Appl Math Optim 31:297–326

Applied Mathematics & Optimization

Optimality, Stability, and Convergence in Nonlinear Control*

A. L. Dontchev,[1] W. W. Hager,[2] A. B. Poore,[3] and Bing Yang[4]

[1]Mathematical Reviews, 416 Fourth Street,
Ann Arbor, MI 48107, USA

[2]Department of Mathematics, University of Florida,
Gainesville, FL 32611, USA

[3]Department of Mathematics, Colorado State University,
Fort Collins, CO 80523, USA

[4]Department of Mathematics, West Virginia Tech,
Montgomery, WV 25136, USA

Communicated by I. Lasiecka

Abstract. Sufficient optimality conditions for infinite-dimensional optimization problems are derived in a setting that is applicable to optimal control with endpoint constraints and with equality and inequality constraints on the controls. These conditions involve controllability of the system dynamics, independence of the gradients of active control constraints, and a relatively weak coercivity assumption for the integral cost functional. Under these hypotheses, we show that the solution to an optimal control problem is Lipschitz stable relative to problem perturbations. As an application of this stability result, we establish convergence results for the sequential quadratic programming algorithm and for penalty and multiplier approximations applied to optimal control problems.

*This research was supported by the U.S. Army Research Office under Contract Number DAAL03-89-G-0082, by the National Science Foundation under Grant Number DMS 9404431, and by Air Force Office of Scientific Research under Grant Number AFOSR-88-0059. A. L. Dontchev is on leave from the Institute of Mathematics, Bulgarian Academy of Sciences, Sofia, Bulgaria.

Key Words. Sufficient optimality conditions, Stability, Sensitivity, Sequential quadratic programming, Penalty/multiplier methods.

AMS Classification. 49K40, 49M30.

1. Introduction

This paper begins with a study of sufficient optimality conditions for infinite-dimensional optimization problems. The theory and assumptions are formulated so that they apply to optimal control problems. This involves the introduction of three norms with properties analogous to those of the L^1-, L^2-, and L^∞-norms. Papers most closely related to this work on sufficient optimality conditions include the analysis [15] of Maurer and Zowe for general optimization problems, and Maurer's analysis [14] of sufficient optimality conditions in optimal control. Maurer's analysis [14] is formulated using two norms with properties analogous to those of the L^2- and L^∞-norms. Recently [4, Lemma 8], we presented an abstract sufficient optimality result that weakens the coercivity assumption in [15] by treating certain inequality constraints as equality constraints. This weakening of the coercivity assumption was at the expense of an interiority assumption that does not hold in the optimal control setting. In Section 2 we replace this interiority assumption by a weaker L^1 complementarity condition which is better suited for the control problem.

In Section 3 we apply the abstract sufficient optimality conditions to nonlinear control problems with endpoint constraints on the state and with set-valued control constraints. Under a uniform independence assumption for the gradients of active constraints, a controllability assumption, and a relatively weak coercivity assumption, we show that any point satisfying the Minimum Principle is a strict local minimizer for the optimal control problem. In Section 4 we consider a more special problem in which the set-valued control constraints are replaced by a system of inequalities. By considering ε-relaxations of the original inequality constraints, we obtain a problem to which the theory of Section 3 is applicable. Since this relaxed problem has fewer constraints than the original problem, we conclude that a local minimizer for the relaxed problem, that is feasible in the original problem, is also a local minimizer for the original problem. In this way we are able to obtain a sufficient optimality result for the original optimal control problem with inequality control constraints. In a related paper [6] Dunn and Tian established a sufficient optimality result for an optimal control problem with a single inequality control constraint $u \geq 0$, with a quasi-quadratic type cost function, with quasi-linear system dynamics, and with a free terminal state. In their paper Dunn and Tian utilize three norms, the L^1-, L^2-, and L^∞-norms similar to our treatment of sufficient optimality. Although the problem studied in [6] has a very special structure, the way the authors handle the interplay between complementarity, integral coercivity, and pointwise coercivity are a significant sharpening of earlier approaches to sufficient conditions for nonconvex infinite-dimensional constrained optimization problems, and were influential in motivating our analysis of the general control problem with endpoint constraints on the state, and with equality and inequality constraints on the control.

Sufficient optimality conditions in optimal control have also been developed by Zeidan *et al.*—see [16], [23], [24], and the references therein. The techniques employed in Zeidan's analysis of sufficient optimality make use of a strengthened form of the control minimum principle and a Jacobi-type condition. Here we exploit an integral coercivity condition. It is known that coercivity is related to the existence of solutions of a corresponding Riccati equation (see [17] and [25]), however, the precise connection between these two conditions, for the setting of this paper, has not been fully investigated yet.

In Section 5 we analyze the stability of the solution to an optimal control problem relative to a parameter. Under the sufficient optimality conditions, we obtain a Lipschitz stability result for the optimal control problem. This result generalizes the corresponding result in Section 6 of [4] by replacing the convex control constraints in [4] by general nonconvex constraints, by allowing the parameter to appear in the constraints, and by including endpoint constraints. Other papers related to stability and convergence analysis of infinite-dimensional optimization problems include those of Alt [1]–[3], Malanowski [11]–[13], and Ito and Kunisch [10]. One of the differences between these papers and ours relates to the abstract formulation and the underlying assumptions. In our paper we develop an abstract setting that more nearly models the optimal control setting by working in three different norms with properties analogous to the L^1-, L^2-, and L^∞-norms. In this way we obtain a coercivity assumption that is weaker than the earlier ones since the active control constraints are taken into account.

In Section 6 we use the stability theory to establish convergence of the sequential quadratic programming algorithm in optimal control, while in Section 7 we establish a convergence result for penalty and multiplier approximations. These estimates for penalty and multiplier approximations extend the analysis of [8] by treating possibly nonconvex control constraints as well as endpoint constraints. In the thesis of Yang [22], penalty and multiplier approximations are analyzed for problems with equality control constraints. The analysis in this paper is more general in that both equality and inequality control constraints are permitted.

2. A Sufficient Optimality Condition

Let us consider the following abstract optimization problem:

$$\text{minimize } C(z) \qquad \text{subject to} \quad g(z) \in K_g, \quad h(z) \in K_h, \tag{1}$$

where $g: Z \to W_g$ and $h: Z \to W_h$, W_g, and W_h are normed linear spaces, and K_g and K_h are closed, convex cones with vertices at the origin of their respective spaces. The norms of Z, W_g, and W_h, denoted $\|\cdot\|_\infty$, correspond to the L^∞-norm for the optimal control problems considered in the succeeding sections. The Lagrangian H associated with (1) is given by

$$H(z, \chi, \psi) = C(z) - \langle \chi, g(z) \rangle - \langle \psi, h(z) \rangle,$$

where $\chi \in W_g^*$ and $\psi \in W_h^*$ (a * is used to denote the dual space and $\langle \cdot, \cdot \rangle$ is the duality map). Let z_* be a point that is feasible in (1) and assume that C, g, and h are twice Fréchet differentiable at z_*. The Kuhn–Tucker conditions associated with (1) have the following form: $\chi \in K_g^+$ and $\psi \in K_h^+$ exist such that

$$\nabla_z H(z_*, \chi, \psi) = 0, \qquad \langle \chi, g(z_*) \rangle = 0, \qquad \langle \psi, h(z_*) \rangle = 0, \tag{2}$$

where the $^+$ is used to denote the polar cone;

$$K_g^+ = \{ \chi \in W_g^* : \langle \chi, k \rangle \geq 0 \text{ for every } k \in K_g \}.$$

In order to obtain a sufficient optimality result applicable to a broad class of optimal control problems, we introduce two additional norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in the space W_h, and the norm $\|\cdot\|_2$ in Z and W_g, which are related in the following way:

$$\lim_{w \in W_h, \|w\|_\infty \to 0} \frac{\|w\|_2^2}{\|w\|_1} = 0 \quad \text{and} \quad \lim_{w \in W_h, \|w\|_\infty \to 0} \|w\|_2 = 0. \tag{3}$$

To be concrete, identify these three norms with those of L^1, L^2, and L^∞. The ∞-norm corresponds to the norm in which the optimization problem (1) is formulated. It is also the norm in which C, g, and h are twice Fréchet differentiable at z_*. We assume that the Taylor remainder terms can be bounded in the 2-norm. More precisely, we assume that

$$\begin{aligned} H(z, \chi, \psi) &= H(z_*, \chi, \psi) + \langle \nabla_z H(z_*, \chi, \psi), z - z_* \rangle \\ &\quad + \tfrac{1}{2} \langle \nabla_{zz}^2 H(z_*, \chi, \psi)(z - z_*), z - z_* \rangle + R(z), \\ h(z) &= h(z_*) + \nabla h(z_*)(z - z_*) + R_h(z), \quad \text{and} \\ g(z) &= g(z_*) + \nabla g(z_*)(z - z_*) + R_g(z), \end{aligned}$$

where, for each $\varepsilon > 0$, a δ exists such that

$$\begin{aligned} &|R(z)| \leq \varepsilon \|z - z_*\|_2^2, \qquad \|R_h(z)\|_2 \leq \varepsilon \|z - z_*\|_2, \quad \text{and} \\ &\|R_g(z)\|_2 \leq \varepsilon \|z - z_*\|_2 \end{aligned} \tag{4}$$

whenever $\|z - z_*\|_\infty \leq \delta$. Moreover, a constant $\beta > 0$ exists such that

$$\langle \nabla_{zz}^2 H(z_*, \chi, \psi) y, z \rangle \leq \beta \|y\|_2 \|z\|_2 \tag{5}$$

for each y and $z \in Z$.

In [4] we present a sufficient optimality result in which it is assumed that the multiplier ψ lies in the interior of the set K_h^+. This assumption is too strong for many control problems since the set K_h^+ may not have an interior. In this paper we assume that the multiplier ψ has the following property: $\gamma > 0$ exists such that

$$\langle \psi, k \rangle \geq \gamma \|k\|_1 \qquad \text{for each} \quad k \in K_h. \tag{6}$$

Note that the condition $\langle \psi, h(z_*) \rangle = 0$ in (2) along with (6) imply that $h(z_*) = 0$.

