MAA 4212—Spring 2014—Some facts about normed vector spaces

In this set of notes we prove several facts listed in Rosenlicht's Chapter VI exercises 22abc and 23, and a few other facts, but we package the information somewhat differently.

It is assumed that the student is familiar with the definition of *norm* and *normed vector space* (see Rosenlicht p. 63/ 22)[1]. Properties of norms will be used without explicitly referring to these definitions. Also, given any normed vector space $(V, \| \ \|)$, in any reference to metric-related features of V, the metric is assumed to be the one associated with the norm (unless otherwise specified).

To avoid clutter, the zero-element of any vector space, including $\mathbf{R}$, will usually be denoted 0 (rather than with a sub- or superscript that would indicate which vector space in which that particular 0 lives). It should always be clear from context what "0" means; in any expression that we write, there will be only one meaning of 0 that makes any sense. For example, if V and V' are vector spaces and $T : V \to V'$ is a linear transformation, the statement "$T(0) = 0$" means "$T(0_V) = 0_{V'}$", and the "0" in the (not necessarily true) statement "T is continuous at 0" is 0_V.

Definition 0.1 Let $(V, \| \ \|)$, $(V', \| \ \|')$ be normed vector spaces, and let $T : V \to V'$ be a linear transformation. We say that T is *bounded* (*in the sense of linear transformations*) if the set

$$\left\{ \frac{\|T(x)\|'}{\|x\|} \ : \ 0 \neq x \in V \right\} \tag{1}$$

is bounded.

Note that the set in (1) is trivially bounded below by 0, so "bounded" is equivalent to "bounded above" for this subset of $\mathbf{R}$. We will use this fact later without explicit mention.

Note also that for $T : (V, \| \ \|) \to (V', \| \ \|')$ to be bounded *in the sense of linear transformations* is not the same as for T to be bounded as a *function* from one metric space to another. The latter is not a useful concept for linear transformations, since it requires that the range of T be a bounded subset of V'—which cannot happen for a linear transformation T unless T is the identically-zero linear transformation. (Otherwise, if $v \in V$ is any element for which $T(v) \neq 0$, $\|T(\lambda v)\| = |\lambda| \ \|T(v)\| \to \infty$ as $\lambda \to \infty$.) So, in these notes, whenever we say that a linear transformation is bounded, we mean bounded *in the sense of linear transformations.*

Remark 0.2 For T, x as in (1), and nonzero $\lambda \in \mathbf{R}$, we have

$$\frac{\|T(\lambda x)\|'}{\|\lambda x\|} = \frac{\|\lambda T(x)\|'}{|\lambda| \ \|x\|} = \frac{\lambda \|T(x)\|'}{|\lambda| \ \|x\|} = \frac{\|T(x)\|'}{\|x\|} \ .$$

[1] In these notes, "vector space" always means *real* vector space.

If $\lambda = \frac{1}{\|x\|}$, then $\|\lambda x\| = 1$. Hence every element of the set in (1) equals $\|T(x)\|'$ for some *unit* vector x. This set also *contains* $\|T(x)\|'$ for every unit vector x. It follows that

$$\left\{ \frac{\|T(x)\|'}{\|x\|} \ : \ x \in V, x \neq 0 \right\} = \{ \ \|T(x)\|' \ : \ x \in V, \|x\| = 1\}. \tag{2}$$

Some authors define "T is bounded" in terms of boundedness of the set on the right-hand side of (2) instead of the set on the left-hand side, but since the sets are equal, this makes no difference.

Proposition 0.3 *Let $(V, \| \ \|)$, $(V', \| \ \|')$ be normed vector spaces, and let $T : V \to V'$ be a linear transformation. Then the following are equivalent:*

1. *T is continuous at one point.*
2. *T is continuous at 0.*
3. *T is bounded.*
4. *T is Lipschitz continuous at 0.*
5. *T is Lipschitz continuous.*
6. *T is uniformly continuous.*
7. *T is continuous.*

Proof: We prove "(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1)."

(1) $\Rightarrow$ (2). Assume that T is continuous at $x_0 \in V$. Let $\epsilon > 0$, and let $\delta > 0$ be such that if $\|x - x_0\| < \delta$ then $\|T(x) - T(x_0)\|' < \epsilon$.

