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Convex Control and Dual Approximations. Part II

by

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In the Part I of the paper, we introduced a convex control problem along with its Lagrange dual, we discussed the evaluation of the dual functional, and we developed three finite element approximations to the dual problem. We now obtain error estimates for these finite element schemes. These estimates are based on the minimum principles developed in Part I and the convergence rates for piecewise polynomial restricted range approximation proven below. Section 4 presents a general theory for estimating the error in dual approximations; Section 5 bounds the error in approximating functions with restricted range; and Section 6 applies the results of the previous sections to estimate the error in dual approximations to control problems.

4. Abstract Error Estimates

Our error estimates are based on the four lemmas given below. Suppose we have functions $f: R^n \to R$ and $g: R^n \to R^m$ and a set $E \subset R^m$. For $(z, \eta) \in R^n \times E$, let us define:

$$h(z, \eta) \equiv f(z) + \eta^T g(z) \text{ and} \tag{4.1}$$

$$h(\eta) \equiv \inf \{h(z, \eta): z \in R^n\}. \tag{4.2}$$

We assume the following:

(i) $f, g \in C^2$ and

(ii) there exists $\alpha > 0$ such that $\nabla_1^2 h(z, \eta) > \alpha I$ for all $z \in R^n$ and $\eta \in E$.

By Lemma 2.2, there exists a unique $z(\eta) \in R^n$ such that $h(z(\eta), \eta) = h(\eta)$. Given $\eta_1, \eta_2 \in E$, define the vectors $z_k = z(\eta_k)$ for $k = 1, 2$, and the matrices:

$$H_0 = \int_0^1 \nabla_1^2 h(z_2 + s(z_1 - z_2), \eta_1)\, ds, \text{ and} \tag{4.3}$$

$$H_1 = \int_0^1 (1-s) \nabla_1^2 h(z_2 + s(z_1 - z_2), \eta_1)\, ds. \tag{4.4}$$

LEMMA 4.1. *For all* $\eta_1, \eta_2 \in E$, *we have*:

$$H_0(z_1-z_2)=\nabla g(z_2)^T(\eta_2-\eta_1). \tag{4.5}$$

Proof. Since z_k is the unconstrained minimum of $h(\cdot, \eta_k)$, it follows that

$$\nabla_1 h(z_k, \eta_k)=0 \quad \text{for} \quad k=1,2. \tag{4.6}$$

Subtracting these equalities and observing that

$$0=\nabla_1 h(z_2,\eta_2)=\nabla_1 h(z_2,\eta_1)+\nabla_1 g(z_2)^T(\eta_2-\eta_1), \tag{4.7}$$

we get:

$$\nabla_1 h(z_1,\eta_1)-\nabla_1 h(z_2,\eta_1)=\nabla_1 g(z_2)^T(\eta_2-\eta_1). \tag{4.8}$$

Hence we have (4.5). ■

LEMMA 4.2. *For all* $\eta_1, \eta_2 \in E$, *we have*:

$$h(\eta_1)-h(\eta_2)-g(z_2)^T(\eta_1,\eta_2)\leqslant -\frac{\alpha}{2}|z_1-z_2|^2. \tag{4.9}$$

Proof. Replacing $h(\eta_k)$ by $f(z_k)+\eta_k^T g(z_k)$, we see that

$$h(\eta_1)-h(\eta_2)-g(z_2)^T(\eta_1-\eta_2)=h(z_1,\eta_1)-h(z_2,\eta_1). \tag{4.10}$$

Expanding the right side of (4.10) in a Taylor series about $z=z_2$ and using the integral form for the remainder term, we get:

$$(4.10)=\nabla_1 h(z_2,\eta_1)(z_1-z_2)+(z_1-z_2)^T H_1(z_1-z_2). \tag{4.11}$$

Substituting for $\nabla_1 h(z_2,\eta_1)$ from (4.7) and using identity (4.5), gives us:

$$(4.10)=(z_1-z_2)^T(H_1-H_0)(z_1-z_2). \tag{4.12}$$

Substracting (4.3) from (4.4) and utilizing assumption (ii), we obtain:

$$(4.10)\leqslant -\alpha|z_1-z_2|^2\int_0^1 s\,ds=-\frac{\alpha}{2}|z_1-z_2|^2. \tag{4.13}$$

■

LEMMA 4.3. *For all* $\eta_1, n_2 \in E$, *we have*:

$$h(\eta_1)-h(\eta_2)-g(z_2)^T(\eta_1-\eta_2)\geqslant -|\nabla g(z_2)^T(\eta_1-\eta_2)|^2/\alpha, \tag{4.14}$$

$$h(\eta_1)-h(\eta_2)-g(z_2)^T(\eta_1-\eta_2)\geqslant -|H_0|\,|z_1-z_2|^2. \tag{4.15}$$

Proof. For any positive definite matrix P, observe that $-sP\geqslant -P$ for all $s\in[0,1]$. Combing this relation with (4.12) and (4.3)–(4.5) gives us:

$$\begin{aligned}(4.10)&=(z_1-z_2)^T\Big[\int_0^1 -s\nabla_1^2 h(z_2+s(z_1-z_2),\eta_1)\,ds\Big](z_1-z_2)\\ &\geqslant -(z_1-z_2)^T H_0(z_1-z_2)\\ &=-(\eta_1-\eta_2)^T\nabla g(z_2)H_0^{-1}\nabla g(z_2)^T(\eta_1-\eta_2).\end{aligned} \tag{4.16}$$

Since the largest eigenvalue of H_0^{-1} is bounded by $1/\alpha$ (assumption (ii)), these last two inequalities lead to (4.14)–(4.15). ■