The next assumption is a stability-type condition for the linearized constraints: For each δg and δh in an ∞-norm neighborhood of the origin in W_g and W_h, respectively, we assume that a solution $\delta z \in Z$ to the equations

$$\nabla h(z_*)\delta z + \delta h = 0 \quad \text{and} \quad g(z_*) + \nabla g(z_*)\delta z + \delta g \in K_g$$

exists; hence, by the Robinson–Ursescu theorem (see [18] and [21]), δz can be chosen so that

$$\|\delta z\|_\infty \leq \beta\|\delta h\|_\infty + \beta \inf\{\|g(z_*) + \delta g - k\|_\infty : k \in K_g\}, \tag{7}$$

where β is a generic constant, independent of δg and δh. In addition, we assume that this δz can be chosen so that

$$\|\delta z\|_2 \leq \beta\|\delta h\|_2 + \beta \inf\{\|g(z_*) + \delta g - k\|_2 : k \in K_g\}. \tag{8}$$

Our final assumption is a coercivity assumption relative to the linearized constraints: Constants α, $\beta > 0$ exist such that

$$\langle \nabla^2_{zz} H(z_*, \chi, \psi)(z - z_*), z - z_*\rangle \geq \alpha\|z - z^*\|_2^2 \tag{9}$$

whenever $g(z_*) + \nabla g(z_*)(z - z_*) \in K_g$, $\nabla h(z_*)(z - z_*) = 0$, and $\|z - z_*\|_\infty \leq \beta$. Note that the constraint $h(z) \in K_h$ of (1) is a cone-type constraint while the coercivity assumption (9) is expressed relative to a null space constraint.

Theorem 1. *If z_* is feasible in* (1), $\chi \in K_g^+$, $\psi \in K_h^+$, *the Kuhn–Tucker conditions* (2) *hold, and conditions* (4)–(9) *are satisfied, then z_* is a strict local minimizer for* (1). *That is, an ∞-norm neighborhood of z_* exists such that, for each feasible point z in this neighborhood, we have $C(z) > C(z_*)$. Moreover, we will see that, for each $\varepsilon > 0$, an associated ∞-norm neighborhood U of z_* exists such that*

$$C(z) \geq C(z_*) + \tfrac{1}{2}(\alpha - \varepsilon)\|z - z_*\|_2^2$$

for every $z \in U$ that is feasible in (1) *where α is specified in* (9).

Proof. Throughout the proof, ε denotes a *generic* positive constant that is independent of z in a neighborhood of z_* and which can be chosen arbitrarily small for z sufficiently close to z_*, and β denotes a *generic* constant that is uniformly bounded from above for z near z_*. Qualitative phrases such as z near z_* should always be interpreted relative to the ∞-norm.

The Taylor expansion for H combined with (2) implies that

$$C(z) = C(z_*) + M(z) + R(z),$$

where

$$M(z) = \langle \chi, g(z)\rangle + \langle \psi, h(z)\rangle + \tfrac{1}{2}\langle \nabla^2_{zz} H(z_*, \chi, \psi)(z - z_*), z - z_*\rangle.$$

Since $\chi \in K_g^+$ and (6) holds, it follows that, for z feasible in (1), we have

$$\langle \chi, g(z) \rangle + \langle \psi, h(z) \rangle \geq \gamma \|h(z)\|_1. \tag{10}$$

Given z near z_* and defining $\delta h = \nabla h(z_*)(z - z_*)$ and $\delta g = \nabla g(z_*)(z - z_*)$, there is δz satisfying (8) with $y = \delta z + z$ satisfying the following relations:

$$\nabla h(z_*)(y - z_*) = 0 \quad \text{and} \quad g(z_*) + \nabla g(z_*)(y - z_*) \in K_g.$$

If z is both near z_* and feasible in (1), it follows from (4) and (8) that

$$\|\delta z\|_2 \leq \beta \|h(z)\|_2 + \varepsilon \|z - z_*\|_2.$$

Applying the triangle inequality yields

$$\|\delta z\|_2 \leq \beta \|h(z)\|_2 + \varepsilon \|y - z_*\|_2. \tag{11}$$

In a similar fashion, from the ∞-norm analogues of (4) and from (7), we have

$$\|\delta z\|_\infty \leq \beta \|h(z)\|_\infty + \varepsilon \|z - z_*\|_\infty.$$

Hence, y is near z_* when z is both near z_* and feasible in (1).

By the coercivity assumption (9), with z replaced by y, and the bound (5) for the second derivative operator, we conclude that

$$\begin{aligned}\langle \nabla_{zz}^2 H(z_*, \chi, \psi)(z - z_*), z - z_* \rangle \\ \geq \alpha \|y - z_*\|_2^2 - \beta \|y - z_*\|_2 \|\delta z\|_2 - \beta \|\delta z\|_2^2.\end{aligned} \tag{12}$$

Combining (10), (11), and (12) yields

$$M(z) \geq \gamma \|h(z)\|_1 + \tfrac{1}{2}(\alpha - \varepsilon) \|y - z_*\|_2^2 - \beta \|h(z)\|_2^2 \tag{13}$$

for z near z_* with z feasible in (1). The triangle inequality and (11) gives

$$\|z - z_*\|_2 \leq \|z - y\|_2 + \|y - z_*\|_2 \leq (1 + \varepsilon) \|y - z_*\|_2 + \beta \|h(z)\|_2.$$

Utilizing the inequality $ab \leq \varepsilon a^2 + b^2/4\varepsilon$, it follows that

$$\|z - z_*\|_2^2 \leq (1 + \varepsilon) \|y - z_*\|_2^2 + \beta \left(1 + \frac{1}{\varepsilon}\right) \|h(z)\|_2^2.$$

Hence, we have

$$\|y - z_*\|_2^2 \geq (1 - \varepsilon) \|z - z_*\|_2^2 - \beta \left(1 + \frac{1}{\varepsilon}\right) \|h(z)\|_2^2. \tag{14}$$

Utilizing the bound $|R(z)| \leq \varepsilon \|z - z_*\|_2^2$ for the remainder term and referring to (13), we see that, for z near z_* with z feasible in (1),

$$\begin{aligned} C(z) - C(z_*) &= M(z) + R(z) \\ &\geq \gamma \|h(z)\|_1 + \tfrac{1}{2}(\alpha - \varepsilon) \|y - z_*\|_2^2 - \varepsilon \|z - z_*\|_2^2 - \beta \|h(z)\|_2^2. \end{aligned}$$

Combining this with (14) yields

$$C(z) - C(z_*) \geq \gamma \|h(z)\|_1 + \tfrac{1}{2}(\alpha - \varepsilon)\|z - z_*\|_2^2 - \beta\left(1 + \frac{1}{\varepsilon}\right)\|h(z)\|_2^2.$$

Finally, exploiting (3) and the fact that $h(z)$ approaches 0 as z approaches z_*, we conclude that, for z near z_* with z feasible in (1),

$$C(z) - C(z_*) \geq \tfrac{1}{2}(\alpha - \varepsilon)\|z - z_*\|_2^2,$$

which completes the proof. □

3. The Optimal Control Problem

Let us consider the following nonlinear optimal control problem with control constraints:

$$\begin{aligned} &\text{minimize} \quad \Psi(x(0), x(1)) + \int_0^1 \varphi(x(t), u(t))\, dt \\ &\text{subject to} \quad \dot{x}(t) = f(x(t), u(t)) \quad \text{and} \quad u(t) \in U(t) \quad \text{a.e. } t \in [0,1], \\ &\qquad\qquad \Phi(x(0), x(1)) = 0, \quad x \in W^{1,\infty}, \quad u \in L^{\infty}, \end{aligned} \tag{15}$$

where f: $R^{n+m} \to R^n$, φ: $R^{n+m} \to R$, Ψ: $R^{2n} \to R$, Φ: $R^{2n} \to R^k$, and $W^{1,\alpha}$ is the usual Sobolev space consisting of absolutely continuous functions with the first derivative in L^α. We consider control constraints in (15) that can be expressed in the following way: Set-valued maps I and J, mapping $[0,1]$ to the subsets of $\{1, \ldots, l\}$, and associated functions g^i: $R^m \to R$ exist such that

$$U(t) = \{u \in R^m\text{: } g^i(u) \leq 0 \text{ and } g^j(u) = 0 \text{ for each } i \in I(t) \text{ and } j \in J(t)\}.$$

We assume that the following sets I^i and J^j are measurable for each i and j:

$$I^i = I^{-1}(i) = \{t \in [0,t]\text{: } i \in I(t)\}$$

and

$$J^j = J^{-1}(j) = \{t \in [0,1]\text{: } j \in J(t)\}.$$

We use Theorem 1 to formulate conditions under which a feasible point (x_*, u_*) for (15) is a local minimizer. To this end, suppose that a closed set $\Delta \subset R^n \times R^m$ and a $\delta > 0$ exist such that $(x_*(t), u_*(t))$ lies in Δ for almost every $t \in [0,1]$, the distance from $(x_*(t), u_*(t))$ to the boundary of Δ is at least δ for almost every $t \in [0,1]$, the first two derivatives of $f(x,u)$, $\varphi(x,u)$, and $g^i(u)$, with respect to x and u exist on Δ, and these derivatives along with the functions $f(x,u)$, and $g^i(u)$ are continuous with respect to $(x,u) \in \Delta$. Similarly, we assume that the first two derivatives of Φ and Ψ exist and are continuous near $(x_*(0), x_*(1))$.

