Graphs

- A graph on $X$ is a symmetric, irreflexive $G \subseteq X^2$
- A graph $G$ on a Polish space $X$ is Borel if it is a Borel subset of $X^2 \setminus \{(x, x) : x \in X\}$.
- $A \subseteq X$ is an anticlique ($G$-free) if $A^2 \cap G = \emptyset$
- $C \subseteq X$ is a clique if for all distinct $x, y \in C$ we have $xGy$. 
Theorem (Geschke)

Let $G$ be a closed graph on $\omega^\omega$ without perfect cliques. Then there is a ccc forcing extension $V[H]$ such that $(\omega^\omega)^V$ is covered by countably many compact $G$-anticliques.
Definition

The Borel chromatic number of a graph $G$ on a Polish space $X$, written $\chi_B(G)$, is the least size of a Polish space $Y$ such that there is a Borel function $c : X \to Y$ where $x_0 G x_1 \Rightarrow c(x_0) \neq c(x_1)$. 

Definition

The weak Borel chromatic number of a graph $G$ on a Polish space $X$ is the least size of a family of pairwise disjoint Borel anticliques which cover $X$. 

Francis Adams
Definable Graphs and Dominating Reals
Definition

The Borel chromatic number of a graph $G$ on a Polish space $X$, written $\chi_B(G)$, is the least size of a Polish space $Y$ such that there is a Borel function $c : X \to Y$ where $x_0 G x_1 \Rightarrow c(x_0) \neq c(x_1)$.

Definition

The weak Borel chromatic number of a graph $G$ on a Polish space $X$ is the least size of a family of pairwise disjoint Borel anticliques which cover $X$. 
Theorem (Geschke)

Let $G$ be a closed graph on a Polish space $X$. Then either $G$ has a perfect clique or there is a ccc forcing extension where the weak Borel chromatic number of $G$ is $\aleph_1 < c$. 
On \( \omega^\omega \) say \( x \leq^* y \) if \( x(n) \leq y(n) \) for all but finitely many \( n \).

Say \( V[H] \) has a dominating real if there is some \( y \in (\omega^\omega)^{V[H]} \) such that \( x \leq^* y \) for all \( x \in (\omega^\omega)^V \).
Dominating Reals

Topological Rephrasing:
In $\omega^\omega$, sets of the form $\{x : x \leq^* y\}$ are $K_\sigma$ and every $K_\sigma$ set is contained in such a set.
Dominating Reals

Topological Rephrasing:
In $\omega^\omega$, sets of the form $\{x : x \leq^* y\}$ are $K_\sigma$ and every $K_\sigma$ set is contained in such a set.

So adding a dominating real is equivalent to covering $(\omega^\omega)^V$ by a $K_\sigma$ set.
Question

For a graph $G$ on a Polish space $X$, when can we cover $X^V$ by countably many compact $G$-anticliques without adding a dominating real?
If $X = \omega^\omega$, we must add a dominating real.
Adding Dominating Reals?

If $X = \omega^\omega$, we must add a dominating real. If $X$ has a closed copy of $\omega^\omega$ in it, we must also add a dominating real.
Adding Dominating Reals?

If $X = \omega^\omega$, we must add a dominating real.
If $X$ has a closed copy of $\omega^\omega$ in it, we must also add a dominating real.

**Theorem (Hurewicz)**

*For $X$ Polish, $X$ has a closed subspace homeomorphic to $\omega^\omega$ iff $X$ isn’t $K_\sigma$.***
Adding Dominating Reals.

Define the $F_\sigma$ graph $D$ on $2^\omega$ by $xDy$ if $x$ has finitely many 1’s and $y$ agrees with $x$ up to its last 1 (or vice versa).
Adding Dominating Reals.

Define the $F_\sigma$ graph $D$ on $2^\omega$ by $xDy$ if $x$ has finitely many 1’s and $y$ agrees with $x$ up to its last 1 (or vice versa).

Let $A \subseteq 2^\omega$ be the sequences with finitely many ones and $B = A^c$.

If $C \subseteq 2^\omega$ is a closed $D$-anticlique, then $C \cap B$ is closed in $2^\omega$. 
Adding Dominating Reals.

