MAA 4212—Matrices, Power Series, and Functions of Matrices

Before the discussion gets serious, make sure you understand the difference between a linear transformation (a function) and a matrix (a bunch of numbers arranged in a rectangular array). A matrix can be used for many things, one of which is to **represent** a linear transformation, but the matrix does not **equal** the linear transformation. In the 1×1 case, the function $T(x) = 3x$ is a linear transformation, which, once we have agreed we're talking about linear transformations, can be represented simply by the number 3, a 1×1 matrix. This representation distinguishes T from any other linear transformation $\mathbf{R} \to \mathbf{R}$, but there is still a difference between the number 3 and the function $x \mapsto 3x$.

Exercise.
1. In class we proved that "a linear transformation is its own derivative". More precisely (the statement in quotes is literal nonsense since the derivative of a function is a function of twice as many variables), we proved that if $T : V \to W$ is a linear transformation from one finite-dimensional normed vector space to another, then T is differentiable at every point of V, and $DT|_p = T$ for all $p \in V$. Reconcile this fact with the following statement: "The exponential function $\exp : \mathbf{R} \to \mathbf{R}$ is its own derivative." Is it true that $D(\exp) = \exp$? Is it true that $D(\exp)|_p = \exp$? (No, and no; explain why neither of these could possibly be true.) In terms of the "grown-up" derivative D, what is the correct version of "The exponential function $\exp : \mathbf{R} \to \mathbf{R}$ is its own derivative"?

Recall that, for any $n \in \mathbf{N}$, all norms on $\mathbf{R}^n$ (or any n-dimensional vector space) are equivalent, and therefore determine the same open sets. Unless otherwise specified, whenever we say that a subset of $\mathbf{R}^n$ is open, we mean "open with respect to some (hence any) norm."

Exercise.
2. Let V, W be finite-dimensional vector spaces, and let $U \subset V$ be open. Prove in detail the following fact that was stated in class:

> For any function $F : U \to W$, and $p \in V$, the answer to the question "Is F is differentiable at p?" is the same regardless of what norms are used on V and W, and that, in the differentiable case, the linear map $DF|_p$ is the same regardless of what norms are used on V and W.

For the rest of these notes, let $M_{n \times n}$ denote the set of $n \times n$ matrices whose entries are real numbers. Recall that, under matrix addition, and multiplication of a matrix by a scalar (= real number), $M_{n \times n}$ becomes a vector space, whose zero element is the matrix all of whose entries are 0. The dimension of $M_{n \times n}$ is n^2, the number of entries. Be careful when reading the rest of these notes that you do not confuse the vector space $M_{n\times n}$ (which is isomorphic to $\mathbf{R}^{n^2}$) with the smaller[1] space $\mathbf{R}^n$.

We define the operator norm on $M_{n\times n}$ by

$$\begin{aligned}\|A\|_{\mathrm{op}} &= \sup\left\{\frac{\|A\mathbf{v}\|_2}{\|\mathbf{v}\|_2} : \mathbf{0} \neq \mathbf{v} \in \mathbf{R}^n\right\} \\ &= \sup\{\|A\mathbf{v}\|_2 : \mathbf{v} \in \mathbf{R}^n, \|\mathbf{v}\|_2 = 1\},\end{aligned}$$

where $\|\ \|_2$ is the Euclidean norm on $\mathbf{R}^n$. (We treat elements of $\mathbf{R}^n$ as column vectors, so $A\mathbf{v}$ defined by matrix multiplication.) Said another way, $\|A\|_{\mathrm{op}}$ is the operator norm of the linear map $\mathbf{E}^n \to \mathbf{E}^n$ defined by $\mathbf{v} \mapsto A\mathbf{v}$. The fact that $\|\ \|_{\mathrm{op}}$ is, indeed, a norm on $M_{n\times n}$, can be proven directly or can be derived as a consequence of problem B1(a) in Homework Assignment 7.)

Exercises.

3. Let $A, B \in M_{n\times n}$. Prove that $\|AB\|_{\mathrm{op}} \leq \|A\|_{\mathrm{op}}\|B\|_{\mathrm{op}}$. (You may either prove this directly, or derive it as a consequence of problem B1(b) in Homework Assignment 7.)