As in Section 2, we assume that certain first-order conditions hold, analogous to (2), and we introduce additional conditions to guarantee that (x_*, u_*) is a local minimizer. In the optimal control context, the first-order conditions are often called the Minimum Principle. Let H denote the Hamiltonian defined by

$$H(x, u, \lambda) = \varphi(x, u) + \lambda^T f(x, u).$$

If (x_*, u_*) is a solution of (15) satisfying the Minimum Principle, then there is an absolutely continuous function $\lambda_*: [0,1] \to R^n$ satisfying the following conditions:

Adjoint Equation.

$$\dot{\lambda}_*(t)^T = -\nabla_x H(x_*(t), u_*(t), \lambda_*(t)) \qquad \text{for a.e.} \quad t \in [0,1].$$

Transversality Condition.

$$\begin{pmatrix} -\lambda(0) \\ \lambda(1) \end{pmatrix} - \nabla\Psi(x_*(0), x_*(1))^T \in \text{Range } \nabla\Phi(x_*(0), x_*(1))^T.$$

Control Minimality.

$$\begin{aligned} &H(x_*(t), u_*(t), \lambda_*(t)) \\ &\quad = \text{minimum}\{H(x_*(t), u, \lambda_*(t)) : u \in U(t)\} \qquad \text{for a.e.} \quad t \in [0,1]. \end{aligned}$$

The Transversality Condition is equivalent to the existence of a vector β_* such that

$$(-\lambda_*(0)^T, \lambda_*(1)^T) = \beta_*^T \nabla\Phi(x_*(0), x_*(1))^T + \nabla\Psi(x_*(0), x_*(1))^T.$$

When the gradients of the control constraints are linearly independent for almost every t, Control Minimality implies that ν_* exists such that

$$\begin{aligned} &\nu_*^i \geq 0 \quad \text{for each } i \in I, \qquad \nu_*^T g(u_*) = 0, \\ &\qquad \text{and} \quad \nabla_u \hat{H}(x_*, u_*, \lambda_*, \nu_*) = 0, \end{aligned} \tag{16}$$

where $\hat{H}$ is the extended Hamiltonian defined by

$$\hat{H}(x, u, \lambda, \nu) = H(x, u, \lambda) + \nu^T g(u).$$

In (16) and elsewhere, equalities and inequalities on $[0,1]$ are interpreted "almost everywhere." To summarize, x_*, u_*, λ_*, ν_*, and β_* are solutions of the following equations and inequalities:

$$\begin{cases} \dot{x} = f(x, u), \qquad \Phi(x(0), x(1)) = 0, \\ \dot{\lambda}^T = -\nabla_x H(x, u, \lambda), \\ \qquad (-\lambda(0)^T, \lambda(1)^T) = \beta^T \nabla\Phi(x(0), x(1)) + \nabla\Psi(x(0), x(1)), \\ \nabla_u \hat{H}(x, u, \lambda, \nu) = 0, \\ g(u) \in N_U(\nu), \end{cases} \tag{17}$$

where $N_U(\nu)$ is the usual normal cone: $h \in N_U(\nu)$ if and only if, for almost every t, we have $h^i(t) \le 0$, $\nu^i(t) \ge 0$, $h^i(t)\nu^i(t) = 0$, and $h^j(t) = 0$ for each $i \in I(t)$ and $j \in J(t)$.

In order to show that a given (x_*, u_*) is a local minimizer, we need additional assumptions related to the stability condition (8) and the coercivity condition (9) of Section 2. In stating these assumptions, a $_*$ attached to any function means that it is evaluated at $x = x_*$, $u = u_*$, $\lambda = \lambda_*$, $\nu = \nu_*$, and $\beta = \beta_*$. These assumptions are stated in terms of a general map A from $[0,1]$ to the subsets of $\{1, \dots, l\}$ where A has the property that the sets A^i defined by

$$A^i = A^{-1}(i) = \{t \in [0,1]: i \in A(t)\}$$

are measurable for each i. Later we consider various choices for A. For compactness, a set-valued superscript such as A is attached to a vector to denote the subvector associated with indices $i \in A(t)$. Since the linearized differential equation and endpoint constraints appear frequently in what follows, we introduce the linear operators L and E defined by

$$L(x,u) = \dot{x} - \nabla_x f_* x - \nabla_u f_* u, \qquad E(x) = \nabla \Phi_* \begin{pmatrix} x(0) \\ x(1) \end{pmatrix}.$$

The following assumptions are utilized:

Independence. The gradients of the control constraints satisfy the following uniform independence condition for some constant $\alpha > 0$:

$$\left| \sum_{i \in A(t)} \nu^i \nabla g_*^i(t) \right| \ge \alpha |\nu| \qquad \text{for every } \nu \text{ and for a.e.} \quad t \in [0,1]. \tag{18}$$

Controllability. For each $e \in R^k$, there are $x \in W^{1,\infty}$ and $u \in L^\infty$ satisfying the following equations:

$$L(x,u) = 0, \qquad E(x) = e, \qquad \nabla g_*^A u = 0.$$

Coercivity. If B is the quadratic form defined by

$$B(z) = \langle \nabla_{zz}^2 \hat{H}_* z, z \rangle + \begin{pmatrix} x(0) \\ x(1) \end{pmatrix}^T \nabla_{xx}^2 (\beta^T \Phi + \Psi)_* \begin{pmatrix} x(0) \\ x(1) \end{pmatrix}, \qquad z = (x,u), \tag{19}$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product, a scalar $\alpha > 0$ exists such that

$$B(x,u) \ge \alpha[\langle x, x \rangle + \langle u, u \rangle] \qquad \text{for every} \quad x \in H^1 \quad \text{and} \quad u \in L^2$$
$$\text{satisfying} \quad L(x,u) = 0, \quad E(x) = 0, \quad \text{and} \quad \nabla g_*^A u = 0.$$

Above, H^1 is the usual abbreviation for $W^{1,2}$.

These assumptions have the following implications:

Lemma 1. *If the Controllability Assumption holds, then* $\operatorname{rank}(\nabla\Phi_*) = k$. *If, in addition, the Independence Assumption holds, then, for each fixed* $\alpha \in [1, \infty]$, *there is a bounded linear map from the data* $s \in L^\alpha$, $e \in R^k$, *and* $c \in L^\alpha$ *to a solution* $x \in W^{1,\alpha}$ *and* $u \in L^\alpha$ *of the following linear system*:

$$L(x, u) = s, \qquad E(x) = e, \qquad \nabla g_*^A u = c. \tag{20}$$

Moreover, if $\beta \in [1, \alpha]$, *then this map from* $L^\alpha \times R^k \times L^\alpha$ *to* $W^{1,\alpha} \times L^\alpha$ *is bounded relative to the norms of* $L^\beta \times R^k \times L^\beta$ *and* $W^{1,\beta} \times L^\beta$.

Proof. The first part of the lemma is a trivial consequence of the Controllability Assumption. By the Independence Assumption, a $\bar{u} \in L^\alpha$ exists such that $\nabla g_*^A \bar{u} = c$. For example, we can take

$$\bar{u}(t) = \nabla g_*^A(t)^T(\nabla g_*^A(t)\nabla g_*^A(t)^T)^{-1}c(t). \tag{21}$$

(This function is measurable by the analysis given in the Appendix). Let $\bar{x} \in W^{1,\alpha}$ be such that $L(\bar{x}, \bar{u}) = s$ and $\bar{x}(0) = 0$. By the Controllability Assumption, E is surjective on the set N defined by

$$N = \{(x, u) \in W^{1,\infty} \times L^\infty \colon L(x, u) = 0, \text{ and } \nabla g_*^A u = 0\}.$$

If $e_1, e_2, \ldots, e_k$ is a basis for R^k, then $x^j \in W^{1,\infty}$ and $u^j \in L^\infty$ exist such that $L(x^j, u^j) = 0$, $E(x^j) = e_j$, and $\nabla g_*^A u^j = 0$ for each j. Hence, a bounded linear map M taking any e to a pair $(x, u) \in N$ for which $E(x) = e$ exists. Defining $(\tilde{x}, \tilde{u}) = M(e - E(\bar{x}))$, it follows that $x = \bar{x} + \tilde{x}$ and $u = \bar{u} + \tilde{u}$ is a solution of (20) that depends linearly on the data. Given $\beta \in [1, \alpha]$, it can be checked that the $(W^{1,\beta} \times L^\beta)$-norm of (x, u) is bounded in terms of the $(L^\beta \times R^k \times L^\beta)$-norm of (s, e, c). □

In what follows, the superscript c denotes complement and $\mu(S)$ denotes the Lebesgue measure of the set S.

Lemma 2. *If the Coercivity Assumption holds, then a scalar* $\alpha > 0$ *exists such that*

$$B(x, u) \geq \alpha[\langle x, x\rangle + \langle u, u\rangle + \langle \dot{x}, \dot{x}\rangle] \tag{22}$$

for every $x \in H^1$ *and* $u \in L^2$ *satisfying* $L(x, u) = 0$, $E(x) = 0$, *and* $\nabla g_*^A u = 0$. *If in addition the Independence and Controllability Assumptions hold, then, for a.e.* $t \in [0, 1]$, *we have*

$$v^T \nabla_{uu}^2 \hat{H}_*(t) v \geq \alpha v^T v \qquad \textit{whenever} \quad \nabla g_*^A(t) v = 0. \tag{23}$$

Proof. If x satisfies the differential equation $\dot{x} = \nabla_x f_* x + \nabla_u f_* u$, then the L^2-norm of $\dot{x}$ is bounded in terms of the L^2-norms of x and u. Hence, the Coercivity Assumption implies the lower bound for B in (22).

