A terrible way to solve exact equations.

Suppose that the equation $M(x, y)dx + N(x, y)dy = 0$ is exact (i.e. $\partial M/\partial y = \partial N/\partial x$). In the book, and in class, we discussed a method for finding an implicit solution of the form $F(x, y) =$ constant. Here is a **wrong** method for trying to find F. Make sure it's **not** the method you're using.

Step 1. Integrate M with respect to x, omitting constants of integration.
Step 2. Integrate N with respect to y, omitting constants of integration.
Step 3. Add the results from steps 1 and 2 together, except that if the same term (say $3xy$) appears in the answers to both steps 1 and 2, only count it once instead of twice. The result of this "addition" is what you plan to call $F(x, y)$.

The trouble is that unlike the method discussed in class and in the book, which *always* gives a solution (we proved it!), the method above *sometimes* gives a solution, and *sometimes* doesn't. (If you think that it always gives the right answer, try to prove it. Part of the problem you'll come across is deciding what you mean by a "term" in steps 1 and 2.) Would you drive a car that sometimes sped up when you hit the brakes?

Here is an example in which the method above fails:

$$(y \sin x \cos x + y)\ dx + (-\frac{1}{2}\cos^2 x + 3y^2 + x)dy = 0. \tag{1}$$

Note that $M_y = N_x = \sin x \cos x + 1$, so the equation is exact. Let's try to solve it the wrong way given above.

Step 1. Using the substitution $u = \sin x$, we find $\int M dx = \frac{1}{2} y \sin^2 x + xy$.
Step 2. We find $\int N dy = -\frac{1}{2} y \cos^2 x + y^3 + xy$.
Step 3. The answers to Steps 1 and 2 have only the term xy in common, so when we combine them we obtain $F(x, y) = \frac{1}{2} y(\sin^2 x - \cos^2 x) + xy + y^3$. This gives us the non-solution

$$\frac{1}{2} y(\sin^2 x - \cos^2 x) + xy + y^3 = c. \tag{2}$$

If $y(x)$ is a function determined implicitly by (2), then implicitly differentiating and solving for y' we find

$$y' = -\frac{2y \sin x \cos x + y}{\frac{1}{2}(\sin^2 x - \cos^2 x) + 3y^2 + x}. \tag{3}$$

However, a solution of (1) satisfies

$$y' = -\frac{y \sin x \cos x + y}{-\frac{1}{2}\cos^2 x + 3y^2 + x},$$

which does not equal the expression in (3).

I leave it as an exercise for you to solve (1) correctly.