LEMMA 4.4. *Suppose that* $G: Z \times M \to R^1$, $H: M \to R^1$, $M^h \subset M$, *and for some* μ^*, $\mu^h \in M$ *and* $z^* \in Z$, *the following ralations hold:*

$$G(z^*, \mu) - G(z^*, \mu^*) \leqslant 0 \text{ for all } \mu \in M, \tag{4.17}$$

$$H(\mu^h) = \sup \{H(v^h) : v^h \in M^h\}. \tag{4.18}$$

Defining the functional

$$D(z, \mu_1, \mu_2) = H(\mu_1) - H(\mu_2) - G(z, \mu_1) + G(z, \mu_2), \tag{4.19}$$

we have:

$$D(z^*, \mu^h, \mu^*) \geqslant G(z^*, \gamma^h) - G(z^*, \mu^*) + D(z^*, \gamma^h, \mu^*) \tag{4.20}$$

for all $\gamma^h \in M^h$.

Proof. Using the identity (4.19) and relation (4.17), we get:

$$\begin{aligned} H(\mu^h) &= H(\mu^*) + G(z^*, \mu^h) - G(z^*, \mu^*) + D(z^*, \mu^h, \mu^*) \\ &\leqslant H(\mu^*) + D(z^*, \mu^h, \mu^*). \end{aligned} \tag{4.21}$$

On the other hand, (4.18) implies that for all $\gamma^h \in M^h$:

$$\begin{aligned} H(\mu^h) &\geqslant H(\gamma^h) \\ &= H(\mu^*) + G(z^*, \gamma^h) - G(z^*, \mu^*) + D(z^*, \gamma^h, \mu^*). \end{aligned} \tag{4.22}$$

Combining (4.21) and (4.22) gives us (4.20). ■

5. Restricted Range Approximation

The following approximation results are needed for the error estimates in Section 6.

LEMMA 5.1. *Consider the space* P_1^h *of piecewise constant functions and suppose that* $f, g \in W^1$ *with* $fg \equiv 0$. *Then we have*

$$\langle f^I - f, g \rangle = 0(h^2). \tag{5.1}$$

Proof. Express the grid intervals of P_1^h as $J_0 \cup J_1$ where the elements of J_0 are all the grid intervals T such that $f|_T = 0$. If $T \in J_0$, then $f^I = f = 0$ on T. Introducing the notation

$$\langle f, g \rangle_{J_0} = \sum_{T \in J_0} \langle f, g \rangle_T,$$

we conclude that:

$$\langle f^I - f, g \rangle_{J_0} = 0. \tag{5.2}$$

If $T \in J_1$, the condition $fg \equiv 0$ implies the existence of $\sigma \in T$ such that $g(\sigma) = 0$. Since $g \in W^1$, a Taylor expansion gives us:

$$|g|_T \leqslant h\, |g^{(1)}|_T. \tag{5.3}$$

On the other hand, $f \in W^1$ and by (3.7), we have:

$$\| f-f^I \|_T \leq ch \, \| f^{(1)} \|_T . \tag{5.4}$$

Combining (5.3) and (5.4), we get:

$$\langle f^I - f, g \rangle_T \leq ch^2 \text{meas}\,(T) \quad \text{for all } T \in J_1, \text{ and} \tag{5.5}$$

$$\langle f^I - f, g \rangle_{J_1} \leq ch^2 . \tag{5.6}$$

Finally (5.2) and (5.6) imply (5.1). ■

LEEMA 5.2. *Consider the space P_1^h of piecewise constant functions and suppose that $f \in W^1$, $g \in W^2$, $g \leq 0$, and $f' g \equiv 0$. Then we have:*

$$[f^I - f, g] = 0\,(h^2) . \tag{5.7}$$

Proof. Express the grid intervals of P_1^h as $J_0 \cup J_1$ where the elements of J_0 are all grid intervals T such that $f'|_T = 0$ almost everywhere.

If $T = [s, t] \in J_0$, then $f^I = f$ on (s, t) and

$$\int_{s+}^{t-} g(\sigma)\, d\,(f^I(\sigma)) = 0 . \tag{5.8}$$

Let G_1 be the union of all end points of grid intervals in J_1. Relation (5.8) and the identity $[f, g] = 0$ implies that:

$$[f^I - f, g] = \sum_{t \in G_1} (f^I(t^+) - f^I(t^-))\, g(t) . \tag{5.9}$$

Since $f' g = 0$, we conclude that for all $T \in J_1$, there exists $\sigma \in T$ such that $g(\sigma) = 0$. Moreover, $g'(\sigma) = 0$ since $g \leq 0$ and $g \in W^2$. Therefore, by a Taylor expansion about σ, we get:

$$\| g \|_T \leq \frac{1}{2} \text{meas}\,(T)^2 \, \| g^{(2)} \|_T \quad \text{for all } T \in J_1 . \tag{5.10}$$

On the other hand, by (3.7) we have:

$$| f^I(t^+) - f^I(t^-) | \leq | f^I(t^+) - f(t) | + | f^I(t^-) - f(t) | \leq ch \, \| f^{(1)} \| . \tag{5.11}$$

Finally (5.9)–(5.11) gives us (5.7). ■

We say that $f \in \tilde{W}^k$ if there exist $l < \infty$ and scalars $0 = s_0 < s_1 \ldots \leq s_l = 1$ such that $f^{(k)}$ is essentially bounded on (s_j, s_{j+1}) for all j.