Define the $F_\sigma$ graph $D$ on $2^\omega$ by $xDy$ if $x$ has finitely many 1’s and $y$ agrees with $x$ up to its last 1 (or vice versa).

Let $A \subseteq 2^\omega$ be the sequences with finitely many ones and $B = A^c$.

If $C \subseteq 2^\omega$ is a closed $D$-anticlique, then $C \cap B$ is closed in $2^\omega$. By covering $2^\omega$ by countably many closed $D$-anticliques, $2^\omega = \bigcup C_n$, we also have $B = \bigcup(B \cap C_n)$, so $B$ is $K_\sigma$. But $B$ is homeomorphic to $\omega^\omega$. 
Loose Graphs

Definition

Let $G$ be a graph on a Polish space $X$. Say $B \subseteq X$ is $G$-loose if there is no \{x_n\} $\subseteq B$ such that $x_n \to x$ and $x_n G x$ for all $n \in \omega$. Say $G$ is loose if $X = \bigcup B_n$ where each $B_n$ is $G$-loose.
Loose Graphs

Definition

Let $G$ be a graph on a Polish space $X$. Say $B \subseteq X$ is $G$-loose if there is no $\{x_n\} \subseteq B$ such that $x_n \to x$ and $x_n G x$ for all $n \in \omega$.

Francis Adams
Definable Graphs and Dominating Reals
**Loose Graphs**

**Definition**

Let $G$ be a graph on a Polish space $X$. Say $B \subseteq X$ is $G$-loose if there is no $\{x_n\} \subseteq B$ such that $x_n \to x$ and $x_n G x$ for all $n \in \omega$. Say $G$ is loose if $X = \bigcup B_n$ where each $B_n$ is $G$-loose.
Loose Graphs

**Theorem**

Let $G$ be a closed, loose graph on a $K_\sigma$ Polish space $X$. Then there is a ccc poset $P(G)$ such that in $V[H]$ there are no dominating reals and $X^V$ is covered by countably many compact $G$-anticliques.
Loose Graphs

Question

Which graphs are loose?
Closure Properties

- If $G$ on $X$ is loose and $F \subseteq G$, then $F$ is loose.
- If $G_1$, $G_2$ are loose graphs on $X$, $Y$ respectively, then $G_1 \times G_2$ is loose on $X \times Y$.
- If $G$ on $X$ is loose, $H$ is a graph on $Y$ and $f : Y \to X$ is a continuous homomorphism, then $H$ is loose.
Nonexamples

Claim

The graph $D$ on $2^\omega$ where $xDy$ if $x$ has finitely many 1’s and $y$ agrees with $x$ up to its last 1 (or vice versa) is not loose

Suppose $2^\omega = \bigcup B_n$ where the $B_n$ are all $D$-loose. We may assume the $B_n$ are contained in $B$ or are a singleton containing one element of $A$. 
Nonexamples

Claim

The graph $D$ on $2^\omega$ where $x D y$ if $x$ has finitely many 1's and $y$ agrees with $x$ up to its last 1 (or vice versa) is not loose.

Suppose $2^\omega = \bigcup B_n$ where the $B_n$ are all $D$-loose. We may assume the $B_n$ are contained in $B$ or are a singleton containing one element of $A$. For the $B_n \subseteq B$, we know $\overline{B_n} \cap A = \emptyset$. 

Francis Adams
Definable Graphs and Dominating Reals
Nonexamples

Claim
The graph $D$ on $2^\omega$ where $xDy$ if $x$ has finitely many 1’s and $y$ agrees with $x$ up to its last 1 (or vice versa) is not loose

Suppose $2^\omega = \bigcup B_n$ where the $B_n$ are all $D$-loose. We may assume the $B_n$ are contained in $B$ or are a singleton containing one element of $A$. For the $B_n \subseteq B$, we know $\overline{B_n} \cap A = \emptyset$. So $\overline{B_n} \subseteq B$ and $B = \bigcup \overline{B_n}$, hence $B$ is $K_\sigma$, a contradiction.
Proposition

If $G$ on a $K_\sigma$ space $X$ has a perfect clique, then $G$ is not loose.

Proof.

