MAA 4212, Spring 2014—Assignment 6's non-book problems

B1. In class we proved the "alternating-series test" theorem: if the real-valued sequence $\{a_n\}$ strictly alternates in sign, and $|a_n|$ decreases *monotonically* to zero, then $\sum a_n$ converges. Give a example showing that the monotonicity assumption in this theorem cannot be removed. (I.e. find a counterexample to the following statement: if the sequence $\{a_n\}$ strictly alternates in sign, and $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.)

B2. Let $\{a_{(m,n)} \mid (m,n) \in \mathbf{N}\times\mathbf{N}\}$ be a "doubly indexed sequence" in $\mathbf{R}$—a map $A : \mathbf{N}\times\mathbf{N} \to \mathbf{R}$, where $a_{(m,n)} = A(m,n)$. It is sometimes useful to picture $\{a_{(m,n)}\}$ as an "infinity-by-infinity matrix". In this problem we are interested in attaching meaning to the notation "$\sum_{m,n} a_{(m,n)}$," also written "$\sum_{m,n=1}^{\infty} a_{(m,n)}$". (Our notation "$a_{(m,n)}$" can also be replaced by any other notation for the values of a function $\mathbf{N}\times\mathbf{N}\to\mathbf{R}$, e.g. $a_{m,n}$ or $A(m,n)$.)

Definition. The doubly-indexed series $\sum_{m,n} a_{(m,n)}$ is *absolutely convergent* (or *converges absolutely*) if there exists a bijection $f : \mathbf{N} \to \mathbf{N}\times\mathbf{N}$ such that $\sum_{j=1}^{\infty} a_{f(j)}$ is absolutely convergent. (Said more loosely, we are calling the doubly-indexed series is absolutely convergent if there is some order in which we can add up the entries of the "infinite matrix" $\{a_{(m,n)}\}$ as the terms of an absolutely convergent singly-indexed series.)

(a) Prove that if $\sum_{m,n} a_{(m,n)}$ converges absolutely and $f, g : \mathbf{N} \to \mathbf{N}\times\mathbf{N}$ are bijections, then $\sum_{j=1}^{\infty} a_{f(j)} = \sum_{j=1}^{\infty} a_{g(j)}$. Hence if $\sum_{m,n} a_{(m,n)}$ converges absolutely, we can unambiguously define

$$\sum_{m,n} a_{(m,n)} = \sum_{j=1}^{\infty} a_{f(j)}$$

where f is *any* bijection $\mathbf{N} \to \mathbf{N}\times\mathbf{N}$.

(b) Explain why we should not attach any numerical value (in $\mathbf{R}$) to the notation "$\sum_{m,n} a_{(m,n)}$" if this doubly-indexed series is *not* absolutely convergent.

(c) Prove that if $\sum_{m,n} a_{(m,n)}$ is absolutely convergent then $\sum_{m=1}^{\infty} a_{(m,n)}$ converges for all $n \in \mathbf{N}$, $\sum_{n=1}^{\infty} a_{(m,n)}$ converges for all $m \in \mathbf{N}$, and

$$\sum_{m,n} a_{(m,n)} = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{(m,n)}\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{(m,n)}\right).$$

(d) Let $\sum_{n=1}^{\infty} b_n, \sum_{n=1}^{\infty} c_n$ be absolutely convergent. Prove that $\sum_{m,n} b_m c_n$ is absolutely convergent, and that

$$\sum_{m,n} b_m c_n = \left(\sum_{n=1}^{\infty} b_n\right)\left(\sum_{n=1}^{\infty} c_n\right).$$

Remark. In the absolutely convergent case, enumerating $\mathbf{N} \times \mathbf{N}$ in the order

$$\begin{array}{lll} (1,1) & & \\ (1,2) & (2,1) & \\ (1,3) & (2,2) & (3,1) \\ & & \dots \end{array}$$

leads us to

$$\sum_{m,n} a_{(m,n)} = \sum_{k=1}^{\infty} \left(\sum_{n+m=k} a_{(m,n)} \right). \tag{1}$$

One of the main reasons that the conclusions of problem B2 are important is the following application to power series, in which the enumeration scheme in (1) appears naturally. (For power series, we still index the terms using $\mathbf{N} \bigcup \{0\}$ rather than $\mathbf{N}$, but aside from the slight bookkeeping change this clearly makes no difference in the conclusions of B2.) Suppose you are multiplying two polynomials together, say $a_0 + a_1 x + \cdots + a_N x^N$ (i.e. $\sum_{n=0}^{N} a_n x^n$) and $b_0 + b_1 x + \cdots + b_M x^M$ (i.e. $\sum_{m=0}^{M} b_m x^m$). After multiplying out, you generally rewrite the result by grouping together all the terms with a given power of x, which is the finite-series statement

$$\left(\sum_{n=0}^{N} a_n x^n \right) \left(\sum_{m=0}^{M} b_m x^m \right) = \sum_{k=0}^{N+M} \left(\sum_{n+m=k} a_n b_m \right) x^k.$$

Since power series are absolutely convergent on the interiors of their intervals of convergence, parts (a) and (d) imply that on the interior of the smaller of the intervals of convergence of two power series centered at 0, you can multiply the series together just as if they were polynomials (with infinitely many terms). For fun, you might try to show the identity $\sin^2 x + \cos^2 x = 1$ or $\sin x \cos x = \frac{1}{2} \sin(2x)$ or $(e^x)^2 = e^{2x}$ this way.