LEMMA 5.3. *Consider the piecewise linear space $S_2^h = C^0 \cap P_h^2$ and suppose that $f, g \in W^1$, $fg \equiv 0$, and $f \in \tilde{W}^2$. Then we have:*

$$\langle f^I - f, g \rangle = 0\,(h^3) . \tag{5.12}$$

Proof. Express the gird intervals of S_2^h as $J_d \cup J$ where J_d is the set of grid intervals T such that $T \cap \{s_k : k = 0, \ldots, l\} \neq \varnothing$, and let J_0 and J_1 be the sets defined in the proof of Lemma 5.1. Since $f = f^I$ on J_0 and (5.5) holds on $J_d \cap J_1$, we get:

$$\langle f^I - f, g \rangle_{J_d} \leq ch^2 \sum_{T \in J_d} \text{meas}\,(T) = 0\,(h^3) . \tag{5.13}$$

On the other hand, for all $T \in J_C$ we have $f''|_T \in L^\infty$ and for piecewise linear interpolation, we obtain:

$$|f - f^I|_T \leqslant ch^2 \, |f^{(2)}|_T \text{ for all } T \in J \cap J_1 . \tag{5.14}$$

Since $f^I = f$ on J_0, (5.14) and (5.3) give us:

$$\langle f^I - f, g \rangle_{J_C} = 0 \, (h^3) . \tag{5.15}$$

Combining Relations (5.13) and (5.15), we get (5.12). ■

LEMMA 5.4. *Consider the pieecewise linear space $S_2^h = C^0 \cap P_2^h$ and suppose that $f \in W^1 \cap \cap \tilde{W}^2$, $g \in W^2$, $g \leqslant 0$, and $f' g \equiv 0$. Then we have:*

$$[f^I - f, g] = 0 \, (h^3) . \tag{5.16}$$

Proof. Let (J_0, J_1) and (J_C, J_d) be the sets defined in Lemmas 5.2 and 5.3 respectively. Since $f, f^I \in W^1$, we see that:

$$[f^I - f, g] = \langle (f^I - f)', g \rangle \text{ and} \tag{5.17}$$

using the identity $f^I = f$ on J_0 leads to:

$$\langle (f^I - f)', g \rangle_{J_0} = 0 . \tag{5.18}$$

The interpolation estimate (3.7) gives us:

$$|(f^I - f)'|_T \leqslant c \text{ for all } T \in T_1 \cap J_d, \text{ and} \tag{5.19}$$

$$|(f^I - f)'|_T \leqslant ch \text{ for all } T \in J_1 \cap J_C . \tag{5.20}$$

Combining relations (5.19) and (5.10), we get:

$$\langle (f^I - f)', g \rangle_{J_1 \cap J_d} \leqslant ch^2 \sum_{T \in J_d} \text{meas } (T) \leqslant ch^3 . \tag{5.21}$$

Similarly, (5.20) and (5.10) imply that:

$$\langle (f^I - f)', g \rangle_{J_1 \cap J_C} \leqslant ch^3 \sum_{T \in J_C} \text{meas } (T) \leqslant ch^3 . \tag{5.22}$$

Relations (5.18), (5.21), and (5.22) complete the proof. ■

Let P_k be the space of polynomials with degree at most $k-1$. Given $f \in C^k$, we say that $p \in P_k$ interpolates f if the number of zeroes of $f - p$ is at least k (counting multiplicities). The following lemma appeared in [5], but the proof was never published:

LEMMA 5.5. *Suppose that $f, g \colon [0, 1] \to R$, $f \geqslant g$, $f \in C^k$, and there exists $p \in P_k$ such that $f \geqslant p \geqslant g$. Then f has an interpolate $f^I \in P_k$ sush that $f \geqslant f^I \geqslant g$.*

REMARK 5.6. If f has an interpolate $f^I \in P_k$ and $\{t_1, \ldots, t_n\}$ are zeroes of $f - f^I$ with associated multiplicities $\{m_1, \ldots, m_n\}$ satisfying

$$\sum_{j=1}^{n} m_j = k ,$$

recall [18, p. 244] that for all $t \in [0, 1]$, there exists $\xi(t) \in [0, 1]$ such that

$$f(t) = f^I(t) + \frac{f^{(k)}(\xi(t))}{k!} \prod_{j=1}^{n} (t-t_j)^{m_j}. \tag{5.23}$$ ■

Proof of Lemma 5.5. Define the set

$$F = \{p \in P_k : f \geqslant p \geqslant g\}. \tag{5.24}$$

The lemma is an immediate consequence of the following assertion:

If $p \in F$, the number of zeroes of $f-p$ (counting multiplicities) is $l-1$, and $l-1<k$, then there exists $q \in P_l$ such that $(p+q) \in F$ and the number of zeroes of $(f-p-q)$ (counting multiplicities) is at least l. (5.25)

Proof of assertion 5.25. Let $\{t_1, \ldots, t_n) \subset [0, 1]$ be the zeroes of $(f-p)$ and let $\{m_1, \ldots, m_n\}$ be the associated multiplicities. Since $f-p \geqslant 0$ on $[0, 1]$, we find by a Taylor expansion about t_j that:

(i) If $t_j<1$, then m_j is even.

(ii) For all j, $(-1)^{m_j}(f-p)^{(m_j)}(t_j)>0$.

