MAT 4930, Spring 2014—some non-book problems

1. Let $\{E_1, E_2\}$ be a tangent frame-field defined on some open set U in a surface M, let $\{\theta_1, \theta_2\}$ be the dual co-frame, and let $E_3 = N = E_1 \times E_2$. Let $\{\omega_{ij}\}$ be the connection 1-forms defined by these data. In class we proved that

$$\begin{aligned} d\theta_1 &= \omega_{12} \wedge \theta_2, && (1) \\ d\theta_2 &= -\omega_{12} \wedge \theta_1 \quad (\text{since } -\omega_{12} = \omega_{21}). && (2) \end{aligned}$$

Show that ω_{12} is the *unique* 1-form on U satisfying (1)–(2).

2. Notation as in problem 1. Let ϕ be a real-valued function on U, and define vector fields $\tilde{E}_1, \tilde{E}_2$ on U by

$$\begin{aligned} \tilde{E}_1 &= \cos\phi\ E_1 + \sin\phi\ E_2\ , \\ \tilde{E}_2 &= -\sin\phi\ E_1 + \cos\phi\ E_2\ . \end{aligned}$$

(a) Show that $\{\tilde{E}_1, \tilde{E}_2\}$ is also a tangent frame-field defined on U, and that $\tilde{E}_1 \times \tilde{E}_2 = E_1 \times E_2$. Define $\tilde{E}_3 = E_3$.

(b) Let $\{\tilde{\theta}_1, \tilde{\theta}_2\}$ be the coframe-field dual to $\{\tilde{E}_1, \tilde{E}_2\}$. Express the $\{\tilde{\theta}_i\}$ in terms of the $\{\theta_i\}$. Once you get the final answer: would the answer look as similar relation between $\{\tilde{E}_i\}$ and $\{E_i\}$ if $\{\tilde{E}_1, \tilde{E}_2\}$ were replaced by a general pair of pointwise-linearly-independent vector fields $\{V_1, V_2\}$, and $\{\tilde{\theta}_1, \tilde{\theta}_2\}$ replaced by the basis of 1-forms dual to $\{V_1, V_2\}$?

(c) Show that $\tilde{\theta}_1 \wedge \tilde{\theta}_2 = \theta_1 \wedge \theta_2$.

(d) Let $\{\tilde{\omega}_{ij}\}$ be the connection 1-forms determined by the adapted frame-field $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$, where $\tilde{E}_3 = E_3$. Show that $\tilde{\omega}_{12} = \omega_{12} + d\phi$.

(e) Show, in two different ways, that $d\tilde{\omega}_{12} = d\omega_{12}$: (i) using part (d); (ii) using part (c) and the Gauss equation.