4. For $A \in M_{n\times n}$, define $L_A : M_{n\times n} \to M_{n\times n}$ and $R_A : M_{n\times n} \to M_{n\times n}$ by $L_A(B) = AB$, $R_A(B) = BA$ (the "L" and "R" stand for "left" and "right"). Check that, for all A, the maps L_A and R_A are linear, and find the directional derivatives

$$(D_B L_A)(C), \quad (D_B R_A)(C).$$

Make sure you understand that A is not "the matrix of L_A" (or the matrix of R_A). Given a linear transformation T of an m-dimensional vector space V to itself, *and a basis* $\{v_j\}$, one defines the matrix of T *with respect to that basis* using the coefficients that are needed to express $T(v_i)$, for each i, as a linear combination of the $\{v_j\}$. For $V = M_{n\times n}$, the dimension is n^2. Were we to choose a basis for V (the most obvious one being $\{\mathbf{e}_{i,j}\}_{i,j=1}^n$, where $\mathbf{e}_{i,j}$ is the matrix whose $(i,j)^{\mathrm{th}}$ entry is 1 and whose other entries are all 0), the matrices of L_A and R_A with respect to that basis would be $n^2 \times n^2$ matrices, not $n \times n$ matrices, and the matrices of L_A and R_A would be different from each other. You should work out these matrices by hand for the 2×2 case to make sure you understand this.

We will define various functions $M_{n\times n} \to M_{n\times n}$ and consider their differentiability properties. In view of Exercise 2, these properties do not depend on which norm we use on $M_{n\times n}$, so we are free to use whatever norm we find the most convenient. The "sub-multiplicativity" of the operator norm (the inequality in Exercise 3) makes this norm the most convenient one for our purposes here (you'll start to see why in Exercise 6 below).[2]

[1] Except when $n = 1$.

[2] Another commonly used norm on $M_{n\times n}$, the "Euclidean norm" ($\|A\|_{euc} = (\sum_{i,j} A_{ij}^2)^{1/2}$)—also known as the *Frobenius norm*—is much less convenient for these purposes.

The fact that products and sums of $n \times n$ matrices are again $n \times n$ matrices enables us to make sense out of polynomials and power series whose variables are matrices. For example, if $p(x) = x^3 + 4x + 4$, we can define an analogous function $\tilde{p} : M_{n\times n} \to M_{n\times n}$ by

$$\tilde{p}(A) = A^3 + 4A + 4I,$$

where I is the $n \times n$ identity matrix, and where $A^3 = AAA$. More generally, given any power series $\sum_{k=0}^{\infty} c_k x^k$—polynomials are the special case in which $c_k = 0$ for all but finitely many k—we can consider the power series $\sum_{k=0}^{\infty} c_k A^k$, where A is a $M_{n\times n}$-valued variable, and where we treat "A^0" in such a sum as I (the identity matrix again)[3]. If the real-valued power series $\sum_{k=0}^{\infty} c_k x^k$ has radius of convergence $\rho > 0$, thereby defining a function $f : (-\rho, \rho) \to \mathbf{R}$, and $A \in M_{n\times n}$ is any matrix for which the corresponding matrix-valued power series converges, it is customary to denote the value of the matrix-valued series by $f(A)$ (i.e. not to bother with the tilde used in the polynomial example above).

Exercises.

5. (a) Define $s : M_{n\times n} \to M_{n\times n}$ by $s(A) = A^2$. Compute the directional derivatives $(D_A s)(B)$ (warning: remember that matrix multiplication is non-commutative), and prove that s is differentiable. Generalize to higher exponents.

(b) Let $p : \mathbf{R} \to \mathbf{R}$ be any polynomial function. Prove that the associated function $p : M_{n\times n} \to M_{n\times n}$ is differentiable.

6. (a) Show that for $A \in M_{n\times n}$ and $k \in \mathbf{N}$, $\|A^k\|_{\text{op}} \le (\|A\|_{\text{op}})^k$.

(b) Suppose the real-valued series $\sum c_k x^k$ has positive radius of convergence ρ. Prove that for any n the associated matrix-valued series has radius of convergence at least ρ (i.e. that $\sum c_k A^k$ converges if $\|A\|_{op} < \rho$). (I'm not using our usual "R" for radius of convergence because I've already used "R" in "R_A".)

(c) Prove that if $\|A\|_{op} < 1$, then (i) $\sum_{k=0}^{\infty} (-1)^k A^k$ converges to some matrix $f(A)$ (hence the power series defines the function $f : (B_1(0) \subset M_{n\times n}) \to M_{n\times n}$); (ii) $(I + A)f(A) = I$; and therefore (iii) $I + A$ is invertible and $(I + A)^{-1}$ equals the infinite series in (i).

7. (a) For $A \in M_{n\times n}$, define

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

Prove that this series converges for all A.