Let τ be chosen so that it is a Lebesgue point of $\hat{H}_*(\cdot)$, of $|\nabla g^i_*(\cdot)|^2$, and of $\nabla g^i_*(\cdot)$ for each $i \in A(\tau)$, and a point of density in the set $(A^j)^c$ for each $j \in A(\tau)^c$. This implies that

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{\tau-\varepsilon}^{\tau+\varepsilon} |\nabla g^i_*(s) - \nabla g^i_*(\tau)|^2\, ds = 0$$

and

$$\lim_{\varepsilon\to 0} \frac{\mu\{(\tau-\varepsilon, \tau+\varepsilon) \cap (A^j)^c\}}{2\varepsilon} = 1.$$

Almost every $\tau \in [0,1]$ has this property (see [20]). Roughly speaking, most points near τ are also elements of $(A^j)^c$ for each $j \in A(\tau)^c$. Hence, if D is the set defined by

$$D = \{t\colon A(t) \subset A(\tau)\},$$

then τ is also a point of density for D. Let $v \in R^m$ be any point for which $\nabla g^A_*(\tau)v = 0$. By the Independence Assumption, a solution $w(t)$ to the equation

$$\nabla g^A_*(t)w(t) = -\nabla g^A_*(t)v \qquad \text{with} \quad |w(t)| \le C \sum_{i\in A(t)} |\nabla g^i_*(t)v|,$$

where C denotes a generic constant, exists. Since $A(t) \subset A(\tau)$ when $t \in D$, we conclude that $\nabla g^i_*(\tau)v = 0$ for each $i \in A(t)$ and $t \in D$, from which it follows that

$$|w(t)| \le C \sum_{i\in A(\tau)} |(\nabla g^i_*(t) - \nabla g^i_*(\tau))v| \qquad \text{for every} \quad t \in D. \tag{24}$$

Given any $\varepsilon > 0$ such that the set $O_\varepsilon = [\tau-\varepsilon, \tau+\varepsilon] \subset [0,1]$, we integrate (24) with respect to $t \in O_\varepsilon$ to obtain

$$\int_{O_\varepsilon} |w(t)|^2 \le C|v|^2\left(\int_{D\cap O_\varepsilon} \sum_{i\in A(\tau)} |\nabla g^i_*(t) - \nabla g^i_*(\tau)|^2 + \int_{D^c\cap O_\varepsilon} |\nabla g^A_*(t)|^2\right). \tag{25}$$

The first term in (25) is $o(\varepsilon)$ since τ is a Lebesgue point of the integrand, while the second term is $o(\varepsilon)$ since τ is a point of density for the set D. Hence, we have

$$\int_{O_\varepsilon} |w(t)|^2 = C|v|^2 o(\varepsilon).$$

Let v_ε be the control defined by

$$v_\varepsilon(t) = \begin{cases} v + w(t) & \text{for} \quad \tau-\varepsilon \le t \le \tau+\varepsilon, \\ 0 & \text{otherwise}. \end{cases}$$

Observe that $\nabla g^A_*(t)v_\varepsilon(t) = 0$. Let y_ε be the solution to $L(y_\varepsilon, v_\varepsilon) = 0$, $y_\varepsilon(0) = 0$. Since the solution to a linear differential equation can be bounded in terms of the forcing term, we have $|y_\varepsilon(t)| \le \varepsilon|v|C$ where C is independent of $t \in [0,1]$. By the

Controllability Assumption, $x_\varepsilon \in W^{1,\infty}$ and $u_\varepsilon \in L^\infty$ exist such that $L(x_\varepsilon, u_\varepsilon) = 0$, $E(x_\varepsilon) = -E(y_\varepsilon)$, and $\nabla g_*^A u_\varepsilon = 0$, with $\|x_\varepsilon\|_{L^\infty} + \|u_\varepsilon\|_{L^\infty} \le \varepsilon |v| C$. Finally, inserting $x = x_\varepsilon + y_\varepsilon$ and $u = u_\varepsilon + v_\varepsilon$ in the Coercivity Assumption and letting ε tend to zero, we obtain (23). $\square$

Within the framework developed above, we have the following sufficient optimality result:

Theorem 2. *Suppose that* $x_* \in W^{1,\infty}$, $u_* \in L^\infty$, $\lambda_* \in W^{1,\infty}$, $\nu_* \in L^\infty$, *and* $\beta_* \in R^k$ *satisfy* (17). *If the Independence and Controllability Assumptions hold for* $A = I \cup J$, *and the Coercivity Assumption holds with* $A = J$, *then* (x_*, u_*) *is a strict local minimizer for* (15).

Proof. We apply Theorem 1 with the following identifications: z is the pair (x, u), Z is $W^{1,\infty} \times L^\infty$, the constraint "$g(z) \in K_g$" corresponds to all the constraints of (15) while the constraint "$h(z) \in K_h$" of (1) is vacuous, W_g is $L^\infty \times L^\infty \times R^k$, the multiplier χ associated with (1) corresponds to the multipliers λ, ν, and β associated with (15), and the p-norm of Section 2 is the norm associated with L^p. The inequality (8) and the coercivity condition (9) follow from Lemmas 1 and 2, respectively. $\square$

Corollary 1. *If, in addition to the hypotheses of Theorem* 2, *the multiplier* ν_* *has the property that, for some* $\alpha > 0$, $\nu_*^i(t) \ge \alpha$ *for each* $i \in J(t)$ *and for a.e.* $t \in [0, 1]$, *then the local minimizer* (x_*, u_*) *of Theorem* 2 *is also a local minimizer for the problem*

$$\begin{aligned} &\textit{minimize} \quad \Psi(x(0), x(1)) + \int_0^1 \varphi(x(t), u(t))\, dt \\ &\textit{subject to} \quad \dot{x}(t) = f(x(t), u(t)), \qquad a.e. \quad t \in [0,1], \\ &\qquad g^I(u(t)) \le 0 \quad and \quad g^J(u(t)) \le 0 \qquad a.e. \quad t \in [0,1], \\ &\qquad \Phi(x(0), x(1)) = 0, \qquad x \in W^{1,\infty}, \qquad u \in L^\infty. \end{aligned} \tag{26}$$

Proof. The proof is identical to that of Theorem 2 except that the constraint $h(z) \in K_h$ of (1) is identified with the constraint $g^J(u(t)) \le 0$ of (26) while the constraint $g(z) \in K_g$ of (1) is identified with all the other constraints of (26), and W_h is the space of L^∞-functions whose jth component is defined on the set J^j. As indicated earlier, the 1-norm of (1) is identified with the L^1-norm in the optimal control context. Condition (6) holds with $\psi = \nu_*^J$ since the components of ν_*^J are bounded from below by a positive constant. $\square$

Remark. The Independence Assumption in Theorem 2 can be weakened to an interior point condition for the linearized inequality constraints.

4. Perturbed Assumptions

In this section we weaken the assumptions of Section 3 by considering ε-active constraints. Let us focus on the control problem of Section 3, but with a different control constraint:

$$\begin{aligned} &\text{minimize} \quad \Psi(x(0), x(1)) + \int_0^1 \varphi(x(t), u(t))\, dt \\ &\text{subject to} \quad \dot{x}(t) = f(x(t), u(t)) \quad \text{and} \quad g(u(t)) \le 0 \quad \text{a.e. } t \in [0,1], \\ &\qquad\qquad \Phi(x(0), x(1)) = 0, \quad x \in W^{1,\infty}, \quad u \in L^\infty. \end{aligned} \tag{27}$$

The only difference between this problem and (15) is that the sets I and J used to formulate (15) do not appear in (27); alternatively, in (27) J is vacuous and $I = \{1, \ldots, l\}$. We assume that (x_*, u_*) is feasible in (15), and that the functions appearing in (27) satisfy precisely the same differentiability assumptions employed in Section 3. Again, if the Minimum Principle holds, and the gradients of the active control constraints are linearly independent, then ν_* exists such that

$$\nu_* \ge 0, \qquad \nu_*^T g(u_*) = 0, \quad \text{and} \quad \nabla_u \hat{H}(x_*, u_*, \lambda_*, \nu_*) = 0.$$

At this point, we introduce the sets I and J of Section 3. Given a fixed (small) positive parameter $\varepsilon > 0$, we define

$$\begin{aligned} &J_\varepsilon(t) = \{i\colon \nu_*^i(t) > \varepsilon\}, \qquad I_\varepsilon(t) = \{i\colon \nu_*^i(t) \le \varepsilon \text{ and } g_*^i(t) \ge -\varepsilon\}, \quad \text{and} \\ &\quad A_\varepsilon(t) = I_\varepsilon(t) \cup J_\varepsilon(t) = \{i\colon g_*^i(t) \ge -\varepsilon\}. \end{aligned} \tag{28}$$

Observe that A_ε is the set of constraints associated with u_* that are ε-active, while J_ε is the set of constraints that are active with uniformly positive multipliers. Taking $I = I_\varepsilon$ and $J = J_\varepsilon$ in (15), the control constraints of (27) that are inactive by at least ε are neglected in (15), while the constraints of (27) with associated multiplier at least ε, are enforced as equalities in (15). If the Independence, Controllability, and Coercivity Assumptions hold for some choice of ε, then, by Theorem 2, (x_*, u_*) is a strict local minimizer for (15). By Corollary 1, this local minimizer of (15) is also a local minimizer of (26). Since (27) has more constraints than (26), yet (x_*, u_*) is feasible in (27), we conclude that (x_*, u_*) is a local minimizer for (27). These observations are summarized in

Theorem 3. *Suppose that* $x_* \in W^{1,\infty}$, $u_* \in L^\infty$, $\lambda_* \in W^{1,\infty}$, $\nu_* \in L^\infty$, *and* $\beta_* \in R^k$ *satisfy* (17). *If, for some* $\varepsilon > 0$, *the Independence and Controllability Assumptions hold for* $A = A_\varepsilon$, *and the Coercivity Assumption holds with* $A = J_\varepsilon$, *then* (x_*, u_*) *is a strict local minimizer for* (27).

In Theorem 3 we make assumptions for a perturbation of the original problem, while, in principle, we would like to make assumptions about the original problem itself. We now show that under appropriate hypotheses, these assumptions hold for ε near 0, if they hold for $\varepsilon = 0$.

