MAA 4212, Spring 2014—Assignment 7's non-book problems

B1. Let V, W, and Z be finite-dimensional normed vector spaces, and let $\mathrm{Hom}(V, W)$ denote the set of linear maps $V \to W$. Recall that $\mathrm{Hom}(V, W)$ is itself a vector space (a subspace of the space of all functions $V \to W$).

(a) Show that the operator norm $\| \ \|_{\mathrm{op}}$ on $\mathrm{Hom}(V, W)$ is, in fact, a norm on this vector space.

(b) Let $T : V \to W$ and $S : W \to Z$ be linear maps. Show that $\|S \circ T\|_{\mathrm{op}} \leq \|S\|_{\mathrm{op}}\|T\|_{\mathrm{op}}$.

B2. (a) Let $a, b \in \mathbf{R}, a < b$. Suppose $g : (a, b) \to \mathbf{R}$ is differentiable. Prove that if g' is bounded, then there exists a continuous extension of g to the closed interval $[a, b]$ (i.e. there exists a continuous function $\tilde{g} : [a, b] \to \mathbf{R}$ that coincides with g on (a, b)).

(b) Prove that the integration-by-parts formula (Rosenlicht p. 133/#17) generalizes to the case in which u is replaced by a function g satisfying only the hypotheses in part (a). (In the Rosenlicht problem, u' is assumed to exist and be continuous on an open interval containing $[a, b]$.)

(c) Suppose $g : (0, \pi) \to \mathbf{R}$ is continuously differentiable and has bounded first derivative. Prove that

$$\lim_{n\to\infty} \int_0^{\pi} g(x) \sin(nx)\ dx = 0.$$

(Hint: (a) and (b) come before (c). You may also wish to read part (f) of the next problem; it may keep you from going in the wrong direction.)

B3. The integral test implies that $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$, and can be used to give crude upper and lower bounds on the sum, but cannot give the precise value of the sum. In this problem you will end up computing the actual value of $\sum 1/n^2$ (by roundabout means).

In this problem, you are free to use the conclusions of the previous problem, whether or not you successully proved them.

(a) Let $f : [0, \pi] \to \mathbf{R}$ be a function. Suppose f'' exists and is continuous on $[0, \pi]$, and that $f(0) = f(\pi) = 0$. For $0 < x < \pi$, define $g(x) = f(x)/\sin(x)$. Prove that the limit of g' exists at both endpoints of $[0, \pi]$, and hence that g' extends to a continuous (and therefore bounded) function on $[0, \pi]$.

Note: this problem is one of those rare instances in which even a real mathematician might use l'Hôpital's Rule.

(b) Let f be as in part (a). Prove that

$$\lim_{n\to\infty} \int_0^{\pi} f(x) \frac{\sin(nx)}{\sin(x)}\ dx = 0.$$

(c) Verify that if n is any integer, then

$$\int_0^\pi x(\pi - x)\cos(2nx)\ dx = \begin{cases} -\pi/(2n^2), & n \neq 0 \\ \pi^3/6, & n = 0 \end{cases}.$$

(Note: for $n \neq 0$ the computation is simpler if you do not break the integral up into two pieces, one for $x^2 \cos 2nx$ and $x \cos 2nx$.) Use this to prove that

$$\sum_{n=1}^{\infty} \left(\int_0^\pi x(\pi - x)\cos(2nx)\ dx \right) = -\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(d) Show that for all integers $n \geq 1$,

$$\cos(2x) + \cos(4x) + \cos(6x) + \cdots + \cos(2nx) = \frac{1}{2} \left(\frac{\sin((2n+1)x)}{\sin(x)} - 1 \right).$$

Use this to prove that

$$\sum_{n=1}^{\infty} \left(\int_0^\pi x(\pi - x)\cos(2nx)\ dx \right) = -\frac{1}{2} \int_0^\pi x(\pi - x) dx.$$

(e) Using the work above, determine the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(f) Did any of parts (b) through (d) involve interchanging the order of a limit as $n \to \infty$ and an integral? Would such an interchange even have been possible? Why or why not?