Constructing R from Q: Dedekind cut approach

The treatment below is adapted from the one in Avner Friedman's text *Advanced Calculus*.

Definition 1. A *(Dedekind) cut* is an ordered pair of subsets of **Q**, (A, B), satisfying
(i) A and B are both nonempty;
(ii) A and B are complements of one another (in **Q**); and
(iii) $a < b$ for all $a \in A,\ b \in B$.
If (A, B) is a cut, we will refer to a maximal element of A (if one exists) and a minimal element of B (if one exists) as *extremal* elements.

Examples.
1. $A = \{x \in \mathbf{Q} \mid x \leq 1\},\ B = \{x \in \mathbf{Q} \mid x > 1\}$. In this example, A has an extremal element but B has does not.
2. $A = \{x \in \mathbf{Q} \mid x < 1\},\ B = \{x \in \mathbf{Q} \mid x \geq 1\}$. In this example B has an extremal element but A does not.
3. $A = \{x \in \mathbf{Q} \mid x \leq 0\} \bigcup \{x \in \mathbf{Q} \mid x > 0 \text{ and } x^2 < 2\},\ B = \{x \in \mathbf{Q} \mid x > 0 \text{ and } x^2 \geq 2\}$. In this example neither A nor B has an extremal element.

The first exercise below shows that, with regard to extremal elements, every cut is of one of the types in the three examples above.

<u>Exercises.</u> Be careful in doing the exercises in this handout that you do not use any of the results in Rosenlicht whose proofs were based on the least-upper-bound property of the real numbers. The purpose of this handout is to show that there exists an ordered field with the LUB property; we can't assume that such an object exists in order to prove that such an object exists. However, you may find yourself wanting to use the analogs of some of Rosenlicht's **LUB 1** through **LUB 5** statements, with the reals replaced by the rationals. If so, supply a proof (for the rationals) of any such statement, remembering that **Q** does not have the LUB property.

1. Let (A, B) is a cut. Prove if one of the sets A, B has an extremal element, the other does not.

2. Let (A, B) be a cut. Prove that $A = \{x \in \mathbf{Q} \mid x < b\ \forall b \in B\}$ and that $B = \{x \in \mathbf{Q} \mid x > a\ \forall a \in A\}$.

3. Let (A, B) be a cut. (a) Prove that if $x \in A$, then A contains every rational number $\leq x$. (b) Prove that if $x \in B$, then B contains every rational number $\geq x$.

4. Let (A, B) be a cut. Prove that for all positive $\epsilon \in \mathbf{Q}$, there exist $a \in A, b \in B$ such that $b - a < \epsilon$.

Definition 2. A cut (A,B) is called *normalized* if B does not contain a minimal element. If (A,B) is a cut we define the *normalization* of (A,B) to be the cut $(\widehat{A},\widehat{B})$ defined as follows: (i) if (A,B) is normalized, then $\widehat{A}=A$, $\widehat{B}=B$; and (ii) if (A,B) is not normalized, then $\widehat{A}=A\bigcup\{b_{\min}\}, \widehat{B}=B-\{b_{\min}\}$, where $b_{\min}$ is the minimal element of B.

Definition 3. A *real number* is a normalized cut. The set of real numbers is denoted $\mathbf{R}$. A real number (A,B) is called *rational* if A contains a maximal element, and *irrational* otherwise. (Note: The term "rational number" in these notes will always mean "element of $\mathbf{Q}$". To refer to a rational element of $\mathbf{R}$, we will use the phrase "rational real number" or "rational cut".)

Notation. Let $\iota:\mathbf{Q}\to\mathbf{R}$ be the map defined by $\iota(q)=(A_q,B_q)$, where $A_q=\{x\in\mathbf{Q}\mid x\leq q\}$ and $B_q=\{x\in Q\mid x>q\}$.

Exercise.

5. Prove that ι is a 1-1 correspondence between $\mathbf{Q}$ and the set of rational real numbers.

Definition 4. The real number $\mathbf{0}$ is $\iota(0)$, where ι is as in Exercise 5. The real number $\mathbf{1}$ is $\iota(1)$. A real number (A,B) is called *negative* if $0\in B$, *nonnegative* if $0\in A$, and *positive* if A contains a positive rational number.