Lemma 3. *Suppose that, for each $t \in [0,1]$, $g_*^i(\cdot)$ is continuous for every $i \in A_0(t)^c$ and $\nabla g_*^i(\cdot)$ is continuous for every $i \in A_0(t)$. If the Independence Assumption holds for $A = A_0$, then, for each positive ε near zero, it holds for $A = A_\varepsilon$.*

Proof. Define the parameter $\delta(t)$ by

$$\delta(t) = \underset{i \notin A_0(t)}{\text{minimum}} \{-g_*^i(t): g_*^i(t) \neq 0\}.$$

If all the constraints are binding at t, then we set $\delta(t) = +\infty$. Since $g_*^i(\cdot)$ is continuous at each $t \in [0,1]$ for every $i \in A_0(t)^c$, an open ball O_t, containing t, exists such that

$$g_*^i(s) \leq -\frac{\delta(t)}{2} \qquad \text{for every} \quad s \in O_t \quad \text{and} \quad i \notin A_0(t).$$

Since $\nabla g_*^i(s)$ is continuous at $s = t$ for each $i \in A_0(t)$, it follows from the Independence Assumption that O_t can be chosen such that

$$\left| \sum_{i \in A_0(t)} v_i \nabla g_*^i(s) \right| \geq \frac{\alpha}{2}|v| \qquad \text{for every } v \text{ and for every} \quad s \in O_t.$$

By compactness, this open cover of $[0,1]$ has a finite subcover $O_{t_1}, O_{t_2}, \ldots, O_{t_k}$. Let $\varepsilon \geq 0$ be any number satisfying the condition

$$\varepsilon < \tfrac{1}{2}\,\text{minimum}\{\delta(t_0), \delta(t_1), \ldots, \delta(t_N)\}.$$

Since $A_\varepsilon(s) \subset A_\varepsilon(t_i)$ for each $s \in O_{t_i}$, we conclude that the Independence Assumption holds with $A = A_\varepsilon$. □

Now suppose that $g_*(t)$ and $\nabla g_*(t)$ are piecewise continuous with a finite number of jump discontinuities, and the Independence Assumption holds for $A = A_0$ and for all $t \in [0,1]$—at a point of discontinuity, (18) needs to hold when we take the limit from the right with $A_0(t^+)$ being the indices of active constraints associated with $g_*(t^+)$, and when we take the limit from the left with $A_0(t^-)$ being the indices of active constraints associated with $g_*(t^-)$. By an argument similar to that of Lemma 3, the Independence Assumption holds for $A = A_\varepsilon$ and each positive ε near 0.

Let us define the set

$$A_\varepsilon^i = A_\varepsilon^{-1}(i) = I_\varepsilon^i \cup J_\varepsilon^i = \{t \in [0,1]: g_*^i(t) \geq -\varepsilon\}.$$

Lemma 4. *Suppose that*

$$\lim_{\varepsilon \to 0} \mu\{A_\varepsilon^i \setminus A_0^i\} = 0 \qquad \textit{for each } i. \tag{29}$$

If the Controllability Assumption holds for $A = A_0$, then, for each positive ε near zero, it holds for $A = A_\varepsilon$.

Proof. If $e_1, e_2, \dots, e_k$ is a basis for R^k, then, under the hypotheses of the lemma, $x^j \in W^{1,\infty}$ and $u^j \in L^\infty$ exist such that $L(x^j, u^j) = 0$, $E(x^j) = e_j$, and $\nabla g_*^{A_0} u^j = 0$ for each j. Now define a perturbed control v_ε^j in the following way:

$$v_\varepsilon^j(t) = \begin{cases} 0 & \text{if } \ t \in A_\varepsilon^i \setminus A_0^i \ \text{ for some } i, \\ u^j(t) & \text{otherwise}. \end{cases}$$

The perturbed state y_ε^j is chosen such that $L(y_\varepsilon^j, v_\varepsilon^j) = 0$ and $y_\varepsilon^j(0) = x^j(0)$. Since $u^j \in L^\infty$, (29) implies that

$$\lim_{\varepsilon \to 0} y_\varepsilon^j(1) = x^j(1) \qquad \text{for each } j.$$

Since the vectors $E(y_\varepsilon^j)$ form a basis for R^k when ε is sufficiently small, and since $\nabla g_*^{A_\varepsilon} v_\varepsilon^j = 0$ for each j, we conclude that the Controllability Assumption holds for $A = A_\varepsilon$ and each positive ε near zero. □

In formulating the next perturbation result, we need to consider the following special set of active indices:

$$\bar{A}(t) = \bigcup_{\sigma > 0} \ \bigcap_{s \in [t-\sigma, t+\sigma]} A(s).$$

Observe that $\bar{A}(t) \subset A(t)$ for each t. Since the active indices are chosen from a finite set, a positive σ exists such that

$$\bar{A}(t) = \bigcap_{s \in [t-\sigma, t+\sigma]} A(s).$$

$\bar{A}$ is the set of indices that are active not only at t, but in a neighborhood of t; the indices outside this set will be inactive at times arbitrarily close to t. In establishing a perturbation result for the Coercivity Assumption, we need to assume that the pointwise coercivity assumption (23) holds with A replaced by $\bar{A}$ at those t in the boundary of A^i for some i. For example, this replacement is valid if there are a finite number of boundary points in the A^i, the boundary points of A^i and A^j for $i \neq j$ are distinct, and at each boundary point t, $\nabla_{uu}^2 \hat{H}_*$ and ∇g_*^i for each $i \in \bar{A}(t)$ are continuous. This replacement is also valid if there is only one control constraint, and at each boundary point of A^1, $\nabla_{uu}^2 \hat{H}_*$ is continuous.

Below, int denotes interior, cl denotes closure, and ∂ denotes boundary.

Lemma 5. *Suppose that the following conditions hold:*

(a) *If K is a compact subset of* $\mathrm{int}(A_0^i)$ *for some i, then* $\inf_{t \in K} \nu_*^i(t) > 0$.
(b) $\mu(\partial A_0^i) = 0$ *for each i.*
(c) *For each i, if $t \in \partial A_0^i$ and $j \in \bar{A}_0(t)$, then $\nabla_{uu}^2 \hat{H}_*$ and ∇g_*^j are continuous at t.*

(d) *The Independence, Controllability, and Coercivity Assumptions hold with* $A = A_0$.

(e) *The pointwise coercivity condition* (23) *holds with* A *replaced by* $\overline{A}_0$ *for each* $t \in \bigcup_i \partial A_0^i$.

Then the Coercivity Assumption holds for $A = J_\varepsilon$ *and each positive* ε *near zero.*

The result above is motivated by the analysis of Dunn and Tian [6] of a sufficient optimality result for a problem with a single control constraint $u \geq 0$, with a quasi-quadratic-type cost function and quasi-linear system dynamics, and with a free terminal state. Although the problem in [6] is quite specialized, it turns out that generalizations of some of their assumptions are precisely what is needed to establish the perturbation result above. For example, in [6] it is assumed that a certain derivative is positive (analogous to (a)), that $\nabla_{uu}^2 \hat{H}_*$ is continuous at each $t \in \partial A_0^1$ (analogous to (c)), and that certain additional conditions hold which imply $\mu(\partial A_0^1) = 0$.

Proof of Lemma 5. If $\mu(A_0^i) = 0$ for some i, then $i \notin A_0(t)$ almost everywhere. Hence, the Coercivity Assumption holds with $A = A_0$ if and only if it holds with $A = A_0 \setminus \{i\}$. This shows that there is no loss of generality in assuming that each A_0^i has a positive measure.

Loosely speaking, the proof has the following structure: We first chisel off the fringe of each A_0^i leaving us with a compact set K^i where ν_*^i is bounded away from zero; we let ε be any positive lower bound for the multipliers ν_*^i over the sets K^i. The fringe set, denoted E, is contained in the union of the sets $A_0^i \setminus K^i$. We show that the pointwise coercivity condition (23) holds at each $t \in E$ when A is replaced by J_ε. Given $x \in H^1$ and $u \in L^2$ that satisfy the conditions

$$L(x,u) = 0, \qquad E(x) = 0, \quad \text{and} \quad \nabla g_*^{J_\varepsilon} u = 0, \tag{30}$$

we decompose (x, u) into a sum of three terms:

$$(x,u) = (y,v) + (\Delta y, \Delta v) + (0, u_E), \tag{31}$$

where the quadratic form B is coercive relative to (y, v), the pair $(\Delta y, \Delta v)$ can be made arbitrarily small by making $\mu(A_0^i \setminus K^i)$ sufficiently small, and u_E is zero except on the fringe E where it satisfies the pointwise coercivity condition. Combining the coercivity relative to (y, v) with the pointwise coercivity relative to u_E and the smallness of $(\Delta y, \Delta v)$, we obtain coercivity relative to (x, u).

Step 1 (Construction of the Fringe). Assumptions (c) and (e) imply that if $t \in \bigcup_i \partial A_0^i$, then an open ball O_t centered at t exists such that, for each $s \in O_t$, $\overline{A}_0(t) \subset A_0(s)$ and

$$v^T \nabla_{uu}^2 \hat{H}_*(s) v \geq \beta v^T v \qquad \text{whenever} \quad \nabla g_*^i(s) v = 0 \quad \text{for every} \quad i \in \overline{A}_0(t), \tag{32}$$

where β is a fixed positive constant smaller than α. Choose O_t small enough such that, for each i where $t \notin \partial A_0^i$, we have either $O_t \subset \operatorname{int} A_0^i$ or $O_t \subset \operatorname{int}(A_0^i)^c$. Since the O_t form an open cover of $\bigcup_i \partial A_0^i$, a finite subcover denoted O exists.