Given $\varepsilon>0$, define the function

$$p_\varepsilon(t) = \varepsilon \prod_{j=1}^{n} \frac{|t-t_j|^{m_j}}{m_j!} \quad \text{for } t \in [0, 1]. \tag{5.26}$$

Since m_j is even for $t_j<1$, we see that p_ε is a polynomial on $[0, 1]$; moreover, we have $p_\varepsilon \in P_l$ and $p_\varepsilon \geqslant 0$. Defining

$$\delta_1 = \frac{1}{2} \inf \{|(f-p)^{(m_j)}(t_j)| : j=1, \ldots, n\}, \tag{5.27}$$

(i) and (ii) imply that there exist open intervals $\{I_1, \ldots, I_n\}$ such that $t_j \in I_j$ and

$$(f-p-p_{\delta_1})(t) \geqslant 0 \text{ for all } t \in I_j \text{ and } j=1, \ldots, n. \tag{5.28}$$

Since $p_\varepsilon(t)$ is monotone in ε, (5.28) also gives us: for all $0 \leqslant \varepsilon \leqslant \delta_1$,

$$(f-p-p_\varepsilon)(t) \geqslant 0 \text{ for all } t \in I_j \text{ and } j=1, \ldots, n. \tag{5.29}$$

Define the following:

$$\begin{cases} \Delta = \{t \in [0, 1] : t \notin I_j \text{ for } j=1, 2, \ldots, n\}, \\ \delta_2 = \inf\{f(t)-p(t) : t \in \Delta\}. \end{cases} \tag{5.30}$$

Since f and p are continuous, Δ is compact, and Δ excludes all zeroes of $f-p$, we have $\delta_2>0$. Observe that $|p_\varepsilon| \leqslant \varepsilon$ and consequently

$$(f-p-p_{\delta_2})(t) \geqslant 0 \text{ for all } t \in \Delta. \tag{5.31}$$

Combining (5.29) and (5.31), we get for all $0 \leqslant \varepsilon \leqslant \min\{\delta_1, \delta_2\}$,

$$(f-p-p_\varepsilon)(t) \geqslant 0 \text{ for all } t \in [0, 1]. \tag{5.32}$$

Now define $q = p_{\varepsilon_m}$ where

$$\varepsilon_m = \sup \{\varepsilon : (f - p - p_\varepsilon) \in F\}. \tag{5.33}$$

Since $|p_\varepsilon| \to \infty$ as $\varepsilon \to \infty$, it follows that $\varepsilon_m < \infty$. Moreover, the number of zeroes of $(f - p - q)$ is greater than $l - 1$ (or else $(f - p - q - p_\varepsilon) \in F$ for ε sufficiently small and the extremality of ε_m is violated). This completes the proof of asserion (5.25). ∎

COROLLARY 5.7. *Consider the space $P_p^h (k > 0)$ and suppose that $f \geq 0$ and $f^{(k)}$ is essentially bounded on the interior of all grid intervals for S_k^h; then there exists $p \in S_k^h$ such that $f \geq p \geq 0$ and for all grid intervals T, we have:*

$$|f - p|_T \leq ch^k \; |f^{(k)}|_T. \tag{5.34}$$

Proof. Apply Lemma 5.5 and Remark 5.6 to each grid interval. ∎

COROLLARY 5.8. *Consider the space $S_k^h = C^0 \cap P_k^h \; (k > 1)$ and suppose that $f \in W^1$, $f' \geq 0$ almost everywhere, and $f^{(k)}$ is essentially bounded on the interior of all grid intervals for S_k^h. Then there exists $p \in S_k^h$ such that $p' \geq 0$ and for all grid intervals T, we have*

$$|f - p|_T \leq ch^k \, |f^{(k)}|_T. \tag{5.35}$$

Moreover, p can be chosen so that $p'|_T = 0$ whenever $f'|_T = 0$.

Proof. By Corollary 5.7, there exists $q \in S_{k-1}^h$ such that $f' \geq q \geq 0$ and for all grid intervals T, we have:

$$|f' - q|_T \leq ch^{k-1} \, |f^{(k)}|_T. \tag{5.36}$$

We construct $p \in S_k^h$ on the grid interval $T = [r, s]$ as follows. First set

$$\rho(t) = f(r) + \int_r^t q(s)\, ds; \tag{5.37}$$

then define

$$p(t) = \rho(t) + \frac{(t - r)}{(s - r)} (f(s) - \rho(s)). \tag{5.38}$$

Since $0 \leq q \leq f'$, we have both $\rho' = q \geq 0$ and $f(s) - \rho(s) \geq 0$. Therefore, $p' \geq 0$ and $p'|_T = 0$ whenever $f'|_T = 0$. Furthermore, (5.38) implies that $p(r) = f(r)$ and $p(s) = f(s)$. Hence $p \in C^0 \cap P_k^h$. Subtracting from (5.37) the identity

$$f(t) = f(r) + \int_r^t f'(s)\, ds \tag{5.39}$$

and utilizing (5.36), we get

$$|f - \rho|_T \leq ch^k \, |f^{(k)}|_T. \tag{5.40}$$

But by (5.38), we have

$$|p - \rho|_T \leq |f(s) - \rho(s)|. \tag{5.41}$$

Finally (5.40) and (5.41) imply (5.35). ∎

6. Control and State Error

Associated with any solution to the dual approximations (3.11)–(3.13) is a pair $z^h=(x^h, u^h)$ that achieves the minimum in the dual function. This section studies the error $\|z^h-z^*\|$ where $z^*=(x^*, u^*)$.

First let us describe more precisely the pair (x^h, u^h). If $\theta^h=(p^h, \lambda^h, w^h)$ solves (3.11), we define

$$\eta^h(t)^T=(\dot{p}^h(t)^T, p^h(t)^T, \lambda^h(t)^T, w^h(t)^T) \text{ for all } t \in [0, 1] \tag{6.1}$$

and z^h is the function $\tilde{z}$ given by (2.27) for $\eta=\eta^h$. Similarly, if $\mu^h=(q^h, \lambda^h, v^h)$ solves (3.12), we define

$$\eta^h(t)^T=(\dot{q}^h(t)^T, q^h(t)^T, \lambda^h(t)^T, v^h(t)^T) \text{ for all } t \in [0, 1] \tag{6.2}$$

and z^h is the function $\tilde{z}$ given by (2.32) for $\eta=\eta^h$. Finally, if $\sigma^h=(x^h, \lambda^h, v^h)$ solves (3.10), we choose $(q^h, \tilde{u}=u^h)$ satisfying (2.34)–(2.35) for $(\tilde{x}, \lambda, v)=(x^h, \lambda^h, v^h)$ where $q^h(1)=0$.

