Theorem 0.1 (Theorem 1.10)**.** *Let V be a vector space and suppose $\mathcal{G}$ and $\mathcal{L}$ are finite subsets of V such that*

$$\begin{aligned} &V = Span(\mathcal{G}), && |\mathcal{G}| = n, \\ &\mathcal{L} \text{ is linearly independent, and} && |\mathcal{L}| = m. \end{aligned}$$

Then $m \le n$ and there is a set $\mathcal{H} \subset \mathcal{G}$, such that $|\mathcal{H}| = n - m$ and $Span(\mathcal{H} \cup \mathcal{L}) = V$.

Proof. We proceed by induction on m.

If $m = 0$ then $\mathcal{L} = \phi$ and we let $\mathcal{H} = \mathcal{G}$ so that $m = 0 \le n$ and $\text{Span}(\mathcal{H} \cup \mathcal{L}) = \text{Span}(\mathcal{G}) = V$.

Now suppose the statement is true for $m = \mu$, where μ is a fixed nonnegative integer. We assume that

$$\mathcal{L} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_\mu, \vec{v}_{\mu+1}\}$$

is linearly independent. Then the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_\mu\}$ is linearly independent. So by the induction hypothesis there is a subset $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{n-\mu}\} \subset \mathcal{G}$ such that

$$\text{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{n-\mu}, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_\mu) = V.$$

Hence

$$\vec{v}_{\mu+1} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_{n-\mu}\vec{u}_{n-\mu} + b_1\vec{v}_1 + b_1\vec{v}_2 + \cdots + b_\mu\vec{v}_\mu,$$

for some scalars a_1, a_2, $\dots$, $a_{n-\mu}$, b_1, b_2, $\dots$, b_μ. We note that $n-\mu > 0$ since otherwise $\vec{v}_{\mu+1}$ would be a linear combination of $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_\mu$ would contradict $\mathcal{L}$ being linearly independent.

Therefore $n-\mu \ge 1$ and $n \ge \mu+1$. Similarly at least of the scalars a_1, a_2, $\dots a_{n-\mu}$ must be nonzero since $\mathcal{L}$ is linearly independent. Suppose without loss of generality that $a_1 \ne 0$. Then

$$\begin{aligned} \vec{u}_1 = {} & (-a_2/a_1)\vec{u}_2 + \cdots + (-a_{n-\mu}/a_1)\vec{u}_{n-\mu} \\ & + (-b_1/a_1)\vec{v}_1 + \cdots + (-b_\mu)/a_1)\vec{v}_\mu + (1/a_1)\vec{v}_{\mu+1}). \end{aligned}$$

We let

$$\mathcal{H} = \{\vec{u}_2, \dots, \vec{u}_{n-\mu}\}.$$

Then

$$\vec{u}_1 \in \text{Span}(\mathcal{H} \cup \mathcal{L}),$$

and

$$\{\vec{u}_1, \dots, \vec{u}_{n-\mu}, \vec{v}_1, \dots, \vec{v}_\mu\} \subset \text{Span}(\mathcal{H} \cup \mathcal{L}),$$

$$V = \text{Span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{n-\mu}, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_\mu) \subset \text{Span}(\mathcal{H} \cup \mathcal{L}) \subset V,$$

and

$$V = \text{Span}(\mathcal{H} \cup \mathcal{L}), \quad \mathcal{H} \subset \mathcal{G}, |\mathcal{H}| = (n - \mu) - 1 = n - (\mu + 1),$$

and the theorem is true for $m = \mu + 1$. Hence the theorem is true for all m by induction. $\square$

$\square$