Notation. For $A\subset\mathbf{Q}$, let $-A=\{-a\mid a\in A\}$.

Exercises.

6. Let $(A_1,B_1),(A_2,B_2)$ be cuts. Define $A_3=\{x\in\mathbf{Q}\mid\exists a_1\in A_1,a_2\in A_2$ such that $x\leq a_1+a_2\}, B_3=\mathbf{Q}-A_3$. Prove that (A_3,B_3) is a cut.
7. Prove that if (A,B) is a cut, then $(-B,-A)$ is a cut, and that $(\widehat{A},\widehat{B})$ is positive if and only if $(\widehat{-B},\widehat{-A})$ is negative.
8. Prove that every real number is either $\mathbf{0}$, positive, or negative, and that the cases are mutually exclusive.

We next need to endow $\mathbf{R}$ with the operations of addition and multiplication. Intuitively, we do this by seeing how, for rational numbers q,r, the cuts $\iota(q+r)$ and $\iota(qr)$ are related to the cuts $\iota(q),\iota(r)$. We then turn these relations into the *definitions* of addition and multiplication of arbitrary normalized cuts (as opposed to the just the rational normalized cuts).

Definition 5. Let $x=(A_1,B_1),y=(A_2,B_2)$ be normalized cuts. The real number $x+y$ is defined to be $(\widehat{A_3},\widehat{B_3})$, the normalization of the cut (A_3,B_3) defined in Exercise 1. (The reason for not simply defining $x+y=(A_3,B_3)$ is that for some irrational choices of x,y, but not all, the cut (A_3,B_3) will not be normalized.)

Exercise.

9. Let $x = (A_1, B_1), y = (A_2, B_2)$ be normalized cuts. (a) If x, y are both non-negative, define $A_3 = \{x \in \mathbf{Q} \mid \exists a_1 \in A_1, a_2 \in A_2, a_1 \geq 0, a_2 \geq 0 \text{ such that } x \leq a_1 a_2\}, B_3 = \mathbf{Q} - A_3$. (b) If x is nonnegative and y is negative, define $A_3 = \{x \in \mathbf{Q} \mid \exists b_1 \in B_1, a_2 \in A_2, \text{such that } x \leq b_1 a_2\}, B_3 = \mathbf{Q} - A_3$. (c) If x is negative and y is nonnegative, define $A_3 = \{x \in \mathbf{Q} \mid \exists a_1 \in A_1, b_2 \in B_2, \text{such that } x \leq a_1 b_2\}, B_3 = \mathbf{Q} - A_3$. (d) If x, y are both negative, define $B_3 = \{x \in \mathbf{Q} \mid \exists a_1 \in A_1, a_2 \in A_2, \text{such that } x \geq a_1 a_2\}, A_3 = \mathbf{Q} - B_3$.

   Show that in all four cases, (A_3, B_3) is a cut.

Definition 6. Let $x = (A_1, B_1), y = (A_2, B_2)$ be normalized cuts. Define $x \cdot y = (\widehat{A_3}, \widehat{B_3})$, the normalization of the cut (A_3, B_3) defined in Exercise 9.

Exercises.

10. Prove that for all $a, b \in \mathbf{Q}$, $\iota(a + b) = \iota(a) + \iota(b)$ and $\iota(ab) = \iota(a) \cdot \iota(b)$.

11. Prove that $\mathbf{R}$, with the operations $+, \cdot$, the additive identity $\mathbf{0}$, and the multiplicative identity $\mathbf{1}$, satisfies field properties I-IV on p. 16 of Rosenlicht.

12. For all $x \in \mathbf{R}$ and nonzero $y \in \mathbf{R}$, figure out how define the elements $-x$ and y^{-1} appropriately, and prove that the field property V on p. 16 of Rosenlicht is satisfied.

13. Prove that $\iota(-a) = -\iota(a)$ for all $a \in \mathbf{Q}$ and that $\iota(a^{-1}) = \iota(a)^{-1}$ for all nonzero $a \in \mathbf{Q}$.

