1. (4 pts) For the following differential equations, determine their order and whether they are linear or nonlinear.

   a) \( y^{(6)} - y' x = \frac{y}{x} + \cos x \)
      linear of order 6

   b) \( \frac{d^4 y}{dx^4} + \frac{1}{y} \frac{dy}{dx} + \frac{y}{x} = 0 \)
      nonlinear of order 4

2. (10 pts) Solve the differential equation

\[
\frac{dy}{dx} \cdot \frac{x^2 - 1}{y - 1} = x
\]

This equation is separable, and we can rewrite it as

\[
\frac{dy}{y - 1} = \frac{x \, dx}{x^2 - 1}
\]

Integrating both sides (using a u-substitution of \( u = x^2 - 1 \) on the right side) we get

\[
\ln(y - 1) = \frac{1}{2} \ln(x^2 - 1) + C
\]

We can solve this explicitly for \( y \):

\[
y = C \sqrt{x^2 - 1} + 1
\]
3. (12 pts) Solve the initial value problem

\[ \frac{dy}{dx} + 4y = e^{-x} \quad y(0) = 1 \]

This is a linear equation with \( P(x) = 4 \) and \( Q(x) = e^{-x} \). So we first find the integrating factor

\[ \mu(x) = e^{\int 4 \, dx} = e^{4x} \]

and then find a general solution:

\[ y = e^{-4x} \left[ \int e^{3x} \, dx + C \right] = e^{-4x} \left[ \frac{1}{3} e^{-3x} + C \right] = \frac{1}{3} e^{-x} + Ce^{-4x} \]

To solve the initial value problem, \( y(0) = 1 = \frac{1}{3} + C \) so \( C = \frac{2}{3} \) and the solution is

\[ y = \frac{1}{3} e^{-x} + \frac{2}{3} e^{-4x} \]
4. (10 pts) Solve the differential equation

\[ \frac{dy}{dx} = \frac{-x^2 - y^2}{2xy} \]

If we rewrite the equation as

\[ \frac{dy}{dx} = -\frac{1}{2} \left( \frac{x}{y} + \frac{y}{x} \right) \]

we see that this equation is homogeneous, and we can make the substitution \( v = \frac{y}{x} \), so \( \frac{dv}{dx} = v + x \frac{dy}{dx} \), and the equation becomes

\[ v + x \frac{dv}{dx} = -\frac{1}{2} \left( \frac{1}{v} + v \right) \]

This equation is separable and after separating the variables we get

\[ \frac{v \, dv}{1 + 3v^2} = -\frac{1}{2x} \, dx \]

After integrating both sides and applying the exponential function, it turns into

\[ 3v^2 + 1 = Cx^{-3} \]

Then we have to rewrite the equation in terms of \( y, x \). After rearranging a little, this yields the solution

\[ 3y^2 x + x^3 = C \]

There is another way to find a solution in this problem. If we write the equation as

\[ (x^2 + y^2) \, dx + 2xy \, dy = 0 \]

we can see that the equation is exact. Then integrate both terms to find the solution

\[ y^2 x + \frac{x^3}{3} = C \]

which, after multiplying by 3, is the solution we obtained via substitution.
5. (10 pts) Solve the differential equation

\[ x \cdot \frac{dy}{dx} + 3(y + x^2) = \frac{\sin x}{x^2} \]

This equation is linear, and we can write it in its standard form as

\[ y' + \frac{3y}{x} = \frac{\sin x}{x^3} - 3x \]

So \( P(x) = \frac{3}{x} \) and \( Q(x) = \frac{\sin x}{x^3} - 3x \). So find the integrating factor

\[ \mu(x) = e^{\int \frac{3}{x} \, dx} = x^3 \]

and then we can find the solution

\[ y = x^{-3} \left[ \int \sin x - 3x^4 \, dx + C \right] = x^{-3} \left[ -\cos x + \frac{3x^5}{5} + C \right] = -\frac{\cos x}{x^3} - \frac{3x^2}{5} + C \]
6. (10 pts) Determine whether or not the following differential equations are exact.

a) \((2y^2 + 2x \cos y)dx + (4xy - x^2 \sin y)dy = 0\)

The equation is exact, with both partial derivatives equal to \(4y - 2x \sin y\).

b) \((3x^2 + y)dx + (6xy + \frac{y^2}{2})dy = 0\)

The equation isn’t exact since \(\frac{\partial}{\partial y}(3x^2 + y) = 1\) while \(\frac{\partial}{\partial x}(6xy + \frac{y^2}{2}) = 6y\).
7. (10 pts) Solve the differential equation

\[ 2xy = (3x^2 - y^2) \frac{dy}{dx} \]

Rewrite this as

\[ 2xy \, dx + (y^2 - 3x^2) \, dy \]

and let \( M(x, y) = 2xy \) and \( N(x, y) = (y^2 - 3x^2) \). Then \( \frac{\partial M}{\partial y} = 2x \) and \( \frac{\partial N}{\partial x} = -6x \) so the equation isn’t exact. But we can look for an integrating factor. See that

\[ \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \cdot \frac{1}{M} = \frac{-6x - 2x}{2xy} = -\frac{4}{y} \]

Since this is a function of only \( y \) we can find an integrating factor

\[ \mu(y) = e^{\int \frac{-4}{y} dy} = y^{-4} \]

Multiplying by this \( \mu \) we get an exact equation

\[ \frac{2x}{y^3} \, dx + \left( \frac{1}{y^2} - \frac{3x^2}{y^4} \right) \, dy = 0 \]

Integrating both terms we find the solution

\[ \frac{x^2}{y^3} - \frac{1}{y} = C \]
8. (12 pts) Solve the initial value problem

\[(e^t y + t e^t y) dt + (t e^t + 2) dy = 0 \quad y(0) = -1\]

This equation is exact, with both partial derivatives equal to \(e^t + t e^t\). Now integrate both terms to get

\[\int t e^t + 2 \, dy = t e^t y + 2y + h(t)\]

and

\[\int e^t y + t e^t \, dt = t e^t y + g(y)\]

where the second integral requires integration by parts. So a solution is

\[t e^t y + 2y = C\]

Using the initial conditions \(t = 0, y = -1\) we find that \(C = -2\) so the solution to the IVP is

\[t e^t y + 2y = -2\]
9. (10 pts) Solve the differential equation

\[ 4y'' - 4y' + y = 0 \]

The auxiliary equation is \( 4r^2 - 4r + 1 = 0 \) which has a repeated root of \( r = \frac{1}{2} \). So the general solution is

\[ y = c_1 e^{t/2} + c_2 te^{t/2} \]
10. (12 pts) Solve the initial value problem

\[ y'' + 3y' - 10y = 0 \quad y(0) = 1 \quad y'(0) = 0 \]

The auxiliary equation is \( r^2 + 3r - 10 = 0 \) which has roots \( r = 2, -5 \). So the general solution is

\[ y = c_1 e^{2t} + c_2 te^{-5t} \]

To solve the IVP we need to first take the derivative, \( y' = 2c_1 e^{2t} - 5c_2 e^{-5t} \). Using the initial conditions we get the two equations \( y(0) = 1 + c_1 + c_2 \) and \( y'(0) = 0 = 2c_1 - 5c_2 \). We can solve for the constants to get \( c_1 = 5/7 \) and \( c_2 = 2/7 \) so the solution is

\[ y = \frac{5}{7} e^{2t} + \frac{2}{7} te^{-5t} \]