For all $y \in B_\delta(0) \subset V$, we then have $\|(y + x_0) - x_0\| = \|y\| < \delta$, so $\|T(y + x_0) - T(x_0)\|' < \epsilon$. But since T is linear, $T(y + x_0) - T(x_0) = T(y) = T(y) - 0 = T(y) - T(0)$, so $\|T(y) - T(0)\|' < \epsilon$.

Thus, given any $\epsilon > 0$ there exists $\delta > 0$ such that for $y \in V$, if $d_V(y, 0) < \delta$ then $d_{V'}(T(y), T(0)) < \epsilon$. Hence T is continuous at 0.

(2) $\Rightarrow$ (3). Assume that T is continuous at 0. Note that for all $y \in V$, $d_{V'}(T(y), T(0)) = \|T(y) - T(0)\|' = \|T(y) - 0\|' = \|T(y)\|'$.

Let $\delta > 0$ be such that if $\|y\| < \delta$ (equivalently, $d_V(y, 0) < \delta$) then $\|T(y)\|' < 1$ (equivalently, $d_{V'}(T(y), T(0)) < 1$); δ exists since T is continuous at 0. Let x be any nonzero element of V, and let $y = \frac{\delta}{2\|x\|}x$. Then $\|y\| = \frac{\delta}{2} < \delta$, and $x = \frac{2\|x\|}{\delta}y$, so

$$
\begin{aligned}
\|T(x)\|' &= \left\| T\left(\frac{2\|x\|}{\delta} y\right) \right\|' \\
&= \left\| \frac{2\|x\|}{\delta}\ T(y) \right\|' \\
&= \frac{2\|x\|}{\delta}\ \|T(y)\|' \\
&< \frac{2\|x\|}{\delta} \times 1 \\
&= \frac{2}{\delta}\ \|x\|,
\end{aligned}
$$

implying $\frac{\|T(x)\|'}{\|x\|} < \frac{2}{\delta}$. Hence the set (1) is bounded above by $\frac{2}{\delta}$, hence is bounded, so T is bounded (by definition).

(3) $\Rightarrow$ (4). Assume that T is bounded, and let $M \in \mathbf{R}$ be an upper bound for the set (1). Let $x \in V, x \neq 0$. Then $x = \|x\| \frac{x}{\|x\|}$, so, using the linearity of T,

$$
\begin{aligned}
\|T(x) - T(0)\|' = \|T(x) - 0\|' &= \|T(x)\|' \\
&= \left\| T\left(\|x\| \frac{x}{\|x\|} \right) \right\|' \\
&= \left\| \|x\|\ T\left(\frac{x}{\|x\|} \right) \right\|' \\
&= \|x\|\ \|T\left(\frac{x}{\|x\|} \right) \|' \\
&\leq \|x\| M \\
&= M\|x - 0\|.
\end{aligned}
$$

Hence $d_{V'}(T(x), T(0)) \leq M d_V(x, 0)$, so T is Lipschitz continuous at 0.

(4) $\Rightarrow$ (5). Assume that T is Lipschitz continuous at 0, and let $K, \delta > 0$ be such that for all $y \in V$ with $\|y\| < \delta$ (equivalently, $d_V(y, 0) < \delta$), we have $\|T(y)\|' \leq K\|y\|$ (equivalently, $d_{V'}(T(y), T(0)) \leq K d_V(y, 0)$).

Let $x \in V, x \neq 0$, and let $y = \frac{\delta}{2} \frac{x}{\|x\|}$. Then $\|y\| = \frac{\delta}{2} < \delta$, so $\|T(y)\|' \leq K\|y\|$. Therefore, using the linearity of T,

$$
\begin{aligned}
\|T(x)\| &= \left\| T\left(\frac{2\|x\|}{\delta}\, y \right) \right\|' \\
&= \left\| \frac{2\|x\|}{\delta}\, T(y) \right\|' \\
&= \frac{2\|x\|}{\delta}\, \|T(y)\|' \\
&\leq \frac{2\|x\|}{\delta}\, K\|y\| \\
&= K\|x\|.
\end{aligned}
$$

Now let $x, z \in V$. Then, again using linearity of T,

$$
d_{V'}(T(x), T(z)) = \|T(x) - T(z)\|' = \|T(x - z)\|' \leq K\|x - z\| = K d_V(x, z). \tag{3}
$$

Hence T is Lipschitz continuous.

