{1}{144}}}}{2\sqrt{n - \frac{1}{24}}} + O\left(\frac{e^{\frac{\pi}{2}\sqrt{\frac{n}{6} - \frac{1}{144}}}}{\sqrt{n - \frac{1}{24}}}\right).$$

It is inconceivable that a single ϑ function could be found to cut out the singularities of $f(q)$.

Mock ϑ-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \dots$$

~~These are~~

$$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \dots$$

These are ~~substantia~~ related to $f(q)$ as shown below.

$$2\phi(-q) - f(q) = f(q) + 4\psi(-q) = \frac{1 - 2q + 2q^4 - 2q^9 + \dots}{(1+q)(1+q^2)(1+q^3)\dots}$$

$$4\chi(q) - f(q) = \frac{(1 - 2q^3 + 2q^{12} - \dots)^2}{(1-q)(1-q^2)(1-q^3)\dots}$$

These are of the 3rd order

Mock ϑ-functions (of 5th order)

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q^2)(1+q^3)} + \dots$$

$$\phi(q) = 1 + q(1+q) + q^4(1+q)(1+q^3) + q^9(1+q)(1+q^3)(1+q^5) + \dots$$

$$\psi(q) = q + q^3(1+q) + q^6(1+q)(1+q^2) + q^{10}(1+q)(1+q^2)(1+q^3) + \dots$$

$$\chi(q) = 1 + \frac{q}{1-q^2} + \frac{q^2}{(1-q^3)(1-q^4)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)} + \dots$$

$$= 1 + \left\{\frac{q}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^3}{(1-q^3)(1-q^4)(1-q^5)} + \dots\right\}$$

$$F(q) = 1 + \frac{q^2}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \cdots$$

$$\phi(-q) + \chi(q) = 2F(q).$$

$$f(q) + 2F(q^2) - 2 = \phi(-q^2) + \psi(q)$$

$$= 2\phi(-q^2) - f(q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}$$

$$\psi(q) - F(q^8) + 1 = q \cdot \frac{1 + q^2 + q^6 + q^{12} + \cdots}{(1-q^8)(1-q^{12})(1-q^{28})\cdots}$$

Mock θ-functions (of 5th order)

$$f(q) = 1 + \frac{q^2}{1+q} + \frac{q^6}{(1+q)(1+q^2)} + \frac{q^{12}}{(1+q)(1+q^2)(1+q^3)} + \cdots$$

$$\phi(q) = q + q^4(1+q) + q^9(1+q)(1+q^2) + \cdots$$

$$\psi(q) = 1 + q(1+q) + q^3(1+q)(1+q^2) + q^6(1+q)(1+q^2)(1+q^3) \cdots$$

$$\chi(q) = \frac{1}{1-q} + \frac{q}{(1-q^2)(1-q^3)} + \frac{q^2}{(1-q^3)(1-q^4)(1-q^5)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)(1-q^7)}$$

$$F(q) = \frac{1}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^{12}}{(1-q)(1-q^3)(1-q^5)}$$

have got similar relations as above.

Mock θ-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}$

(ii) $\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^6}{(1-q^3)(1-q^4)(1-q^5)}$

These are not related to each other.

Ever yours sincerely
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