1 Introduction

My main area of research is the study of definable graphs on Polish (separable, completely metrizable) spaces, primarily from the perspective of the set theory of the reals. In order to study the structure of such graphs, I associate certain cardinal invariants to a definable graph $G$ on a Polish space $X$, where $G$ being definable typically means that $G$ as a set of ordered pairs is in the Borel $\sigma$-algebra of the space $X^2$. I aim to prove inequalities among these invariants and other well-known cardinals, or alternatively to find consistency results showing the impossibility of proving such inequalities. Concerning Borel graphs, these consistency results lead to the discovery of combinatorial principles of such graphs. Here we find connections with other areas of set theory such as the study of arbitrary infinite graphs or the study of Borel graphs on Polish spaces from the point of view of descriptive set theory. I have developed interesting examples of interesting Borel graphs coming from a variety of mathematical disciplines including set theory, analysis, and topology.

I have also worked in the area of computable model theory. A central notion in this field is that of a computable structure $A$ being computably categorical. This means that if another computable structure $B$ is isomorphic to $A$, then there is an isomorphism between them that is actually a computable function. The problem of determining which structures are computably categorical has been extensively studied for particular classes of structures like graphs or linear orders. I have investigated the effective categoricity of ultrahomogeneous structures, which are highly symmetric structures with the rational numbers as a linear order $(\mathbb{Q}, <)$ as the canonical example. This provides a novel approach by investigating the effective categoricity of ultrahomogeneous structures of any kind, rather than working solely within a specific class of structures.

2 Definable Graphs

The systemic study of definable graphs was largely introduced in [1]. This paper first developed the idea of the Borel chromatic number of a Borel graph $G$ on a Polish space $X$. The classical chromatic number of a graph $G$ on $X$, $\chi(G)$, is the least (possibly infinite) number of anticliques necessary to cover $X$, where an anticlique is $A \subseteq X$ such that no two elements of $A$ are connected by an edge in $G$. Say a graph $G$ on $X$ has countable (finite) Borel chromatic number if $X$ can be covered by countably many (finitely many) Borel anticliques and the Borel chromatic number of $G$, $\chi_B(G)$, is the least size of such a collection. If $X$ can’t be covered by countably many Borel $G$-anticliques, say that $G$ has uncountable Borel chromatic number. They also investigate possible (classical) chromatic numbers of Borel graphs. This area of descriptive graph combinatorics, as outlined in [2], has developed into a subfield of descriptive set theory that has had great success in finding analogs of finite graph theory results for Borel graphs on Polish spaces. This is often done in cases where the degree of each vertex is finite.

For graphs containing vertices of infinite, even uncountable, degree, methods from higher set theory provide an alternative approach to studying the structure of Borel graphs. One way I do this is, for any graph $G$ on a space $X$, to define $I_G$ to be the $\sigma$-ideal of sets that can be covered by
countably many compact $G$-anticliques. Now define two cardinal numbers $\text{cov}(I_G)$ and $\text{non}(I_G)$, where $\text{cov}(I_G)$ is the least number of compact $G$-anticliques necessary to cover $X$ and $\text{non}(I_G)$ is the least cardinality of a set not in $I_G$. These are called the covering number and uniformity number of $I_G$. Similar cardinals can be defined for any $\sigma$-ideal and the cardinals associated with the collections of meager sets and sets of Lebesgue measure zero are particularly well-studied, as in [3].

A first problem is to compare these cardinals to the cardinal $b$, which is the least size of a subset of $\omega^\omega$ (with the product topology) which can’t be covered by countably many compact sets. In particular, for which graphs $G$ is it consistent that $b < \text{non}(I_G)$? For some graphs this is impossible. For $E_0$ on $2^\omega$, the equivalence relation of eventual agreement of infinite binary sequences, we can prove $\text{non}(I_{E_0}) \leq b$. There are graphs for which there is no such provable inequality. A forcing construction of Geschke in [4] shows that, for closed graphs $G$, it is consistent that $\aleph_1 < \text{non}(I_G) = 2^{\aleph_0}$ as long as $G$ has no perfect clique (a perfect set where any two points are connected in $G$). Analyzing this construction we identify a class of graphs for which we can find our desired consistency result.

**Definition 2.1.** Let $G$ be a graph on a Polish space $X$. Say that a set $A \subseteq X$ is $G$-loose if $A$ contains no sequence $\{x_n : n \in \omega\}$ such that $x_n \rightarrow x$ and $x_n \in G x$ for all $n \in \omega$. Say that $G$ is $\sigma$-loose if we can write $X = \bigcup A_n$ where each $A_n$ is $G$-loose.

With this definition in hand, we can prove the following theorem:

**Theorem 2.2.** Let $G$ be a closed graph on a $\sigma$-compact Polish space $X$ which is $\sigma$-loose in every ccc forcing extension. Then it is consistent that $\aleph_1 = b < \text{non}(I_G) = \aleph_2 = c$.

The preceding definition and theorem point towards the problem of determining which graphs are $\sigma$-loose. One large class of graphs we showed to be $\sigma$-loose are graphs with countable coloring number. The coloring number of a graph, first introduced by Erdős in [5] is a generalization of the chromatic number of a graph in that a graph with countable coloring number has countable chromatic number, but not conversely. This provides a criterion for $\sigma$-looseness that is not topological, and connects the problem with classical infinitary combinatorics.