Since $\mu(\partial A_0^i) = 0$, it follows that $\mu(A_0^i) = \mu(\operatorname{int}(A_0^i))$. Given $\delta > 0$, the regularity properties of Lebesgue measure imply that a closed set $C^i \subset \operatorname{int} A_0^i$ with $\mu(\operatorname{int} A_0^i \setminus C^i) \le \delta$ exists. Let U^i denote the union of the elements of O that intersect ∂A_0^i, and let K^i be the compact subset of A_0^i given by

$$K^i = C^i \cup (A_0^i \setminus U^i).$$

(Since the open set U^i contains all boundary points of A_0^i, the set $A_0^i \setminus U^i$ is closed.) Choose ε so that

$$0 < \varepsilon < \inf_{t \in K^i} \nu_*^i(t), \qquad i = 1, 2, \ldots, l.$$

If $i \in J_\varepsilon(t)$, then $\nu_*^i(t) \ge \varepsilon$, which implies that $g_*^i(t) = 0$, and $i \in A_0(t)$. Hence, $J_\varepsilon(t) \subset A_0(t)$ for every t. Let E be the set where J_ε and A_0 differ:

$$E = \{t \in [0,1]: J_\varepsilon(t) \ne A_0(t)\}.$$

If $i \in A_0(t) \setminus J_\varepsilon(t)$, then $t \in A_0^i \setminus K^i$. Hence, we conclude that

$$E \subset \bigcup_i (A_0^i \setminus K^i).$$

By the choice of the K^i, the measure of E is at most $l\delta$.

Step 2 (Pointwise Coercivity). We now show that, for each $s \in E$, we have

$$v^T \nabla_{uu}^2 \hat{H}_*(s) v \ge \beta v^T v \qquad \text{whenever} \quad \nabla g_*^i(s) v = 0 \quad \text{for every} \quad i \in J_\varepsilon(s). \tag{33}$$

The proof proceeds in the following way: For any $s \in E$, the set of $i \in A_0(s) \setminus J_\varepsilon(s)$ is nonempty. Given $i \in A_0(s) \setminus J_\varepsilon(s)$, we noted earlier that $s \in A_0^i \setminus K^i$. By the definition of K^i, an open set O_t in O exists such that $s \in O_t$, t lies in $\bigcup_i \partial A_0^i$, and (32) holds. We now show that $\bar{A}_0(t) \subset J_\varepsilon(s)$ so that (32) implies (33). If $t \notin \operatorname{int} A_0^i$, then by the choice of O_t, there are points in $(A_0^i)^c$ arbitrarily close to t, which implies that $i \notin \bar{A}_0(t)$. If $t \in \operatorname{int} A_0^i$, then $i \in \bar{A}_0(t)$, and, by the choice of O_t and U^i, $O_t \subset K^i$, which implies that $i \in J_\varepsilon(s)$. Since all the elements of $\bar{A}_0(t)$ are contained in $J_\varepsilon(s)$, we have $\bar{A}_0(t) \subset J_\varepsilon(s)$ and (33) holds.

Step 3 (Decomposition). Now suppose that $x \in H^1$ and $u \in L^2$ satisfy (30). We show that the pair (x, u) has the decomposition (31) where

$$u_E(t) = \begin{cases} u(t) & \text{for} \quad t \in E, \\ 0 & \text{for} \quad t \in E^c, \end{cases}$$

$$\|\Delta y\|_{L^2} + \|\Delta v\|_{L^2} \le C\sqrt{\delta}\|u_E\|_{L^2} \qquad (C \text{ a generic constant}),$$

$$L(y, v) = 0, \qquad E(y) = 0, \quad \text{and} \quad \nabla g_*^{A_0} v = 0.$$

The construction proceeds in the following way. Starting with the function u_E specified above, let δx be such that $L(\delta x, -u_E) = 0$ and $\delta x(0) = 0$. Applying the Schwarz inequality to the usual representation of $\delta x(t)$ in terms of an integral, we conclude that $\delta x(t) \leq C\sqrt{\delta}\|u_E\|_{L^2}$ since $\mu(E) \leq l\delta$. Let $(\bar{x}, \bar{u})$ be a pair given by Lemma 1 for which $L(\bar{x}, \bar{u}) = 0$, $E(\bar{x}) = -E(\delta x)$, and $\nabla g_*^{A_0}\bar{u} = 0$, with

$$\|\bar{x}\|_{L^2} + \|\bar{u}\|_{L^2} \leq C|E(\delta x)| \leq C|\nabla\Phi_*||\delta x(1)| \leq C\sqrt{\delta}\|u_E\|_{L^2}.$$

With the following choices, the decomposition (31) holds:

$$y = x + \delta x + \bar{x}, \qquad \Delta y = -\delta x - \bar{x}, \qquad v = u - u_E + \bar{u}, \qquad \Delta v = -\bar{u}.$$

Step 4 (Integral Coercivity). Applying the quadratic form B to (x, u) and utilizing the decomposition (31), the pointwise coercivity relation (33), the fact that B is coercive when applied to (y, v), the inequality $2ab \leq a^2/\sigma + \sigma b^2$ where σ is an arbitrary positive scalar, and the fact that $v = \bar{u}$ on E while $u_E = 0$ except on E, we conclude that, for δ sufficiently small, a generic constant $\alpha > 0$ exists such that

$$B(x, u) \geq \alpha[\langle y, y\rangle + \langle v, v\rangle + 2\langle u_E, u_E\rangle].$$

After substituting $y = x - \Delta y$ and $v = u - \Delta v - u_E$, we deduce that, for some $\alpha > 0$, we have

$$B(x, u) \geq \alpha[\langle x, x\rangle + \langle u, u\rangle],$$

which completes the proof. □

Remark. With regard to the assumptions of Lemma 5, condition (a) is analogous to the strict complementarity assumption in optimization; loosely speaking, a multiplier associated with an active constraint is positive except possibly at the boundary of the set of times where the constraint is active. Condition (b) holds if there are a countable number of times where constraints change between active and nonactive. Condition (c) holds, for example, if the control and the multipliers are continuous.

5. Stability of Solutions

In this section we examine how the solution to an optimal control problem depends on a parameter. We begin by studying problem (15), with each function depending on a parameter p in a topological space P:

$$\begin{aligned} &\text{minimize} \quad \Psi_p(x(0), x(1)) + \int_0^1 \varphi_p(x(t), u(t))\, dt \\ &\text{subject to} \quad \dot{x}(t) = f_p(x(t), u(t)) \quad \text{and} \quad u(t) \in U_p(t) \quad \text{a.e. } t \in [0, 1], \\ &\qquad\qquad \Phi_p(x(0), x(1)) = 0, \quad x \in W^{1,\infty}, \quad u \in L^\infty, \end{aligned} \tag{34}$$

where

$$U_p(t) = \{u \in R^m \colon g_p^i(u) \le 0 \text{ and } g_p^j(u) = 0 \text{ for each } i \in I(t) \text{ and } j \in J(t)\}.$$

Given a local minimizer $(x, u) = (x_*, u_*)$ of (34) corresponding to $p = p_*$, we wish to show that a nearby local minimizer (x_p, u_p) exists for p in a neighborhood of p_*. Analogous to the assumptions in Section 3, we suppose that a closed set $\Delta \subset R^n \times R^m$ and a $\delta > 0$ exist such that $(x_*(t), u_*(t))$ lies in Δ for almost every $t \in [0,1]$, the distance from $(x_*(t), u_*(t))$ to the boundary of Δ is at least δ for almost every $t \in [0,1]$, the first two derivatives of $f_p(x,u)$, $\varphi_p(x,u)$, and $g_p^i(u)$, with respect to x and u exist on Δ, and these derivatives along with the function values $f_p(x,u)$ and $g_p^i(u)$ are continuous with respect to $(x,u) \in \Delta$ and p near p_*. Similarly, we assume that the first two derivatives of $\Phi_p(y,z)$ with respect to y and z exist near $(y,z) = (x_*(0), x_*(1))$, and that these derivatives along with the function values are continuous in p near p_* and (y,z) near $(x_*(0), x_*(1))$.

Since the functions in (34) depend on p, both the Hamiltonian and the extended Hamiltonian depend on p:

$$H_p(x,u,\lambda) = \varphi_p(x,u) + \lambda^T f_p(x,u)$$

and

$$\hat{H}_p(x,u,\lambda,\nu) = H_p(x,u,\lambda) + \nu^T g_p(u).$$

With these definitions, a local minimizer (x_p, u_p) of (34), and associated multipliers λ_p, ν_p, and β_p that satisfy the Minimum Principle, are solutions to the following system:

$$\begin{cases} \dot{x} = f_p(x,u), \qquad \Phi_p(x(0), x(1)) = 0, \\ \dot{\lambda}^T = -\nabla_x H_p(x,u,\lambda), \\ \qquad (-\lambda(0)^T, \lambda(1)^T) = \beta^T \nabla\Phi_p(x(0), x(1)) + \nabla\Psi(x(0), x(1)), \\ \nabla_u \hat{H}_p(x,u,\lambda,\nu) = 0, \\ g_p(u) \in N_U(\nu). \end{cases} \tag{35}$$

Abstractly, the problem of solving (35) can be thought of in the following way:

Find $\omega \in \Omega$ such that $T_p(\omega) \in F(\omega) \subset W$ where

$$\begin{aligned} &\omega = (x,u,\lambda,\nu,\beta), \quad \Omega = W^{1,\infty} \times L^\infty \times W^{1,\infty} \times L^\infty \times R^k, \\ &T_p = \left[\dot{x} - f_p, \Phi_p, \dot{\lambda} + \nabla_x H_p, (\lambda(0)^T, -\lambda(1)^T) + \beta^T \nabla\Phi_p + \nabla\Psi_p, \nabla_u \hat{H}_p, g_p\right], \\ &F(\omega) = \{0\} \times \{0\} \times \cdots \times \{0\} \times N_U(\nu), \quad \text{and} \qquad (36) \\ &W = L^\infty \times R^k \times L^\infty \times R^{2n} \times L^\infty \times L^\infty. \end{aligned}$$

Our main stability result is the following:

Theorem 4. *Suppose that* $x_* \in W^{1,\infty}$, $u_* \in L^\infty$, $\lambda_* \in W^{1,\infty}$, $\nu_* \in L^\infty$, *and* $\beta_* \in R^k$ *satisfy* (35) *when* $p = p_*$. *If the Independence and Controllability Assumptions hold for* $A = I \cup J$ *and the Coercivity Assumption holds for* $A = J$, *then, for* p *in a neighborhood of* p_*, *a solution* ω_p *of* (35) *exists whose* (x_p, u_p) *component is a strict local minimizer of* (34). *Moreover, a constant* κ *exists such that*

$$\|\omega_p - \omega_q\|_\Omega \le \kappa \|T_p(\omega_p) - T_q(\omega_p)\|_W \tag{37}$$

for each p *and* q *in a neighborhood of* p_*.