(5) $\Rightarrow$ (6). Assume that T is Lipschitz continuous, and let $K > 0$ be such that for all $x, y \in V$, $d_{V'}(T(x), T(y)) \leq K d_V(x, y)$. Let $\epsilon > 0$. Then for all $x, y \in V$ with $d_V(x, y) < \frac{\epsilon}{K}$, we have $d_{V'}(T(x), T(y)) < \epsilon$. Hence T is uniformly continuous.

(6) $\Rightarrow$ (7). Trivial.

(7) $\Rightarrow$ (1). Trivial. ■

Remark 0.4 For any two of the seven properties listed in Proposition 0.3, it is relatively easy to prove directly that those two properties are equivalent; it is not necessary to go through the seven-property cycle given in the proposition. We have listed all these properties just to provide extra information, not because they aid in proving the equivalence of other properties. In most textbooks, the only equivalences you will find stated explicitly (of the ones in Proposition 0.3) are (1) $\iff$ (7) $\iff$ (3).

Recall the following:

Definition 0.5 Two norms $\|\ \|$, $\|\ \|'$ on a vector space V are called *equivalent* if there exist $c_1, c_2 > 0$ such that for all $v \in V$,

$$
\|v\| \leq c_1\|v\|' \quad \text{and} \quad \|v\|' \leq c_2\|v\|. \tag{4}
$$

Note that the zero-dimensional vector space $\{0\}$ has only one norm, so "equivalent norms" is not an interesting concept for this space. For any vector space of dimension greater than zero, the requirement "$c_1, c_2 > 0$" in Definition 0.5 is superfluous: since any norm of any nonzero vector is positive, the inequalities in (4) cannot both be satisfied for all v unless c_1, c_2 are both positive. The only reason we explicitly require c_1, c_2 to be positive in the definition is so that we do not have repeat this argument whenever we want to make use of this guaranteed positivity.

Remark 0.6 There are two other common definitions of "equivalent norms" that are equivalent to Definition 0.5: (i) There exist $c_1, c_2 > 0$ such that for all $v \in V$, $c_1\|v\| \leq \|v\|' \leq c_2\|v\|$, and (ii) There exists $c > 0$ such that for all $v \in V$, $\frac{1}{c}\|v\| \leq \|v\|' \leq c\|v\|$. We leave the student to show that these definitions of "equivalent norms" are equivalent to Definition 0.5.

Recall also the following facts (or prove any that you don't recall, or never learned):

1. For any normed vector space $(V, \| \ \|)$, the function $\| \ \| : V \to \mathbf{R}$ is continuous.

2. "Equivalence of norms" is an equivalence relation on the set of norms on a given vector space.

3. If two norms are equivalent, then their associated metrics are equivalent to each other.

4. Equivalent metrics on a set X determine the same open sets in X, and hence the same closed sets and the same compact sets (since "closed" and "compact" are defined purely in terms of open sets).

5. For any $n \in \mathbf{N}$, the ℓ^1, ℓ^2 (Euclidean), and ℓ^∞ norms on $\mathbf{R}^n$ are equivalent (proven last semester).

6. A subset of $\mathbf{E}^n = (\mathbf{R}^n, \| \ \|_2)$ is compact if and only if it is closed and bounded (the Heine-Borel Theorem). Hence, from the previous two facts it follows that every closed, bounded subset of $(\mathbf{R}^n, \| \ \|_1)$ is compact.

7. Let E be a metric space, V a vector space, and $\| \ \|, \| \ \|'$ equivalent norms on V. Let d, d' be the metrics on V associated with these two norms. Then a function $f : E \to V$ is continuous with respect to d if and only if it is continuous with respect to d', and a function $f : V \to E$ is continuous with respect to d if and only if it is continuous with respect to d'.

The following theorem and subsequent corollary are of fundamental importance.

Theorem 0.7 *Let $n \in \mathbf{N}$. All norms on $\mathbf{R}^n$ are equivalent. (Each is equivalent to any other.)*

Proof: Since "equivalence of norms on $\mathbf{R}^n$" is an equivalence relation, it suffices to show that an arbitrary norm $\| \ \|$ on $\mathbf{R}^n$ is equivalent to the ℓ^1-norm $\| \ \|_1$. (Any two norms equivalent to $\| \ \|_1$ are equivalent to each others, by the properties of equivalence relations.)

Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbf{R}^n$, and let $\| \ \|$ be a norm on $\mathbf{R}^n$. Let $C = \max\{\|e_i\| \mid 1 \le i \le n\}$. Then for any $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ we have

$$\|x\| = \|\sum_{i=1}^n x_i e_i\| \le \sum_{i=1}^n \|x_i e_i\| = \sum_{i=1}^n |x_i| \ \|e_i\| \le \sum_{i=1}^n |x_i| C = C\|x\|_1 \ , \tag{5}$$

so

Define $g : (\mathbf{R}^n, \| \ \|_1) \to \mathbf{R}$ by $g(x) = \|x\|$. Then

$$|g(x) - g(y)| = | \ \|x\| - \|y\| \ | \le \|x - y\| \le C\|x - y\|_1, \tag{6}$$

where we have used (5) in the last step. Hence g is Lipschitz, and therefore continuous.

Now let $S = \{y \in \mathbf{R}^n \mid \|y\|_1 = 1\}$. Then S is closed and bounded in $(\mathbf{R}^n, \| \ \|_1)$, hence compact in this metric space. Let $h = g|_S : S \to \mathbf{R}$. Then h is a continuous real-valued function on a compact set, hence achieves a minimum value, say m. Let $y_0 \in S$ be a point at which h achieves this minimum. Note that $0 \notin S$, since $\|0\| \ne 1$. Hence $m = h(y_0) = \|y_0\| > 0$.

Thus for all $y \in S$, we have $\|y\| \ge m > 0$. Note that for all nonzero $x \in \mathbf{R}^n$, we have

$$\left\| \frac{x}{\|x\|_1} \right\|_1 = 1,$$

so $\frac{x}{\|x\|_1} \in S$ and $\left\| \frac{x}{\|x\|_1} \right\| \ge m$. Hence $m \le \left\| \frac{x}{\|x\|_1} \right\| = \frac{1}{\|x\|_1}\|x\|$, implying $m\|x\|_1 \le \|x\|$. Since $m > 0$, this implies that for all nonzero $x \in \mathbf{R}^n$ we have

$$\|x\|_1 \le \frac{1}{m}\|x\|. \tag{7}$$

Trivially (7) holds also for $x = 0$, hence for all $x \in \mathbf{R}^n$. Since both (7) and (5) hold for all $x \in \mathbf{R}^n$, the norms $\| \ \|$ and $\| \ \|_1$ are equivalent. ■

Corollary 0.8 *Let V be a finite-dimensional vector space. Then all norms on V are equivalent.*

Proof: If $\dim(V) = 0$ there is only one norm, so assume that $n := \dim(V) > 0$. Then V is isomorphic to $\mathbf{R}^n$. Let $L : \mathbf{R}^n \to V$ be an isomorphism. As the student may (and should) easily check, if $\| \ \|$ is a norm on V, then the function $\| \ \|_L : \mathbf{R}^n \to \mathbf{R}$ defined by $\|x\|_L = \|L(x)\|$ is a norm on $\mathbf{R}^n$.

Let $\| \ \|, \| \ \|'$ be norms on V, and let $\| \ \|_L$, $\| \ \|'_L$ be the induced norms on $\mathbf{R}^n$. Since all norms on $\mathbf{R}^n$ are equivalent, there exist $c_1, c_2 > 0$ such that

$$\|x\|_L \leq c_1\|x\|'_L \text{ and } \|x\|'_L \leq c_2\|x\|_L$$

for all $x \in \mathbf{R}^n$. Hence for all $v \in V$,

$$\|v\| = \|L(L^{-1}(v))\| = \|L^{-1}(v)\|_L \leq c_1\|L^{-1}(v)\|'_L = c_1\|L(L^{-1}(v))\|' = c_1\|v\|',$$

and similarly $\|v\|' \leq c_2\|v\|$. Hence $\| \ \|$ and $\| \ \|'$ are equivalent. ■

Remark 0.9 In view of Corollary 0.8, all norms on a finite-dimensional vector space V determine the same open sets. This collection of open sets is called the *norm topology* on V.