Write $X = \bigcup K_m$ for compact $K_n$, let $X = \bigcup B_n$ for $G$-loose $B_n$, and let $C \subseteq X$ be a perfect clique.
### Proposition

*If $G$ on a $K_\sigma$ space $X$ has a perfect clique, then $G$ is not loose.*

### Proof.

Write $X = \bigcup K_m$ for compact $K_n$, let $X = \bigcup B_n$ for $G$-loose $B_n$, and let $C \subseteq X$ be a perfect clique. For some $m, n \in \omega$ the set $S = C \cap B_n \cap K_m$ must be uncountable. Let $x$ be a limit point of some $\{x_i\} \subseteq S$. 

Francis Adams

*Definable Graphs and Dominating Reals*
Nonexamples

Proposition

If $G$ on a $K^\sigma$ space $X$ has a perfect clique, then $G$ is not loose.

Proof.

Write $X = \bigcup K_m$ for compact $K_n$, let $X = \bigcup B_n$ for $G$-loose $B_n$, and let $C \subseteq X$ be a perfect clique. For some $m, n \in \omega$ the set $S = C \cap B_n \cap K_m$ must be uncountable. Let $x$ be a limit point of some $\{x_i\} \subseteq S$. Then $\{x_i\} \subseteq B_n$, $x_i \to x$, and $x_iGx$ for each $i$ since they all come from the clique $C$. □
Proposition

If $G$ on $X$ is locally countable, then $G$ is loose.

Proof.

If $G$ is locally countable, each connected component is countable.
Examples

Proposition

If $G$ on $X$ is locally countable, then $G$ is loose.

Proof.

If $G$ is locally countable, each connected component is countable. Write $X = \bigcup B_n$ where each $B_n$ contains at most one element from each component. Then each $B_n$ is $G$-loose since any element of $X$ can share an edge with at most one element of $B_n$. \qed
Definable Looseness

What if we require the loose sets to be Borel?

**Definition**

Say that $G$ on $X$ is Borel loose if $G$ is loose, witnessed by $X = \bigcup B_n$ for Borel sets $B_n$. 
Fix \( \{s_k\} \subseteq 2^{<\omega} \) dense with \( |s_k| = k \). Define the graph \( G_0 \) on \( 2^\omega \) by \( xG_0y \) iff \( \exists n \in \omega, z \in 2^\omega \) \( x = s_n \upharpoonright 0 \upharpoonright z \) and \( y = s_n \upharpoonright 1 \upharpoonright z \) or vice versa.

\( G_0 \) is closed, locally countable, and has uncountable Borel chromatic number (since every nonmeager Borel set has a \( G_0 \) edge).
Theorem (Kechris-Solecki-Todorcevic)

For an analytic graph $G$, exactly one of the following holds:

- $G$ has countable Borel chromatic number.
- There is a continuous homomorphism of $G_0$ to $G$. 

Borel-looseness

**Theorem**

\( G_0 \) isn’t Borel-loose. Moreover, a nonmeager Borel set isn’t \( G_0 \)-loose.
Theorem

$G_0$ isn’t Borel-loose. Moreover, a nonmeager Borel set isn’t $G_0$-loose.

So if an analytic graph $G$ is Borel-loose, it must have countable Borel chromatic number.
The $F_\sigma$ graph $D$ isn’t loose, but has countable Borel chromatic number.
Borel-looseness

The $F_\sigma$ graph $D$ isn’t loose, but has countable Borel chromatic number.

Even worse the complete bipartite graph with partite sets $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}$ is $\Delta^0_3$ and has Borel chromatic number 2.
Open Questions

To what extent can we extend the forcing result to $F_\sigma$ graphs?

Is there a closed, non-loose graph that has no perfect cliques?

Is there a minimal non-loose graph?
Open Questions

- To what extent can we extend the forcing result to $F_\sigma$ graphs?
Open Questions

- To what extent can we extend the forcing result to $F_\sigma$ graphs?
- Is there a closed, non-loose graph that has no perfect cliques?
Open Questions

- To what extent can we extend the forcing result to $F_\sigma$ graphs?
- Is there a closed, non-loose graph that has no perfect cliques?
- Is there a minimal non-loose graph?