The tables below indicate the convergence rates that will be established for the error $\|z^h-z^*\|$ using various choices for the finite element spaces. The first three columns give the finite element space associated with each dual variable, and the last column gives the exponent in the convergence rate for $\|z^h-z^*\|$. Following the tables, we state the regularity needed for our convergence proofs.

The entries (k,j), $(\tilde{k},j)$, and k in the tables below mean $P_k^h \cap C^j$, $P_{\tilde{k}}^h \cap C^j$, and P_k^h, respectively.

	Method (3.11)			
	p	λ	w	Error
1.	(2, 0)	1	1	.5
2.	$(\tilde{k}+1,0)$	$\tilde{k}$	$\tilde{k}$	k
	Method (3.12) (Affine State Constraints)			
	q	λ	v	Error
3.	(2,0)	1	1	1.0
4.	(3,0)	(2,0)	(2,0)	1.5
5.	$(\tilde{k}+1,0)$	$\tilde{k}$	$(\tilde{k}, 0)$	k
	Method (3.13) (Affine State Constraints)			
	x	λ	v	Error
6.	(2,0)	1	(2,0)	1.0
7.	(2,0)	(2,0)	(2,0)	1.5
8.	$(\tilde{k}, 0)$	$\tilde{k}$	$(\tilde{k}, 0)$	k
	Method (3.13) (General Convex State Constraints)			
	x	λ	v	Error
9.	(2,0)	1	(2,0)	1.0
10.	(3,0)	(2,0)	(2,0)	1.5
11.	$(\tilde{k}+1,0)$	$\tilde{k}$	$(\tilde{k}, 0)$	k

REMARK 6.1. We show later that the convergence rate in cost (such as $\mathscr{L}(\theta^h) - C(z^*)$) is always twice the convergence rate of z^h.

REMARK 6.2. If the state constraints of (C) are vacuous, then programs (3.11) and (3.12) are the same; hence, the error in method (3.11) will satisfy the better estimates given for method (3.12) above.

Assumptions

In all cases, we assume that (1.2)–(1.4) hold; hence, there exist solutions to the primal and the dual problems. In addition, for each of the 11 cases above, we make the corresponding regularity assumptions listed below. The terminology "FA holds" means that there are a finite number of times on [0 ,1] where a state constraint alternates from binding to nonbinding (finite alteration).

1. $(x^*, u^*, \lambda^*, v^*, p^*) \in W^1$ and $(v^*, p^*) \in \tilde{W}^2$.
2. $(v^*, p^*) \in W^1$, $(\lambda^*. \dot{v}^*, \dot{p}^*) \in \tilde{W}^k$, and FA holds.
3. $(\dot{q}^*, \dot{x}^*, u^*, \lambda^*, v^*) \in W^1$.
4. $(\dot{q}^*, \dot{x}^*, u^*, \lambda^*, v^*) \in W^1$ and $(\lambda^*, v^*, \dot{q}^*) \in \tilde{W}^2$.
5. $(q^*, v^*) \in W^1$, $(\dot{q}^*, \lambda^*, v^*) \in \tilde{W}^k$, and FA holds.
6. $(\dot{x}^*, \lambda^*, v^*, u^*) \in W^1$.
7. $(\dot{x}^*, \lambda^*, v^*, u^*) \in W^1$ and $(\lambda^*, v^*) \in \tilde{W}^2$.
8. $(x^*, v^*) \in W^1$, $(x^*, \lambda^*, v^*) \in \tilde{W}^k$, and FA holds.
9. $(\dot{x}^*, \lambda^*, v^*, u^*) \in W^1$ and $K_s \in C^3$.
10. $(x^*, \lambda^*, v^*, u^*) \in W^1$, $(\dot{x}^*, \lambda^*, v^*) \in \tilde{W}^2$, and $K_s \in C^3$.
11. $(x^*, v^*) \in W^1$, $(\dot{x}^*, \lambda^*, v^*) \in \tilde{W}^k$, $K_s \in C^3$, and FA holds.

Proofs are now presented for Results 1, 3, 4, 5, and 9. Using the techniques introduced in these proofs, the remaining results can be established.

Proof of Result 3. Define $\mu=(q, \lambda, v)$, $z=(x, u)$, $\mu^*=(q^*, \lambda^*, v^*)$, $H(\mu)=L(\mu)$ (L given by (2.30)), and

$$G(z, \mu)=\langle S^T v-q, \dot{x}-Ax-Bu\rangle+\langle\lambda, K_c(u)\rangle+[v, K_s(x)]. \tag{6.3}$$

By the complementary slackness conditions, $G(z^*, \mu^*)=0$ and consequently we have:

$$G(z^*, \mu)-G(z^*, \mu^*)\leqslant 0 \text{ for all } \mu \in M. \tag{6.4}$$

Let us apply Lemma 4.4 with $\gamma^h=\mu^I=(q^I, \lambda^I, v^I)$, an interpolate of (q^*, λ^*, v^*) satisfying $q^I(1)=0$.