Recall the following:

Definition 0.10 Let $n \in \mathbf{N}$, let V be an n-dimensional vector space, and let $\mathbf{e} = \{e_1, \ldots, e_n\}$ be a basis of V. The *coordinate functions* determined by $\mathbf{e}$ are the functions $x_i : V \to \mathbf{R}$, $1 \leq i \leq n$, defined by $x_i(\sum_j a_j e_j) = a_i$. (Thus $v = \sum_j x_j(v)e_j$ for all $v \in V$.)

Proposition 0.11 *Let $(V, \| \ \|_V), (W, \| \ \|_W)$ be normed vector spaces, and assume that V is finite-dimensional. Then every linear transformation $V \to W$ is continuous.*

Proof: If $\dim(V) = 0$ then V has only one element, so any function $V \to W$ is continuous. Assume now that $n := \dim(V) > 0$, and let $T : V \to W$ be linear.

Let $\{e_i\}_{i=1}^n$ be a basis of V, let $\{x_i : V \to \mathbf{R}\}_{i=1}^n$ be the corresponding coordinate functions, and define $\| \ \|_{1,\mathbf{e}} : V \to \mathbf{R}$ by $\|v\|_{1,\mathbf{e}} = \sum_{i=1}^n |x_i(v)|$. As the student may (and should) check, $\| \ \|_{1,\mathbf{e}}$ is a norm on V.

Let $C = \max\{\|T(e_i)\|_W \mid 1 \leq i \leq n\}$. Then for all $v \in V$, we have

$$\begin{aligned}
\|T(v)\|_W &= \left\| T\left(\sum_{i=1}^n x_i(v)e_i\right)\right\|_W \\
&= \left\|\sum_{i=1}^n x_i(v)T(e_i)\right\|_W \\
&\leq \sum_{i=1}^n \|x_i(v)T(e_i)\|_W \\
&= \sum_{i=1}^n |x_i(v)| \ \|T(e_i)\|_W \\
&\leq \sum_{i=1}^n |x_i(v)| \ C \\
&= C\|v\|_{1,\mathbf{e}}.
\end{aligned}$$

But all norms on V are equivalent, so there exists $K > 0$ such that $\|v\|_{1,\mathbf{e}} \leq K\|v\|_V$. Hence for all $v \in V$ we have $\|T(v)\|_W \leq CK\|v\|_V$, so T is Lipschitz continuous at 0. By Proposition 0.3, T is continuous. ■

An important special case of Proposition 0.11 is the following.

Corollary 0.12 *Let $n \in \mathbf{N}$ and let $(V, \|\ \|)$ be an n-dimensional vector space. Let $\{e_i\}_{i=1}^n$ be a basis of V, and let $\{x_i\}_{i=1}^n$ be the corresponding coordinate functions. Then $x_i : V \to \mathbf{R}$ is continuous, $1 \leq i \leq n$.*

Proof: Coordinate functions are linear transformations $V \to \mathbf{R}$, hence are continuous by Proposition 0.11. ■

Although we have stated and proved Corollary 0.12 as a corollary of Proposition 0.11, it can be proved directly from Corollary 0.8: with notation as above, we have $|x_i(v)| \leq \sum_{j=1}^n |x_j(v)| = \|v\|_{1,\mathbf{e}} \leq K\|v\|_V$, implying that $x_i : (V, \|\ \|) \to \mathbf{R}$ is a bounded linear transformation, hence continuous.

Remark 0.13 If $V = \mathbf{R}^n$ and $\|\ \|$ is the ℓ^1, ℓ^2, or ℓ^∞ norm, then for $1 \leq i \leq n$ we have

$$\|a_i e_i\| = |a_i| \leq \|(a_1, \ldots, a_n)\|. \tag{8}$$

This makes it easy to prove that the usual coordinate functions $\{x_i\}$ on $\mathbf{R}^n$ (the coordinate functions determined by the standard basis) are continuous with respect to these norms. But (8) is *not* true for *every* norm on $\mathbf{R}^n$. Hence, without Theorem 0.7, it is not obvious that these coordinate functions are continuous with respect to *every* norm on $\mathbf{R}^n$. For the same reason, it is not trivial that the coordinate functions $V \to \mathbf{R}$ determined by an arbitrary basis of an arbitrary n-dimensional normed vector space are continuous.