14. Combining exercises 11 and 12, we have now shown that $\mathbf{R}$ is a field. What is it that exercises 5,10, and 13, together with Definition 4, say about the relationship of $\mathbf{Q}$ to $\mathbf{R}$?

Definition 7. Let $\mathbf{R}_+$ denote the set of positive real numbers, and let $\mathbf{R}_-$ denote the set of negative real numbers.

Exercises.

15. Prove that $\iota(\mathbf{Q}_+) \subset \mathbf{R}_+$ and $\iota(\mathbf{Q}_-) \subset \mathbf{R}_-$.

16. Prove that $\mathbf{R}$ has the order property (as defined on p. 19 of Rosenlicht).

17. Define $<$, $>$, etc. as on p. 19 of Rosenlicht. Prove that if $x = (A_1, B_1) \in \mathbf{R}$ and $y = (A_2, B_2) \in \mathbf{R}$, then $x \leq y$ iff $A_1 \subset A_2$.

Finally, we have the theorem we've been waiting for:

Theorem. $\mathbf{R}$ has the Least Upper Bound property.

Proof. Let $S \subset \mathbf{R}$ be a nonempty set bounded from above. Thus the set $\mathcal{B}$ of upper bounds of S is nonempty. Define $A \subset \mathbf{Q}$ by $A = \bigcap_{(C,D)\in\mathcal{B}} C$. Thus $a \in A$ iff for every $(C, D) \in \mathcal{B}$, $a \in C$. Define $B = \mathbf{Q} - A$ $(= \bigcup_{(C,D)\in\mathcal{B}} D)$. We will show that (i) (A, B) is a cut; (ii) its normalization $(\widehat{A}, \widehat{B})$ is an upper bound for S, and (iii) that there is no smaller upper bound of S.

First, B is nonempty because it is a union of nonempty sets. To see that A is nonempty, let $(C', D') \in S$ (this uses nonemptiness of S) and let $c \in C'$ (this uses nonemptiness of C'). If $(C, D) \in \mathcal{B}$, then by exercise 13, $C' \subset C$, so $c \in C$. Hence c lies in every C for which $(C, D) \in B$, so $c \in A$. Thus, both A and B are nonempty. By definition, they are complements of each other. Next, suppose $a \in A$, $b \in B$. Then $a \in C$ for every $(C, D) \in \mathcal{B}$, and $b \in D_2$ for some $(C, D) \in \mathcal{B}$. Select such a (C, D) with $b \in D$. Then $a \in C$, so, since (C, D) is a cut, $a < b$. Hence (A, B) is a cut, establishing (i) above.

Turning to (ii), the argument above that A is nonempty actually shows that A contains *every* element c for which there exists $(C', D') \in S$ with $c \in C'$. Hence, $\widehat{A} \supset A \supset C'$ for whenever $(C', D') \in S$, which by exercise 13 says $x \leq (\widehat{A}, \widehat{B})$ for every $x \in S$. Thus $(\widehat{A}, \widehat{B})$ is an upper bound for S.

Finally, we establish (iii). Suppose $y = (C, D) \in \mathbf{R}$ is an upper bound for S. Then, by definition of B, $A \subset C$. Suppose that $A \neq \widehat{A}$. Then B has a minimal element $b_{\min}$, which is the maximal element of $\widehat{A}$. Thus if $\widehat{A}$ is not contained in C, then (A, B) is not normalized and $b_{\min} \notin C$. Hence, by exercise 2, C contains no element $\geq b_{\min}$, and hence is contained in $\{a \in \mathbf{Q} \mid a < b_{\min}\}$, which is precisely A (since it is the complement of B). Thus $C \subset A$ and $A \subset C$, implying $C = A$ (hence $D = B$ also), which is a contradiction since (C, D) is normalized and (A, B) is not.

Therefore $\widehat{A} = A$, and hence $\widehat{A} \subset C$, implying $(\widehat{A}, \widehat{B}) \leq y$. Thus $(\widehat{A}, \widehat{B})$ is $\leq$ every upper bound of S, and we are done. ∎