Consider the D terms in (4.20): After an integration by parts, we see that

$$D(z^*, \eta^h, \eta^*)=\int_0^1 [h(z^h(t), \eta^h(t), t)-h(z^*(t), \eta^*(t), t) -(\eta^h(t)-\eta^*(t))^T g(z^*(t), t)]\, dt \tag{6.5}$$

where $h(\cdot, \cdot, \cdot)$ and $g(\cdot, \cdot)$ were defined in (2.31),

$$\eta^h(t)=(\dot{q}^h(t), q^h(t), \lambda^h(t), v^h(t)), \text{ and} \tag{6.6}$$

$$\eta^*(t)=(\dot{q}^*(t), q^*(t), \lambda^*(t), v^*(t)). \tag{6.7}$$

By Theorem 1.1 and Lemma 2.7, $z^*(t)$ achieves the pointwise minimum of $h(\cdot, \eta^*(t), t)$ for almost every $t \in [0, 1]$, and by construction $z^h(t)$ achieves the pointwise minimum of $h(\cdot, \eta^h(t), t)$ for almost every $t \in [0, 1]$.

For fixed t, define

$$\begin{cases} h(z, \eta) = h(z, \eta, t), & g(z) = g(z, t) \\ (z_1, \eta_1) = (z^h(t), \eta^h(t)), & (z_2, \eta_2) = (z^*(t), \eta^*(t)). \end{cases} \tag{6.8}$$

Applying (4.9) and integrating over $t \in [0, 1]$, we get:

$$D(z^*, \mu^h, \mu^*) \leqslant -\frac{\alpha}{2} \|z^h - z^*\|^2. \tag{6.9}$$

Similarly, applying (4.14) with $\eta_1^T = \eta^I(t)^T = (\dot{q}^I(t)^T, q^I(t)^T, \lambda^I(t)^T, \nu^I(t)^T)$, we have:

$$D(z^*, \mu^I, \mu^*) \geqslant -c\,\{\|\dot{q}^I - \dot{q}^*\|^2 + \|q^I - q^*\|^2 + \|\lambda^I - \lambda^*\|^2 + \|\nu^I - \nu^*\|^2\} \tag{6.10}$$

where $\infty > c > 0$ is a constant depending on parameters such as $|\nabla_1 K_C(u^*(\cdot), \cdot)|$, $\|\dot{S}\|$, and α. Combining (6.9), (6.10), and (4.20), we find:

$$\|z^h - z^*\|^2 \leqslant c\,\{\langle \lambda^* - \lambda^I, K_c(u^*)\rangle + [\nu^* - \nu^I, K_s(x^*)] + \|\dot{q}^I - \dot{q}^*\|^2 + \|q^I - q^*\|^2 + \|\lambda^I - \lambda^*\|^2 + \|\nu^I - \nu^*\|^2\}. \tag{6.11}$$

Finally (3.7), Lemma 5.1, Lemma 5.2, and (6.11) give us $\|z^h - z^*\| = 0(h)$. ∎

Proof of Result 4. Since $\mu^I = (q^I, \lambda^I, \nu^I) \in M$, (6.11) again holds. Using the additional regularity $(\dot{q}^*, \lambda^*, \nu^*) \in \tilde{W}^2$, (6.11) and Lemmas 5.3–5.4 give us $\|z^h - z^*\| = 0(h^{3/2})$. ∎

Proof of Result 5. Choose $S_k^h \subset \tilde{S}_k^h$ so that a grid point of S_k^h lies at each point of discontinuity in $(\lambda^*, \dot{\nu}^*, \dot{q}^*)$ and at each point where a state constraint changes from binding to nonbinding. Let q^I be an interpolate of q^* satisfying $q^I(1) = 0$, and let (λ^I, ν^I) be the approximations to (λ^*, ν^*) given by Corollaries 5.7 and 5.8 respectively. Hence $\mu^I = (q^I, \lambda^I, \nu^I) \in M$, and (6.11) holds.

Since $\lambda^* \geqslant \lambda^I \geqslant 0 \geqslant K_C(u^*)$ and $\langle \lambda^*, K_C(u^*)\rangle = 0$, we see that

$$\langle \lambda^* - \lambda^I, K_c(u^*)\rangle = 0. \tag{6.12}$$

Since a grid point is situated wherever a state constraint changes from binding to nobinding and $\dot{\nu}_j^I = 0$ on grid intervals where $\dot{\nu}_j^* = 0$, we conclude that $\dot{\nu}_j^I = 0$ whenever $\dot{\nu}_j^* = 0$ and

$$[\nu^* - \nu^I, K_s(x^*)] = 0. \tag{6.13}$$

Therefore, (6.11), (3.7), (5.34), and (5.35) give us $\|z^h - z^*\| = 0(h^k)$. ∎

Proof of Result 1. Given $\mu = (p, \lambda, w) \in \mathfrak{M}$, let $\delta(p(1))$ denote the value of δ achieving the maximum in (2.10) for $\gamma = p(1)$. Define $z = (x, u)$, $\mu^* = (p^*, \lambda^*, w^* = \dot{\nu}^*)$,

$$H(\mu) = \mathscr{L}_0(\mu) + \mathscr{L}_1(p(1)) - p(0)^T x_0, \text{ and} \tag{6.14}$$

$$G(z, \mu) = \langle p, \dot{x} - Ax - Bu\rangle + \langle \lambda, K_c(u)\rangle + \langle w, K_s(x)\rangle - K_s(x(1), 1)^T \delta(p(1)). \tag{6.15}$$

Since $q^*(1^-)=0$, we see that

$$\nabla_1 K_s(x^*(1), 1)^T v^*(1^-)-p^*(1)=0. \tag{6.16}$$

From the complementary slackness condition $[v^*, K_s(x^*)]=0$ and the normalization $v^*(1)=0$, we obtain

$$K_s(x^*(1), 1)^T v^*(1^-)=0. \tag{6.17}$$

Since the components of $K_s(\cdot, 1)$ are convex and $K_s(x^*(1), 1)\leqslant 0$, (6.16) and (6.17) imply that

$$\delta(p^*(1))=v^*(1^-) \text{ and} \tag{6.18}$$

$$\mathscr{L}_1(p^*(1))=p^*(1)^T x^*(1). \tag{6.19}$$

Therefore, using (6.17), we have $G(z^*, \mu^*)=0$ and

$$G(z^*, \mu)-G(z^*, \mu^*)\leqslant 0 \text{ for all } \mu\in\mathfrak{M}. \tag{6.20}$$

