1. (4 pts) For the following equations, determine their order and whether they are linear or nonlinear.

a) \( y'' \cos(x) + y(x^2 + 1) = 2 + \ln(x) + xy^{(4)} \)

   order 4, linear

b) \( \frac{d^3y}{dx^3} + \frac{dy}{dx}(x^3 - x) = xe^x + \frac{x}{y} \)

   order 3, nonlinear

2. (10 pts) Solve the initial value problem

\[ \frac{dx}{dt} = x^2t \quad x(0) = 2 \]

This equation is separable, so we can rewrite it as

\[ \frac{dx}{x^2} = t \, dt \]

Integrate both sides of this equation to get

\[ -\frac{1}{x} = \frac{t^2}{2} + C. \]

To determine the constant, substitute \( t = 0 \) and \( x = 2 \) into the equation to see that \( C = -\frac{1}{2} \).

So the solution to the IVP is \( -\frac{1}{x} = \frac{t^2}{2} - \frac{1}{2} \). We can solve this for \( x \) to get the explicit solution

\[ x = \frac{2}{1 - t^2} \]

(We could also have solved for \( x \) and then used the initial condition to find \( C \).)
3. (6 pts) Use implicit differentiation to show that $y - \ln(y) = x^2 + 1$ is a solution to the equation $\frac{dy}{dx} = \frac{2xy}{y - 1}$.

Implicitly differentiate $y - \ln(y) = x^2 + 1$ and solve for $y'$:

$$y' - \frac{y'}{y} = 2x$$

$$y'(1 - \frac{1}{y}) = 2x$$

$$y' \cdot \frac{y - 1}{y} = 2x$$

$$y' = 2x \cdot \frac{y}{y - 1} = \frac{2xy}{y - 1}$$

and this is the differential equation we wanted to obtain.

4. (10 pts) Find the general solution for the equation $\frac{dx}{dt} = t^2 + t^2x^2$.

Factoring out $t^2$ from the right side, we see the equation $\frac{dx}{dt} = t^2(1 + x^2)$ is separable, so we can rewrite it as

$$\frac{dx}{1 + x^2} = t^2 \, dt$$

Integrate both sides of the this equation to get

$$\arctan(x) = \frac{t^3}{3} + C$$

Then we can solve explicitly for $x$ to see the general solution for the equation is

$$x = \tan\left(\frac{t^3}{3} + C\right)$$
5. (10 pts) Find the general solution for the differential equation

$$\frac{dx}{dt} = \frac{x + 4}{t}$$

This is a first order linear equation. To use our method for solving such equations, start by rewriting the equation as

$$\frac{dx}{dt} - \frac{x}{t} = \frac{4}{t}$$

So the coefficient functions are \( P(t) = -\frac{1}{t} \) and \( Q(t) = \frac{4}{t} \).

Then the integrating factor is

$$\mu(t) = e^{\int P(t) \, dt} = e^{\int -\frac{1}{t} \, dt} = e^{-\ln t} = t^{-1} = \frac{1}{t}$$

So the solution to the ODE is

$$x = \frac{1}{\mu(t)} \left[ \int Q(t) \cdot \mu(t) \, dt + C \right]$$

$$x = \frac{1}{t^{-1}} \left[ \int \frac{4}{t} \cdot t^{-1} \, dt + C \right]$$

$$= t \left[ \int t \, dt + C \right]$$

$$= t \left[ \frac{4}{t} + C \right]$$

$$= -4 + Ct$$