Proof. First, we show that a solution ω_p of (35) satisfying (37) exists. Then we apply Theorem 2 to show that the (x_p, u_p) component of ω_p is a local minimizer of (34). The existence of ω_p and the bound (37) are based on Robinson's implicit function theorem [19] for generalized equations of the form (36) (see [5] for extensions of this theorem, and see [4] and [7] for applications to optimal control). This theorem states that if the linearized problem

$$\text{find} \quad \omega \in \Omega \qquad \text{such that} \quad \nabla T_0[\omega_*]\omega + \psi \in F(\omega)$$

has a unique solution depending Lipschitz continuously on $\psi \in W$, then the original problem (36) has a solution near the reference point ω_* which is locally unique and which satisfies (37). In the optimal control context, this linearization takes the form:

Given $s \in L^\infty$, $e \in R^k$, $a \in L^\infty$, $b \in R^{2n}$, $c \in L^\infty$, and $d \in L^\infty$,
find $x \in W^{1,\infty}$, $u \in L^\infty$, $\lambda \in W^{1,\infty}$, $\nu \in L^\infty$, and $\beta \in R^k$ such that

$$\begin{cases} \dot{x} = \nabla_x f_* x + \nabla_u f_* u + s, \quad \nabla\Phi_* \begin{pmatrix} x(0) \\ x(1) \end{pmatrix} = e, \\ \dot{\lambda} = -\nabla_{xx}^2 H_* x - \nabla_{xu}^2 H_* u - \nabla_x f_*^T \lambda + a, \\ \quad \begin{pmatrix} -\lambda(0) \\ \lambda(1) \end{pmatrix} = b + \nabla\Phi_*^T \beta + \nabla_{xx}^2 (\beta^T \Phi + \Psi)_* \begin{pmatrix} x(0) \\ x(1) \end{pmatrix}, \\ \nabla g_*^J u = c^J, \quad \nabla g_*^I u + g_*^I \le c^I, \quad \nu^I \ge 0, \\ \quad (\nabla g_*^I u + g_*^I - c^I)^T \nu^I = 0, \\ \nabla_{uu}^2 \hat{H}_* u + \nabla_{xu}^2 H_* x + \nabla_u f_*^T \lambda + \nabla g_*^T \nu = d. \end{cases} \tag{38}$$

As in the earlier work [4] and [7], the Lipschitz properties of the solutions to (38) are analyzed by studying a related quadratic program:

$$\text{minimize} \quad B(x,u) - \langle d, u \rangle - \langle a, x \rangle + b_0^T x(0) + b_1^T x(1)$$

$$\text{subject to} \quad \dot{x} = \nabla_x f_* x + \nabla_u f_* u + s, \quad \nabla\Phi_* \begin{pmatrix} x(0) \\ x(1) \end{pmatrix} = e,$$

$$\nabla g_*^J u = c^J, \quad \nabla g_*^I u + g_*^I \le c^I, \quad x \in H^1, \quad \text{and} \quad u \in L^2. \tag{39}$$

Here B is the quadratic form defined in (19), b_0 is the first n components of b, and b_1 is the second n components of b. Recall that when a mathematical program subject to linear equality and inequality constraints is coercive relative to the null space of the linear operator associated with the equality constraints, the first-order conditions are sufficient for optimality. Hence, if (38) holds for some $\omega \in \Omega$, the (x, u) component of ω is the unique solution of (39). Conversely, by the Coercivity Assumption, there is a unique solution to the program (39) posed in $H^1 \times L^2$, and, by the Independence and Controllability Assumptions, associated multipliers $\lambda \in H^1$, $\nu \in L^\infty$, and $\beta \in R^k$ exist such that (38) holds (for example, see [9]). We now show that under the hypotheses of Theorem 4, this solution of (39) and the associated multipliers satisfying (38) lie in Ω. Thus there is a one-to-one correspondence between a solution of (38) and a solution of (39). By showing that the solution to (39) depends Lipschitz continuously on the data, it follows that the solution to (38) also depends Lipschitz continuously on the data.

The analysis of (39) proceeds in the following way. By Lemma 1, we can translate x by an element of $W^{1,\infty}$ and u by an element of L^∞ to obtain an equivalent problem of the form:

$$\begin{aligned}&\text{minimize} \quad B(x,u) - \langle \bar{d}, u\rangle - \langle \bar{a}, x\rangle + b_0^T x(0) + b_1^T x(1)\\ &\text{subject to} \quad \dot{x} = \nabla_x f_* x + \nabla_u f_* u, \quad \nabla\Phi_* \begin{pmatrix} x(0) \\ x(1)\end{pmatrix} = 0,\\ &\qquad\qquad \nabla g_*^J u = 0, \quad \nabla g_*^I u \le 0, \quad x \in H^1, \quad \text{and} \quad u \in L^2,\end{aligned} \tag{40}$$

where $\bar{d}$ and $\bar{a} \in L^\infty$ are affine functions of s and $d \in L^\infty$ and $e \in R^k$. Since the solution of a coercive quadratic program depends Lipschitz continuously on linear perturbations of the cost function (for example, see Lemma 4 of [4] or Lemma 1 of [8]), we have the following bound for the change δx and δu in the solution to (40) corresponding to a change in the data:

$$\|\delta x\|_{H^1} + \|\delta u\|_{L^2} \le \alpha^{-1}(\|\delta\bar{d}\|_{L^2} + \|\delta\bar{a}\|_{L^1} + |\delta b|). \tag{41}$$

To estimate the change $\delta\lambda$, $\delta\nu$, and $\delta\beta$ in the multipliers (and at the same time establish their uniqueness), we first utilize Lemma 1 to obtain $y \in W^{1,\infty}$ and $v \in L^\infty$ such that

$$L(y,v) = 0, \qquad E(y) = -\delta\beta, \qquad \nabla g_*^I u = 0, \qquad \nabla g_*^J u = 0,$$

where the norm of (y, v) is bounded by $C|\delta\beta|$ for some generic constant C. After integration by parts,

$$\begin{aligned}0 &= \langle \nabla_x f_* y + \nabla_u f_* v - \dot{y}, \delta\lambda\rangle\\ &= \langle y, \delta\dot{\lambda} + \nabla_x f_*^T \delta\lambda\rangle + \langle v, \nabla_u f_*^T \delta\lambda\rangle + \delta\lambda(0)^T y(0) - \delta\lambda(1)^T y(1).\end{aligned} \tag{42}$$

From the boundary condition for λ in (38) and the boundary condition for y, we have

$$\delta\lambda(0)^T y(0) - \delta\lambda(1)^T y(1)$$
$$= |\delta\beta|^2 - \begin{pmatrix} y(0) \\ y(1) \end{pmatrix} \nabla_{xx}^2(\beta^T\Phi + \Psi)_* \begin{pmatrix} \delta x(0) \\ \delta x(1) \end{pmatrix} - y(1)^T\delta b_1 - y(0)^T\delta b_0;$$

by the adjoint equation in (38), we have

$$\langle y, \delta\dot{\lambda} + \nabla_x f_*^T \delta\lambda \rangle = -\langle y, \nabla_{xx}^2 H_* \delta x + \nabla_{xu}^2 H_* \delta u - \delta a \rangle;$$

and by the last relation in (38) and the identity $\nabla g_* v = 0$, we have

$$\begin{aligned} \langle v, \nabla_u f_*^T \delta\lambda \rangle &= -\langle v, \nabla_{uu}^2 \hat{H}_* \delta u + \nabla_{xu}^2 H_* \delta x + \nabla g_*^T \delta\nu - \delta d \rangle \\ &= -\langle v, \nabla_{uu}^2 \hat{H}_* \delta u + \nabla_{xu}^2 H_* \delta x - \delta d \rangle. \end{aligned} \tag{43}$$

Combining (42)–(43) and utilizing the bound for (y, v) in terms of $|\delta\beta|$, we obtain an estimate of the form

$$|\delta\beta| \le C(\|\delta x\|_{H^1} + \|\delta u\|_{L^2} + \|\delta a\|_{L^1} + \|\delta d\|_{L^1} + |\delta b_0| + |\delta b_1|),$$

where C is a generic constant independent of the data. Referring to (41), we conclude that

$$|\delta\beta| \le C(\|\delta s\|_{L^\infty} + \|\delta a\|_{L^\infty} + \|\delta c\|_{L^\infty} + \|\delta d\|_{L^\infty} + |\delta e| + |\delta b|).$$

Since $\|\delta x\|_{L^\infty} \le C\|\delta x\|_{H^1}$, this bound for $\delta\beta$ combined with the adjoint equation in (38) implies that

$$\|\delta\lambda\|_{H^1} \le C(\|\delta s\|_{L^\infty} + \|\delta a\|_{L^\infty} + \|\delta c\|_{L^\infty} + \|\delta d\|_{L^\infty} + |\delta e| + |\delta b|).$$

In order to make the transition from the L^2-bound for δu in (41) to an L^∞-bound, we observe that the last two lines of (38) imply that for the translated problem (40), $v = u(t)$ is the solution to the following quadratic program for almost every $t \in [0, 1]$:

$$\begin{aligned} &\underset{v}{\text{minimize}} \quad \tfrac{1}{2} v^T \nabla_{uu}^2 \hat{H}_*(t) v + v^T \nabla_{ux}^2 H_*(t) x(t) + v^T \nabla_u f_*^T(t) \lambda(t) - v^T \bar{d}(t) \\ &\text{subject to} \quad \nabla g_*^T(t) v = 0, \quad \nabla g_*^I(t) v \le 0. \end{aligned}$$