Let us apply Lemma 4.4 with $\gamma^h=\mu^I=(p^I, \lambda^I, w^I)$, an interpolate of μ^* satisfying $p^I(1)=p^*(1)$. Define the variable η^h by (6.1) and the variables η^* and Ω^h by

$$\eta^*(t)^T=(\dot{p}^*(t)^T, p^*(t)^T, \lambda^*(t)^T, \dot{v}^*(t)^T), \text{ and} \tag{6.21}$$

$$\Omega^h=\mathscr{L}_1(p^h(1))-\mathscr{L}_1(p^*(1))-\{x^*(1)^T(p^h(1)-p^*(1))$$
$$+K_s(x^*(1), 1)^T[\delta(p^h(1))-\delta(p^*(1))]\}. \tag{6.22}$$

After integrating by parts, we obtain:

$$D(z^*, \eta^h, \eta^*)=\int_0^1 [h(z^h(t), \eta^h(t), t)-h(z^*(t), \eta^*(t), t)$$
$$-[(\eta^h(t)-\eta^*(t))^T g(z^*(t), t)]\, dt+\Omega^h \tag{6.23}$$

where $h(\cdot, \cdot, \cdot)$ and $g(\cdot, \cdot)$ were given by (2.5). Combining (6.17)–(6.19), we observe that

$$\Omega^h=\mathscr{L}_1(p^h(1))-x^*(1)^T p^h(1)+K_s(x^*(1), 1)^T \delta(p^h(1))\leqslant 0 \tag{6.24}$$

since

$$\mathscr{L}_1(p^h(1))=\mathscr{L}_1(p^h(1), \delta(p^h(1)))$$
$$=\text{minimum}\{x^T p^h(1)-K_s(x, 1)^T \delta(p^h(1)): x\in R^n\}. \tag{6.25}$$

By Theorem 1.1, (x^*, u^*) achieves the minimum in (1.5) for $(p, \lambda, v)=(p^*, \lambda^*, v^*)$. Hence by Corollary 2.8, we have:

$$h(z^*(t), \eta^*(t), t)=\text{minimum}\{h(z, \eta^*(t), t): z\in R^{n+m}\} \tag{6.26}$$

for almost every $t\in[0, 1]$.

As in the proof of Result 3, we apply Lemma 4.2 to the integrand in (6.23) and combine with (6.24) to get:

$$D(z^*, \eta^h, \eta^*)\leqslant -\frac{\alpha}{2}\|z^h-z^*\|^2. \tag{6.27}$$

Similarly applying (4.14) with $\eta_1^T = \eta^I(t)^T = (\dot{p}^I(t)^T, p^I(t)^T, \lambda^I(t)^T, \dot{v}^I(t)^T)$ leads to:

$$D(z^*, \eta^I, \eta^*) \geqslant -c\{\|\dot{p}^I - \dot{p}^*\|^2 + \|p^I - p^*\|^2 + \|\lambda^I - \lambda^*\|^2 + \|w^I - w^*\|^2\}. \quad (6.28)$$

By our regularity assumption and (3.7), we see that $p^I - p^* = 0(h) = \lambda^I - \lambda^*$ and $\dot{p}^I - \dot{p}^* = 0(h) = w^I - w^*$ except for a fixed set of grid intervals where $\dot{p}^I - \dot{p}^* = 0(1) = w^I - w^*$. Hence (6.28) gives us:

$$D(z^*, \eta^I, \eta^*) \geqslant 0(h). \quad (6.29)$$

Combining (4.20), (6.27), and (6.29), we get:

$$\|z^h - z^*\|^2 \leqslant c\langle \lambda^* - \lambda^I, K_c(u^*)\rangle + c\langle w^* - w^I, K_s(x^*)\rangle + 0(h). \quad (6.30)$$

By Lemma 5.1, $\langle \lambda^* - \lambda^I, K_c(u^*)\rangle = 0(h^2)$; similarly, $\langle w^* - w^I, K_s(x^*)\rangle = 0(h)$ since $w^I - w^* = 0(h)$ except for a fixed set of grid intervals where $w^* - w^I = 0(1)$. Therefore, (6.30) implies that $\|z^h - z^*\| = 0(h^{1/2})$. ∎

Proof of Result 9. Define $\mu = (x, \lambda, v)$, $z = (x, u)$, $\mu^* = (x^*, \lambda^*, v^*)$, and $H(\mu) = l(\mu)$ where l is given by (2.43). For given $\mu = (\tilde{x}, \lambda, v) \in F$, the feasible set (3.3), let $(q(\mu), u(\mu))$ denote the $(q, \tilde{u})$ pair satisfying (2.34)–(2.35) and the initial condition $q(1) = 0$; we set

$$p(\mu)(\cdot) = \nabla_1 K_s(x(\cdot), \cdot)^T v(\cdot) - q(\mu)(\cdot) \text{ and} \quad (6.31)$$

$$G(z^*, \mu) = \langle p(\mu), \dot{x} - Ax - Bu\rangle + \langle \lambda, K_c(u)\rangle + [v, K_s(x)] - K_s(x(1), 1)^T v(1^-). \quad (6.32)$$

Using (6.17) we find:

$$G(z^*, \mu) - G(z^*, \mu^*) \leqslant 0 \text{ for all } \mu \in F. \quad (6.33)$$

