MAA 4211, Fall 2013—Assignment 7's non-book problems

B1. Let d_1 and d_2 be equivalent metrics on a set E. Prove the following directly from the definitions of "compact" and "sequentially compact".

(a) (E, d_1) is compact if and only if (E, d_2) is compact.

(b) (E, d_1) is sequentially compact if and only if (E, d_2) is sequentially compact.

(By "directly from the definitions" I mean: prove both (a) and (b) without using any theorems of the form "Compactness [or sequential compactness] implies, or is implied by, some other property of metric spaces.")

B2. Let (E, d) be a metric space, let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence in E, and assume this sequence has a convergent subsequence $\{p_{n_i}\}_{i=1}^{\infty}$. Let $p = \lim_{i\to\infty} p_{n_i}$. Show that the original sequence $\{p_n\}_{n=1}^{\infty}$ also converges to p.

(Note (E, d) is not assumed to have any properties other than being a metric space; e.g. we are not assuming (E, d) is complete or sequentially compact. The hypotheses say only that *this particular* Cauchy sequence $\{p_n\}_{n=1}^{\infty}$ has a convergent subsequence, not that every sequence has a convergent subsequence, and not that every Cauchy sequence has a convergent subsequence.)

B3. Recall from class that $\ell^\infty(\mathbf{R})$ is the normed vector space $(\mathbf{R}_b^\infty, \| \; \|_\infty)$, where $\mathbf{R}_b^\infty$ is the set of bounded infinite sequences in $\mathbf{R}$ (an element $\vec{a} \in \mathbf{R}_b^\infty$ is an infinite sequence $\{a_m\}_{m=1}^{\infty}$ in $\mathbf{R}$ whose range is a bounded set in $\mathbf{R}$), and where $\|\vec{a}\|_\infty = \sup\{|a_i| \mid i \in \mathbf{N}\}$. As with any normed vector space, when we speak of metric-space properties of $\ell^\infty(\mathbf{R})$, the metric is assumed to be the one associated with the given norm (unless otherwise specified); thus the ℓ^∞ metric on $\mathbf{R}_b^\infty$ is the function $d : \mathbf{R}_b^\infty \times \mathbf{R}_b^\infty \to \mathbf{R}$ given by $d(\vec{a}, \vec{b}) = \sup\{|a_i - b_i| \mid i \in \mathbf{N}\}$.

Since a sequence in $\ell^\infty(\mathbf{R})$ is a sequence of sequences, to avoid confusion in this problem we will use a superscript rather than a subscript to label the terms of a sequence in $\ell^\infty(\mathbf{R})$; we will write such a sequence as $\{\vec{a}^{(n)}\}_{n=1}^{\infty}$. Thus the n^{th} term in such a sequence is a real-valued sequence $\vec{a}^{(n)} = \{a_i^{(n)}\}_{i=1}^{\infty} = \{a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots\}$.

(a) Let $\{\vec{a}^{(n)}\}_{n=1}^{\infty}$ be a Cauchy sequence in $\ell^\infty(\mathbf{R})$. Show that for all $i \in \mathbf{N}$, the real-valued sequence $\{a_i^{(n)}\}_{n=1}^{\infty}$ (the sequence of "i^{th} components" of the $\vec{a}^{(n)}$) is a Cauchy sequence in $\mathbf{R}$. Note that in $\{a_i^{(n)}\}_{n=1}^{\infty}$, the index i is fixed; it is n that varies: $\{a_i^{(n)}\}_{n=1}^{\infty} = \{a_i^{(1)}, a_i^{(2)}, a_i^{(3)}, \dots\}$.

(b) Let $\{\vec{a}^{(n)}\}_{n=1}^{\infty}$ be as in part (a). Since $\mathbf{R}$ is complete, for all $i \in \mathbf{N}$ there exists $c_i \in \mathbf{R}$ such that $\lim_{n\to\infty} a_i^{(n)} = c_i$. Let $\vec{c}$ be the sequence $\{c_i\}_{i=1}^{\infty} \in \mathbf{R}^\infty$—no subscript "$b$", yet. Show that the sequence $\vec{c}$ *is*, in fact, bounded. (So $\{c_i\}_{i=1}^{\infty} \in \mathbf{R}_b^\infty$ after all.)

(c) Let $\{\vec{a}^{(n)}\}_{n=1}^{\infty}$ and $\vec{c}$ be as in part (b). Show that $\{\vec{a}^{(n)}\}_{n=1}^{\infty}$ converges in $\ell^\infty(\mathbf{R})$ to $\vec{c}$. (Note: unlike for sequences in $\mathbf{R}^m$, this CANNOT be deduced just from the fact that $\{a_i^{(n)}\}_{n=1}^{\infty}$ converges to c_i for all i; see part (e) below.) Thus $\ell^\infty(\mathbf{R})$ is complete.

Just FYI: A complete normed vector space is called a *Banach space.*

Notation for the remaining parts of this problem. For $n \in \mathbf{N}$, let $\vec{e}^{(n)} \in \mathbf{R}_b^\infty$ be the sequence whose n^{th} term is 1 and all of whose other terms are zero (e.g. $\vec{e}^{(3)} = \{0, 0, 1, 0, 0, 0, 0, \dots\}$).

(d) Show that for all $i \in \mathbf{N}$, $\{e_i^{(n)}\}_{n=1}^\infty$ converges in $\mathbf{R}$ to 0.

(e) Let $\vec{0}$ be the zero element of $\mathbf{R}_b^\infty$ (the sequence $\{0, 0, 0, 0 \dots\}$). Compute $d(\vec{e}^{(n)}, \vec{0})$ for all n, and use this to show that $\{\vec{e}^{(n)}\}_{n=1}^\infty$ does not converge in $\ell^\infty(\mathbf{R})$ to $\vec{0}$, even though the i^{th}-component sequence $\{e_i^{(n)}\}_{n=1}^\infty$ converges to the i^{th} of $\vec{0}$ for all i.

