MAA 4212, Spring 2014—Assignment 3's non-book problems

Note: Problem B1 logically belongs with the "Exercises on the Mean Value Theorem" that were part of Assignment 2. It was not included there only because that assignment was already long enough.

B1. Let $I \subset \mathbf{R}$ be an open interval, let $x_0 \in I$, and $f : I \to \mathbf{R}$ be a function that is continuous on I and differentiable on $I \setminus \{x_0\}$. Assume that $\lim_{x\to x_0^+} f'(x)$ and $\lim_{x\to x_0^-} f'(x)$ exist and are equal. Prove that f is differentiable at x_0 and that $f'(x_0)$ has the same value as these two limits (and hence that f is continuously differentiable at x_0).

Be careful not to assume that f has any properties not given in the hypotheses. For example, don't assume that f' is continuous on $I \setminus \{x_0\}$.

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Notation. In class, the terminology "pointed partition", and the corresponding Riemann sum for a given function on a given compact interval, were defined defined. Here is better notation for that terminology:

> Let $a, b \in \mathbf{R}, a < b$. A *pointed partition* of $[a, b]$ is a pair $(\mathcal{P}, T)$, where $\mathcal{P} = \{x_i\}_{i=0}^N$ is a partition of $[a, b]$ and $T = \{x_i'\}_{i=1}^N$ is a set of sample-points for the partition $\mathcal{P}$. If f is a real-valued function on $[a, b]$, the Riemann sum for f corresponding to the pointed partition $(\mathcal{P}, \mathcal{T})$ is $S(\mathcal{P}, T, f) := \sum_{i=1}^N f(x_i')(\Delta x)_i$.

B2. Let $a, b \in \mathbf{R}, a < b$, and let $f : [a, b] \to \mathbf{R}$. Prove that if f is integrable on $[a, b]$, then for any sequence $(\mathcal{P}_n, T_n)$ of pointed partitions for which $\|\mathcal{P}_n\| \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} S(\mathcal{P}_n, T_n, f) = \int_a^b f(x)\, dx.$$

(Hence the integral can be evaluated by taking such a limit, *if you know ahead of time that f is integrable.*)

B3. For any real-valued function f, the *positive part of f*, denoted f_+, and *negative part of f*, denoted f_-, are defined by $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = -\min\{f(x), 0\}$. (Thus both f_+ and f_- are non-negative, and $f = f_+ - f_-$ [why?].)

Let $a, b \in \mathbf{R}, a < b$, and let $f : [a, b] \to \mathbf{R}$. Prove that if f is integrable on $[a, b]$, then so are f_+ and f_-.