Let us apply (4.20) with $\gamma^h = \mu^I = (x^I, \lambda^I, v^I)$, an interpolate of (x^*, λ^*, v^*) such that $x^I(0) = x_0$, $x^I(1) = x^*(1)$, and $v^I(1) = v^*(1^-)$. Define $u^h = u(\mu^h)$, $u^I = u(\mu^I)$, $p^h = p(\mu^h)$, η^h by (6.1) with w^h replaced by $\dot{v}^h$, and η^* by (6.21). By our choice of finite element spaces, observe that $p^h \in A$; integrating by parts, we get:

$$D(z^*, \eta^h, \eta^*) = \int_0^1 [h(z^h(t), \eta^h(t), t) - h(z^*(t), \eta^*(t), t) - (\eta^h(t) - \eta^*(t))^T g(z^*(t), t)]\, dt + \Omega^h \quad (6.34)$$

where $h(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot)$ were given in (2.5) and

$$\Omega^h = p^h(1)^T x^h(1) - K_s(x^h(1), 1)^T v^h(1^-) - p^*(1)^T x^*(1) + K_s(x^*(1), 1)^T v^*(1^-) - \{x^*(1)^T(p^h(1) - p^*(1)) - K_s(x^*(1), 1)^T(v^h(1^-) - v^*(1^-))\}. \quad (6.35)$$

Since $(\tilde{x}, \tilde{u}) = (x^h, u^h)$ satisfy (2.34)–(2.35) for $(q, \lambda, v) = (q(\mu^h), \lambda^h, v^h)$, we know that (x^h, u^h) achieves the minimum in the dual function (1.5) for $(p, \lambda, v) = (p^h, \lambda^h, v^h)$. Hence Corollary 2.8 is applicable and (2.39) gives us

$$p^h(1)^T x^h(1) - K_s(x^h(1), 1)^T v^h(1^-) = \mathscr{L}_1(p^h(1), v^h(1^-)). \quad (6.36)$$

Combing (6.17) and (6.36), we obtain $\Omega^h \leqslant 0$. Moreover, Corollary 2.8 implies that $z^h(t)$ and $z^*(t)$ minimize $h(\cdot, \eta^h(t), t)$ and $h(\cdot, \eta^*(t), t)$ respectively for almost every $t \in [0, 1]$.

As in the proof of Result 3, the application of Lemma 4.2 to the integrand in (6.34) leads to:

$$D(z^*, \eta^h, \eta^*) \leqslant -\frac{\alpha}{2}\|z^h - z^*\|^2. \tag{6.37}$$

Similarly, (4.15), (2.47), and (3.7) give us:

$$D(z^*, \eta^I, \eta^*) \geqslant -c\{\|x^I - x^*\|^2 + \|u^I - u^*\|^2\}$$

$$\geqslant -c\{\|\mu^I - \mu^*\|^2 + \|\dot{x}^I - \dot{x}^*\|^2\} = 0(h^2). \tag{6.38}$$

Moreover, by Lemmas 5.1 and 5.2, we have:

$$G(z^*, \mu^I) - G(z^*, \mu^*) = \langle \lambda^I - \lambda^*, K_c(u^*)\rangle + [\nu^I - \nu^*, K_s(x^*)] = 0(h^2). \tag{6.39}$$

Combining (4.20) and (6.37)–(6.39), we get $\|z^h - z^*\| = 0(h)$. ■

Proof of Remark 6.1. Suppose that μ^h satisfies (4.18) and $\mu^* \in M$ satisfies

$$H(\mu^*) = \text{maximum}\{H(\mu): \mu \in M\}. \tag{6.40}$$

Observe that

$$H(\mu^*) \geqslant H(\mu^h) \geqslant H(\mu^I) = H(\mu^*) + G(z^*, \mu^I) - G(z^*, \mu^*) + D(z^*, \mu^I, \mu^*). \tag{6.41}$$

In the proofs above, we worked with the estimate

$$\|z^h - z^*\|^2 \leqslant \frac{2}{\alpha}|G(z^*, \mu^I) - G(z^*, \mu^*)| + \frac{2}{\alpha}|D(z^*, \mu^I, \mu^*)|. \tag{6.42}$$

Hence (6.41) implies that the error bound for $H(\mu^*) - H(\mu^h)$ would be the square of the bound for $\|z^h - z^*\|$. ■

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Wypukłe problemy sterowania i aproksymacje dualne

Korzystając z dualnego sformułowania wypukłych problemów sterowania [2] otrzymano oszacowanie błędu dla aproksymacji elementami skończonymi. Wcześniejsze rezultaty [4] dotyczące metody Ritza-Trefftza dla zadań z kwadratowym wskaźnikiem jakości przy afinicznych ograniczeniach nierównościowych stanu i sterowania zostały rozszerzone na ogólny przypadek wypukły. Również wprowadzono i zbadano dwie nowe wersje metody Ritza-Trefftza.

Выпуклые задачи управления и дуальные аппроксимации

Используя дуальную формулировку выпуклых задач управления [2] получена оценка ошибки аппроксимации конечными элементами. Более ранние результаты [4], касающиеся метода Рица-Трефца для задач с квадратным показателем качества, при аффинных ограничениях типа неравенства на состояние и управление, были расширены для общей выпуклой задачи. Были введены и исследованы также две новые версии метода Рица-Трефца.

Erratum to Part I

Page	Instead of	should be
1_{12}	[11-14], [7-8], [17], [18]	[12-15], [8-9], [19], [20]
11_9	[5]	[6]
12_2	[16, Theorem 2.17]	[17, Theorem 2.17]
20^{10}	$\mathfrak{J}^h$	J^h