The Replacement Theorem

Theorem (Theorem 1.10)

Let V be a vector space and suppose $\mathcal{G}$ and $\mathcal{L}$ are finite subsets of V such that

$$\begin{aligned} &V = Span(\mathcal{G}), && |\mathcal{G}| = n, \\ &\mathcal{L} \text{ is linearly independent, and} && |\mathcal{L}| = m. \end{aligned}$$

Then $m \leq n$ and there is a set $\mathcal{H} \subset \mathcal{G}$, such that $|\mathcal{H}| = n - m$ and $Span(\mathcal{H} \cup \mathcal{L}) = V$.

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Proof

We proceed by induction on m.
If $m = 0$ then $\mathcal{L} = \phi$ and we let $\mathcal{H} = \mathcal{G}$ so that $m = 0 \leq n$ and $\text{Span}(\mathcal{H} \cup \mathcal{L}) = \text{Span}(\mathcal{G}) = V$.
Now suppose the statement is true for $m = \mu$, where μ is a fixed nonnegative integer. We assume that

$$\mathcal{L} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_\mu, \vec{v}_{\mu+1}\}$$

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for some scalars $a_1, a_2, \ldots, a_{n-\mu}, b_1, b_2, \ldots, b_\mu$. We note that $n - \mu > 0$ since otherwise $\vec{v}_{\mu+1}$ would be a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_\mu$ would contradict $\mathcal{L}$ being linearly independent. Therefore $n - \mu \geq 1$ and $n \geq \mu + 1$. Similarly at least of the scalars $a_1, a_2, \ldots a_{n-\mu}$ must be nonzero since $\mathcal{L}$ is linearly independent. Suppose without loss of generality that $a_1 \neq 0$. Then

$$\begin{aligned}\vec{u}_1 = & (-a_2/a_1)\vec{u}_2 + \cdots + (-a_{n-\mu}/a_1)\vec{u}_{n-\mu} \\ & + (-b_1/a_1)\vec{v}_1 + \cdots + (-b_\mu)/a_1)\vec{v}_\mu + (1/a_1)\vec{v}_{\mu+1}).\end{aligned}$$

Proof (continued)

We let

$$\mathcal{H} = \{\vec{u}_2, \ldots, \vec{u}_{n-\mu}\}.$$

Then

$$\vec{u}_1 \in \mathrm{Span}(\mathcal{H} \cup \mathcal{L}),$$

and

$$\{\vec{u}_1, \ldots, \vec{u}_{n-\mu}, \vec{v}_1, \ldots, \vec{v}_\mu\} \subset \mathrm{Span}(\mathcal{H} \cup \mathcal{L}),$$

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$$V = \mathrm{Span}(\mathcal{H} \cup \mathcal{L}), \quad \mathcal{H} \subset \mathcal{G}, |\mathcal{H}| = (n-\mu) - 1 = n - (\mu + 1),$$

and the theorem is true for $m = \mu + 1$. Hence the theorem is true for all m by induction. □

Proof (continued)

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