Another class of graphs under investigation are metric graphs. These are graphs $G = G(X, d, D)$ where $(X, d)$ is a metric space, $D$ is a countable set of distances, and $x G y$ if $d(x, y) \in D$. These graphs have largely been considered on $\mathbb{R}^n$ with the Euclidean metric as in [6]; the problem of determining the exact chromatic number of $G(\mathbb{R}^n, d, \{1\})$ is a notorious open problem. The properties of such graphs vary with the underlying metric space. We show that for $n$-dimensional euclidean space the graphs are $\sigma$-loose for any countable set of distances. In contrast, we obtain a non-$\sigma$-loose graph for any infinite set of distances on $(2^\omega, d)$ where $d(x, y) = 2^{-n}$ when $n$ is least such that $x(n) \neq y(n)$. Moreover, for certain infinite-dimensional metrizable spaces, we prove that the metric graphs are not $\sigma$-loose for any metric or any set of distances whose closure contains 0. Through these examples, metric graphs lead to problems of topology such concerning areas as metric space theory and dimension theory.

I have begun investigating another cardinal invariant $\text{gr}(G)$, defined to be the least cardinality of a $G$-anticlique not contained in a $\sigma$-compact anticlique. While related to $\text{non}(I_G)$, the two do exhibit different behavior. For the very simple graph $V$ on $2^\omega \times \{0, 1\}$ where $(x, i)V(y, j)$ if $x = y$
and \( i \neq j \), we can already prove the inequality \( \text{gr}(V) \leq b \). Since \( V \) is very basic, I showed that many graphs \( G \) embed a copy of \( V \) and for such graphs we have \( \text{gr}(G) \leq \text{gr}(V) \). It is also essentially shown in [4] that \( \text{gr}(G) \geq p \) holds for any \( F_r \) (a countable union of closed sets) graph \( G \), where \( p \) is a cardinal related to Martin’s Axiom, a well-known statement independent of ZFC which has applications in set theory, topology, and analysis. These results provide bounds for the cardinals \( \text{gr}(G) \), but there is room to relate graph characteristics \( \text{gr}(G) \) to many other known cardinal invariants.

3 Computable Model Theory

Computable model theory studies the effectiveness of model-theoretic results and constructions. The objects of study are computable structures, which are structures such that the underlying set, as well as the functions and relations, are computable. As mentioned in the introduction, a central notion in computable model theory is the effective categoricity of a structure, as described in [7]. A computable structure is computably categorical if any isomorphic computable structure is computably isomorphic. More generally, a computable structure is \( \Delta_2^0 \)-categorical if given any isomorphic computable structure, there is an isomorphism which is computable from the halting problem. One can also define even more general notions of a structure being \( \Delta_2^0 \)-categorical for computable ordinals \( a \). There is a large body of literature determining the computably categorical or \( \Delta_2^0 \)-categorical structures within certain classes of structures. This has been done for many classes, such as linear orders [8], equivalence structures (sets equipped with an equivalence relation) [9], and injection structures (sets equipped with an injection) [10].

A structure \( A \) is ultrahomogeneous if every isomorphism of finitely generated substructures extends to an automorphism of the entire structure. We define a generalization of this notion by saying that a structure \( A \) is weakly ultrahomogeneous if it can be made ultrahomogeneous after specifying finitely many elements of the universe of \( A \). In [11] we proved the following general theorem about the effective categoricity of weakly ultrahomogeneous structures is:

**Theorem 3.1.** Let \( A \) be a computable weakly ultrahomogeneous structure. Then \( A \) is \( \Delta_2^0 \)-categorical. If \( A \) is relational, i.e. has only relations and not functions, then \( A \) is in fact computably categorical.

With this result in mind, we also study how these notions compare for specific classes of structures.

For linear orders or equivalence structures, we showed the computable weakly ultrahomogeneous structures are exactly the computably categorical structures. For more complicated structures, like graphs or partial orders, it is not hard to show the collection of computably categorical structures properly contains the weakly ultrahomogeneous structures. We find more interesting behavior by looking at injection structures. Here the collection of computable weakly ultrahomogeneous structures lies properly between the computably categorical and \( \Delta_2^0 \)-categorical structures.

In light of other work on the effective categoricity certain classes of structures, my future work includes finding the weakly ultrahomogeneous structures within these classes. Possible classes to examine include algebraic examples like boolean algebras or abelian p-groups, or other combinatorial structures like partial orders or rooted trees equipped with a predecessor function. Another
direction for my future research is locating where weakly ultrahomogeneous structures can fall in the Ershov hierarchy, a complexity hierarchy refining $\Delta^0_2$-categoricity. This has been done for equivalence structures in [12] and is particularly relevant for the class of injection structures, which is a relatively simple class of structures where the weakly ultrahomogeneous structures extend beyond the computably categorical structures.

4 Conclusion

A prominent theme in my research is to use techniques and examples from several areas of mathematics to inform a question, and to find problems that lend themselves to such an approach by connecting various mathematical disciplines. Computable model theory works on the boundary of computability and model theory to find effective content of model-theoretic notions using methods and ideas from both areas. The study of Borel graphs is also inherently interdisciplinary. Borel graphs are objects born of descriptive set theory. Higher set theory has been fruitfully used to study properties of the reals, and can similarly be used to provide insights into the structure of Borel graphs. The rich literature of infinite graph theory is also relevant, and the fact that the infinite graphs under consideration are presented as Borel graphs on a Polish space adds even more structure. Examples of Borel graphs come from within set theory, like from the areas of Borel equivalence relations or Polish group actions, as well as from areas outside of set theory such as analysis and topology. The fact that so many perspectives are brought to bear makes this area very interesting to me and offers many avenues for further research.
References


