One-to-one and onto:
What you are really doing when you solve equations

When you solve an equation, you go through some steps and get an answer, say $x = 0$. Does this mean that $x = 0$ is a solution of the equation you started with? Not necessarily. The reason is that "if" is not the same as "only if" or as "if and only if".

Example 1. Show that if a real number x is a solution of the equation $x = x + 1$, then $x = 0$.

Method. Suppose x is a real number satisfying $x + 1 = x$. Then

$$\begin{array}{rrcll} & x+1 & = & x & \\ \Rightarrow & x^2+x & = & x^2, & \text{(multiply both sides of equation above by } x\text{)} \\ \Rightarrow & x & = & 0 & \text{(after subtracting } x \text{ from both sides).} \quad \blacksquare \end{array}$$

Here "$\Rightarrow$" means "implies" (or "which implies", if preceded by a comma, as in the transition from the second to the third line above). There is no contradiction above: *if* there is a real number x satisfying $x = x + 1$, *then* $x = 0$. This is quite different from saying that *if* $x = 0$, *then* $x + 1 = x$. The latter is what the (false) assertion "0 is a solution of $x + 1 = x$" means.

Example 2. Find all real solutions of $\ln(2x + 6) = \ln(9 - x^2)$.

Method. First suppose that x is a real number satisfying the equation above. Then

$$\begin{array}{rrcl} & \ln(2x+6) & = & \ln(9-x^2) \\ \Rightarrow & 2x+6 & = & 9-x^2, \\ \Rightarrow & x^2+2x-3 & = & 0, \\ \Rightarrow & (x-1)(x+3) & = & 0, \\ \Rightarrow & x=1 & \text{or} & x=-3. \end{array}$$

Does this mean that the solutions of the original equation are 1 and -3? No. Plugging $x = 1$ into $\ln(2x + 6) = \ln(9 - x^2)$, we get a true statement ($\ln 8 = \ln 8$), but plugging in $x = -3$ we get nonsense, since $\ln(0)$ is not defined. So 1 is the only solution, despite the fact that our algebra going from top to bottom was completely valid. $\blacksquare$

The problem in both examples is that not all the steps are reversible. In each example, if every "$\Rightarrow$" could be replaced by "$\Leftarrow$" (meaning: what's on the right implies what's on the left) then we'd get a logical implication that starts at the bottom and ends at the top, implying that the numbers at the bottom are solutions of the equations at the top. As written, the examples above say only that *if* there are any solutions of the starting equations, those solutions must be *among* the numbers written at the bottom. One can figure out which (if any) of the numbers at the bottom are solutions of the equations at the top just by plugging in. But another way to be sure that the answers at bottom are solutions of the equation at top is if every step is reversible.

Example 3. Find all solutions of the equation $2x + 1 = 7$.
Method.
Part A. First suppose that x is a solution of $2x + 1 = 7$. Then

$$\begin{array}{rrcl} & 2x+1 & = & 7 \\ \Rightarrow & 2x & = & 6, \\ \Rightarrow & x & = & 3. \end{array}$$

Part B. Reading the following implications from the bottom up:

$$\begin{array}{rrcl} & 2x+1 & = & 7. \\ \Leftarrow & 2x & = & 6, \\ \Leftarrow & x & = & 3 \end{array}$$

Part A shows that the only possible solution is $x = 3$; Part B shows that $x = 3$ is indeed a solution. Thus 3 is the only solution of $2x + 1 = 7$. ∎

We could have combined steps in Example 3 by using " $\Longleftrightarrow$ " arrows: "Statement 1 $\Longleftrightarrow$ Statement 2" means that each statement implies the other. That's the case with the statements (equations) in Example 3:

$$\begin{array}{rrcl} & 2x+1 & = & 7 \\ \Longleftrightarrow & 2x & = & 6, \\ \Longleftrightarrow & x & = & 3. \end{array}$$

What this has to do with "one-to-one" and "onto". Suppose you are given a function f with domain G. If, for general y, solving the equation $f(x) = y$ by going "down the stack" (as in Part A of Example 3) produces a single value for x, that means that the equation $f(x) = y$ has *at most* one solution. In other words, no two x's can give the same y, because if $f(x) = y$, you've written down a formula for what x must be in terms of y. In mathematical terminology, f is *one-to-one*.

On the other hand, if, for any y in a given set S, you can reverse all the steps in your chain of equations (going "up the stack", as in Part B of Example 3), that shows that for each y in S there is a x such that $f(x) = y$. In mathematical terminology, f is *onto* S.

Rough summary.

- Going "down the stack" is related to "one-to-one".
- Going "up the stack" is related to "onto".

Example 4. Show that the function $f(x) = 2x + 1$, with domain $\mathbf{R}$ (the real line) is one-to-one and onto $\mathbf{R}$.

Method.

Part A. Let y be a general real number. If x is a solution of $f(x) = y$, then

$$\begin{array}{rrcl} & 2x+1 & = & y \\ \Rightarrow & 2x & = & y-1, \\ \Rightarrow & x & = & \frac{y-1}{2}. \end{array}$$

Thus f is one-to-one, because the *only possible* x satisfying $f(x) = y$ is $\frac{y-1}{2}$.

Part B. Let y be a general real number. Then (reading from the bottom up)

$$\begin{array}{rrcl} & 2x+1 & = & y. \\ \Leftarrow & 2x & = & y-1, \\ \Leftarrow & x & = & \frac{y-1}{2} \end{array}$$

I.e. $f(\frac{y-1}{2}) = y$, so for any y there is an x satisfying $f(x) = y$. Hence f is onto $\mathbf{R}$.

As in Example 3, we could have shortened our writing simply by doing part A first, then systematically noting that every "$\Rightarrow$" in Part A could have been replaced by "$\iff$". We could then just have inserted the extra decoration on the arrows without rewriting the whole stack of equations. Of course in general, you can't just replace "$\Rightarrow$" by "$\iff$" if you feel like it; you have to check that the implication goes both ways! In Example 1, the "$\Rightarrow$" on the middle line can't be replaced by "$\iff$" (because of division by 0); in example 3, the "$\Rightarrow$" on the second line can't be replaced by "$\iff$" (because the domain of ln is smaller than the domain of the functions on line 2).

A wonderful property of linear equations.

Most techniques for solving linear equations involve only reversible steps, as in Example 4 above. In a linear algebra class, you will learn some of the important consequences of this fact.

In practice, you will be faced with much more difficult equations or systems of equations than the ones in the examples above! The examples above were kept simple to emphasize the logic rather than the algebra. Try applying this logic the next few times you solve harder equations.