[3] We *define* $A^0 = I$ if $\det(A) \ne 0$; if $\det(A) = 0$ we do not define A^0, just as we do not define 0^0 in the 1×1 case. Just as in the 1×1 case, treating A^0 as I in "$\sum_{k=0}^{\infty} c_k A^k$" is a definition of *the notation* "$\sum_{k=0}^{\infty} c_k A^k$", not a definition of A^0.

(b) Prove that *if A and B commute* (i.e. if $AB = BA$), then $e^{A+B} = e^A e^B$ (hint: Homework Assignment 6 problem B2(d)).

(c) Compute $(D_A \exp)(B)$ (for arbitrary A and B). As a check, see what your answer reduces to in the 1-dimensional case.

(d) Let $n = 2$, and let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Check that $J^2 = -I$. Using this fact to simplify your work, compute e^{tJ} and e^{xI+yJ}, where $t, x, y \in \mathbf{R}$. Then say something deep (and has something to do with what you just showed).

(e) Let $y, a, b, c \in \mathbf{R}$, and let $A = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Let A^t, B^t denote the transposes of A, B respectively. Compute $e^A, e^{A^t}, e^B, e^{B^t}$. Speculate about how much computation is involved in exponentiating strictly upper-triangular and strictly lower-triangular $n \times n$ matrices for general n.

(f) Demonstrate that the assumption "if A and B commute" in part (b) cannot be removed, by finding non-commuting 2×2 matrices A, B for which $e^{A+B} \neq e^A e^B$. (Hint: there is a reason I placed this problem after part (e).)

(g) Define $C = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. Check that $C^3 = -\Delta^2 C$, where $\Delta = \sqrt{a^2 + b^2 + c^2}$. Using this to simplify your work, compute e^C. In your computation, you should see some familiar-looking series of real numbers coming up. Replace these familiar series by the functions they converge to, so that your final answer has no infinite series left in it.

(h) Using part (e) above and exercise 6(b) as your guide, name at least two conditions on a nonzero matrix A either of which guarantees that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} A^k$$

converges. When it converges, call the sum $\log(I + A)$, since that's what it reduces to in the 1×1 case. For the specific matrix A in part (e), compute $\log(I + A)$, $\exp(\log(I + A))$, and $\log(e^A)$.

8. Let $G_n \subset M_{n \times n}$ denote the subset of invertible matrices, and let $\iota : G_n \to G_n$ denote the inversion map ($\iota(A) = A^{-1}$).

(a) Use the fact $\det(A)$ is a polynomial in the entries of A to show that the determinant function $\det : M_{n \times n} \to \mathbf{R}$ is continuous, hence that $\det^{-1}(0)$ is a closed subset of $M_{n \times n}$, hence that G_n is an open subset of $M_{n \times n}$. (Therefore the notion of "differentiable function on G_n" is defined.)

(a) Prove that $(D_I\ \iota)(B) = -B$.

(b) More generally, if $A \in G_n$, prove that

$$(D_A\iota)(B) = -A^{-1}BA^{-1}.$$

(Hint: $(A+tB)^{-1} = A^{-1}(I+tBA^{-1})^{-1}$, plus problem 6c.) How does this fact fit in with the one-dimensional case?

(c) Prove that ι is differentiable.

Definition. The *Hessian* of a function $f : \mathbf{R}^n \to \mathbf{R}^m$ at a point $\mathbf{a} \in \mathbf{R}^n$ is the function $H_{\mathbf{a}}f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^m$ defined by

$$(H_{\mathbf{a}}f)(\mathbf{v}, \mathbf{w}) = D_{\mathbf{a}}(\mathbf{x} \mapsto (D_{\mathbf{x}}f)(\mathbf{v}))(\mathbf{w}),$$

provided all these directional second derivatives exist. (If the formula above is confusing, this is what it says: Fix a vector $\mathbf{v}$. The directional derivative $(D_{\mathbf{x}}f)(\mathbf{v})$ is then a function of the base point $\mathbf{x}$. Take the directional derivative of this new function in the direction $\mathbf{w}$, at the point $\mathbf{a}$. The result is defined to be $H_{\mathbf{a}}f(\mathbf{v}, \mathbf{w})$. If we were to restrict $\mathbf{v}, \mathbf{w}$ to be the standard basis unit vectors, the Hessian would just be the collection of second partials.)

(d) Prove that $H_A\iota$ exists for all $A \in G_n$, and compute $(H_A\iota)(B, C)$. As a check on your answer, see what your formula reduces to when $n = 1$.