(f) Compute $d(\vec{e}^{(n)}, \vec{e}^{(m)})$ for all $m, n \in \mathbf{N}, m \neq n$. Use your answer to show that no subsequence of $\{\vec{e}^{(n)}\}_{n=1}^\infty$ can be Cauchy. Use this to deduce that no subsequence of $\{\vec{e}^{(n)}\}_{n=1}^\infty$ can converge.

Note: since any sequence is trivially a subsequence of itself, the last conclusion implies that $\{\vec{e}^{(n)}\}_{n=1}^\infty$ does not converge in $\ell^\infty(\mathbf{R})$ to *anything*, so, in particular, it does not converge to $\vec{0}$. But I still want you to do part (e) by the method indicated in part (e).)

(g) Use part (f) to deduce that the closed unit ball $\overline{B}_1(\vec{0}) \subset \ell^\infty(\mathbf{R})$ is not sequentially compact.

> **Remark**. Thus, by parts (c) and (g) this ball is a closed, bounded subset of a complete normed vector space, yet is not sequentially compact, hence is not compact (or totally bounded). The Heine-Borel Theorem (closed, bounded subsets of $\mathbf{R}^m$—with metric given by any norm—are compact) does not extend to infinite-dimensional vector spaces.

B4. Let (E, d) be a metric space, and let $U \subset Y \subset E$. Assume that Y is closed in E and that U is closed in Y (i.e. that U is closed as a subset of the subspace $(Y, d|_{Y \times Y})$). Prove that U is closed as a subset of (E, d).

B5. Let (E, d) be a metric space, $S \subset E$. Call S *disconnected* if S is not connected. Prove that S is disconnected if and only if $S = A \bigcup B$ for some nonempty sets $A, B \subset S$ for which $\overline{A} \bigcap B = \emptyset = A \bigcap \overline{B}$. (Here $\overline{A}$ and $\overline{B}$ denote the closures of A and B in E, not in the subspace S.)

B6. Let (E, d) be a metric space, $S \subset E$ a nonempty subset, and $p \in E$. The *distance from p to S*, written $\text{dist}(p, S)$, is defined to be $\inf\{d(p, q) \mid q \in S\}$.

(a) Prove that $\text{dist}(p, S) = 0$ if and only if $p \in \overline{S}$.

(b) For $(E, d) = \mathbf{E}^2$, give an example of each of the following.

(i) A subset S and a point $p \notin S$ for which the infimum defining $\text{dist}(p, S)$ *is not* achieved.

(ii) A subset S and a point $p \notin S$ for which the infimum defining $\text{dist}(p, S)$ *is* achieved. (Note that "The infimum defining $\text{dist}(p, S)$ is achieved" is equivalent to "There is a point $q \in S$ that, among all points in S, minimizes distance to p.")

(c) Using B5, prove that S is disconnected if and only if $S = A \bigcup B$ for some nonempty sets $A, B \subset S$ for which every point of each set is a positive distance from the other set (i.e. $\text{dist}(p, B) > 0 \ \forall p \in A$ and $\text{dist}(p, A) > 0 \ \forall p \in B$).

Motivation for part (c): Recall that, heuristically, we wanted "S is not connected" to mean that S cannot be partitioned into two nonempty disjoint subsets that "don't touch each other". There is no official definition of one subset of a metric space *touching*, or not touching, another. However, were we (not unreasonably) to define "A does not touch B" to mean "every point of A is a positive distance from B", then the characterization of disconnectedness in this problem would turn the heuristic characterization of "not connected" into a precise one that agrees with the mathematical definition.

B7. Let (E, d) be a metric space.

(a) Let $p \in E$. Show that the singleton set $\{p\}$ is connected.

(b) Let $p \in E$, and let $\mathcal{F}_p = \{S \subset E \mid S \text{ is connected and } p \in S\} \subset P(E)$. Let

$$C_p = \bigcup_{S \in \mathcal{F}_p} S.$$

Prove that C_p is connected.

(Do not re-invent the wheel to prove this. You should need no more than a couple of sentences, if you apply a couple of facts already proven.)

The set C_p defined above is called the *connected component of* p *in* E (or in (E, d)). We will use the notation "C_p" with this meaning for the rest of this problem. A subset $C \subset E$ is called a *connected component* of E if $C = C_p$ for some $p \in E$.

(c) For $p \in E$, prove that C_p is the largest connected set containing p, in the following sense: if $S \subset E$ is connected and $p \in S$, then $S \subset C_p$.

(d) Define a relation $\sim$ on E by declaring $p \sim q$ if and only if $q \in C_p$. Prove that $\sim$ is an equivalence relation, and that the equivalence classes are exactly the connected components of E.

Recall that, for any equivalence relation on a set S, the equivalence classes partition S into pairwise disjoint subsets. (For the relation above, "pairwise disjointness" means that for any $p, q \in E$, either $C_p = C_q$ or $C_p \bigcap C_q = \emptyset$.) Thus a metric space is always the disjoint union of its connected components.

(e) (E, d) is called *totally disconnected* if the only nonempty connected subsets of E are the singleton sets. Prove that $\mathbf{Q}$, with its usual metric, is totally disconnected.

Note: in Assignment 4, Problem B1, you proved that $\mathbf{Q}$ is not connected (but "connected" was not in our mathematical vocabulary at the time). Now you are proving something much stronger.

(f) Prove that every connected component of (E, d) is a closed subset of E. (Here (E, d) is a general metric space again, not totally disconnected.)

(g) Use part (f) to prove that if (E, d) has only finitely many connected components, then each connected component is both open and closed.