MAA 4211, Fall 2013—Assignment 8's non-book problems

B1. In problem 3 on p. 91 of the textbook, suppose you remove the hypothesis that the sets S_1, S_2 are both closed. Is the conclusion still true? (Prove your answer, of course.)

B2. Let $E = (0,1) \times (0,1) = \{(x,y) \in \mathbf{R}^2 \mid 0 < x < 1 \text{ and } 0 < y < 1\}$, and give E the Euclidean metric. Prove that for any $p \in E$, the complement of p is arcwise connected, and hence connected. (See Rosenlicht p. 93/29a for definition of "arcwise connected", if this has not yet been defined in class.)

B3. Let E be as in the problem B2. Prove that no continuous real-valued function on E can be injective. (Hint: There is a reason I had you do problem B2 first.) Remark: Continuity is essential in this statement. It can be shown that E and the interval $(0,1)$ have equal cardinality (the same cardinality as $\mathbf{R}$), so there *is* a bijective function from E to $(0,1)$ (hence an injective function from E to $\mathbf{R}$)—it's just that such a function can't be continuous.

Definition. Let E be a metric space and let $p \in E$. The *path component of* p (or the *path component of E containing* p) is the set

$$\{q \in E \mid \exists \text{ a continuous function} f : [0,1] \to E \text{ with } f(0) = p \text{ and } f(1) = q\}.$$

A *path component of E* is any set that is the path component of some point in E. (Note that, by this definition, a path component is never empty. Also note that a metric space is arcwise connected if and only if it has exactly one path component, i.e. if the path component of any point is the entire metric space).

B4. Prove that every metric space is the disjoint union of its path components.

B5. Let $n \geq 1$ and let A be an open subset of $\mathbf{E}^n$, with the Euclidean metric inherited from $\mathbf{E}^n$. Prove that every path component of A is an open subset of $\mathbf{E}^n$.