Again, exploiting the fact that the solution to a quadratic program depends Lipschitz continuously on a linear perturbation, we obtain the estimate

$$|\delta u(t)| \le C(|\delta x(t)| + |\delta\lambda(t)| + |\delta\bar{d}(t)|)$$

for almost every $t \in [0, 1]$. Since the L^∞-norm is bounded in terms of the H^1-norm, we utilize the H^1-bounds for δx and $\delta\lambda$ to obtain the estimate

$$\|\delta u\|_{L^\infty} \le C(\|\delta s\|_{L^\infty} + \|\delta a\|_{L^\infty} + \|\delta c\|_{L^\infty} + \|\delta d\|_{L^\infty} + |\delta e| + |\delta b|).$$

The last equation in (38) coupled with the Independence Assumption yields an analogous estimate for $\delta\nu$:

$$\|\delta\nu\|_{L^\infty} \le C(\|\delta s\|_{L^\infty} + \|\delta a\|_{L^\infty} + \|\delta c\|_{L^\infty} + \|\delta d\|_{L^\infty} + |\delta e| + |\delta b|).$$

Finally, the first two equations in (38) yield L^∞-bounds for the derivatives of δx and $\delta\lambda$. To summarize:

$$\begin{aligned}&\|\delta x\|_{W^{1,\infty}} + \|\delta\lambda\|_{W^{1,\infty}} + \|\delta u\|_{L^\infty} + \|\delta\nu\|_{L^\infty} + |\delta\beta| \\ &\quad \le C(\|\delta s\|_{L^\infty} + \|\delta a\|_{L^\infty} + \|\delta c\|_{L^\infty} + \|\delta d\|_{L^\infty} + |\delta e| + |\delta b|). \end{aligned} \tag{44}$$

As a by-product of (44), we conclude that the solution of (40) and the associated multipliers lie in Ω. Applying Corollary 2 of [4] it follows that, for p in a neighborhood of p_*, there is a solution ω_p of (35) satisfying an estimate of the form (37) where κ is any constant larger than the Lipschitz constant C in (44).

We now show that since ω_p depends continuously on p, the analogues of the Independence, Controllability, and Coercivity Assumptions obtained by replacing the "*" by p hold when p is near p_*. Consequently, by Theorem 2 the (x_p, u_p) component of ω_p is a local minimizer for (34). The Independence Assumption holds for p near p_* since bounded, uniformly independent vectors remain uniformly independent after small changes. The Controllability Assumption holds for p near p_* by an argument similar to that used to prove Lemma 4; that is, let $e_1, e_2, \ldots, e_k$ be a basis for R^k, and choose $x^j \in W^{1,\infty}$ and $u^j \in L^\infty$ such that $L(x^j, u^j) = 0$, $E(x^j) = e^j$, and $\nabla g_*^A u^j = 0$ for each j. By the Independence Assumption, $v_p^j \in L^\infty$ exists such that

$$\nabla g_p^A(u_p) v_p^j = 0 \quad \text{and} \quad \|v_p^j - u^j\|_{L^\infty} \le C\|\nabla g_p^A(u_p) - \nabla g_*^A\|_{L^\infty},$$

where C is a generic constant. Let y_p^j be the solution to the equation

$$\dot{y}_p^j = \nabla_x f_p(x_p, u_p) y_p^j + \nabla_u f_p(x_p, u_p) v_p^j, \qquad y_p^j(0) = x^j(0).$$

For each j, $y_p^j(1)$ approaches $x^j(1)$ as p tends to p_*. Hence, the vectors

$$\nabla\Phi_p(x_p(0), x_p(1)) \begin{pmatrix} y_p^j(0) \\ y_p^j(1) \end{pmatrix}$$

form a basis for R^k when p is near p_*. Finally, Lemma 1 (which proves surjectivity of the linearized constraints) coupled with Lemma 6 of [4] implies that the Coercivity Assumption holds for p near p_*. □

As a Section 3, we next consider a special form of the parametric optimization problem (34):

$$\begin{aligned} &\text{minimize} \quad \Psi_p(x(0), x(1)) + \int_0^1 \varphi_p(x(t), u(t))\, dt \\ &\text{subject to} \quad \dot{x}(t) = f_p(x(t), u(t)) \quad \text{and} \quad g_p(u(t)) \le 0 \quad \text{a.e. } t \in [0,1], \\ &\qquad\qquad \Phi_p(x(0), x(1)) = 0, \quad x \in W^{1,\infty}, \quad u \in L^\infty. \end{aligned} \tag{45}$$

Suppose that at a local minimizer $(x, u) = (x_*, u_*)$ of (45) corresponding to $p = p_*$, the Minimum Principle holds and the gradients of the active constraints are linearly independent. Thus multipliers λ_*, ν_*, and β_* associated with x_* and u_* satisfying (35) exist. Given $\varepsilon > 0$, let I_ε, J_ε, and A_ε be the sets defined in (28).

Corollary 2. *Suppose that $x_* \in W^{1,\infty}$, $u_* \in L^\infty$, $\lambda_* \in W^{1,\infty}$, $\nu_* \in L^\infty$, and $\beta_* \in R^k$ satisfy* (35) *when $p = p_*$. If, for some $\varepsilon > 0$, the Independence and Controllability Assumptions hold for $A = A_\varepsilon$ and the Coercivity Assumption holds for $A = J_\varepsilon$, then, for p in a neighborhood of p_*, there is a solution ω_p of* (35) *whose (x_p, u_p) component is a strict local minimizer for* (45). *Moreover, a constant κ exists such that*

$$\|\omega_p - \omega_q\|_\Omega \le \kappa \|T_p(\omega_p) - T_q(\omega_p)\|_W \tag{46}$$

for each p and q in a neighborhood of p_*.

Proof. Taking $I = I_\varepsilon$ and $J = J_\varepsilon$ and applying Theorem 4, there is a solution ω_p of (35) whose (x_p, u_p) component is a strict local minimizer for (34); and, by (37), ω_p depends continuously on p. Hence, $\nu_p(t) \ge \varepsilon/2$ for every $t \in J_\varepsilon$ and p near p_*. By Corollary 1, (x_p, u_p) is a strict local minimizer for the problem

$$\begin{aligned}
&\text{minimize} \quad \Psi(x(0), x(1)) + \int_0^1 \varphi_p(x(t), u(t))\, dt \\
&\text{subject to} \quad \dot{x}(t) = f_p(x(t), u(t)) \quad \text{and} \quad g_p^{A_\varepsilon}(u(t)) \le 0 \quad \text{a.e. } t \in [0,1], \\
&\qquad\qquad\quad \Phi_p(x(0), x(1)) = 0, \quad x \in W^{1,\infty}, \quad u \in L^\infty.
\end{aligned} \tag{47}$$

A neighborhood of p_* and of (x_*, u_*) in $W^{1,\infty} \times L^\infty$ exist such that each feasible point for (47) is also feasible in (45). Since (47) has fewer constraints than (45), we conclude that (x_p, u_p) is a strict local minimizer of (45) for p near p_*. □

6. Sequential Quadratic Programming

In this section and in the following one we apply Corollary 2 to numerical algorithms in optimal control. We first consider a sequential quadratic programming (SQP) algorithm for the problem (27). If x_k, u_k, λ_k, ν_k, and β_k denote the current iterates in the SQP algorithm, then the new iteratives x_{k+1} and u_{k+1} are obtained by computing a local minimizer of the following linear-quadratic problem:

$$\begin{aligned}
&\text{minimize} \quad \langle \nabla\Psi_k, x - x_k \rangle + \langle \nabla_x \varphi_k, x - x_k \rangle + \langle \nabla_u \varphi_k, u - u_k \rangle \\
&\qquad\qquad + \tfrac{1}{2} B_k(x - x_k, u - u_k) \\
&\text{subject to} \quad L_k(x - x_k, u - u_k) = f_k - \dot{x}_k, \quad E_k(x - x_k) = -\Phi_k, \\
&\qquad\qquad g_k + \nabla g(u_k)(u - u_k) \le 0,
\end{aligned} \tag{48}$$

where B_k, L_k, and E_k are defined like B, L, and E except that all the functions are evaluated at x_k, u_k, λ_k, ν_k, and β_k instead of at x_*, u_*, λ_*, ν_*, and β_*, respectively. Similarly, the functions φ_k, Ψ_k, f_k, Φ_k, and g_k stand for $\varphi(x_k, u_k), \Psi(x_k(0), x_k(1))$, $f(x_k, u_k)$, $\Phi(x_k(0), x_k(1))$, and $g(u_k)$, respectively. The new multipliers λ_{k+1}, ν_{k+1}, and β_{k+1} are those associated with (48). In the local quadratic convergence result that follows, the continuity assumptions are slightly stronger than those appearing previously; instead of requiring continuity of second derivatives, we need Lipschitz continuity. More precisely, we suppose that a closed set $\Delta \subset R^n \times R^m$ and a $\delta > 0$ exist such that $(x_*(t), u_*(t))$ lies in Δ for almost every $t \in [0, 1]$, the distance from $(x_*(t), u_*(t))$ to the boundary of Δ is at least δ for almost every $t \in [0, 1]$, the first two derivatives of $f(x, u)$, $\varphi(x, u)$, and $g^i(u)$, with respect to x and u, exist on Δ, and these derivatives along with the function values $f(x, u)$ and $g^i(u)$ are *Lipschitz* continuous with respect to $(x, u) \in \Delta$. Similarly, we assume that the first two derivatives of Φ and Ψ exist and are Lipschitz continuous near $(x_*(0), x_*(1))$.

Theorem 5. *Suppose that* $x_* \in W^{1,\infty}$, $u_* \in L^\infty$, $\lambda_* \in W^{1,\infty}$, $\nu_* \in L^\infty$, *and* $\beta_* \in R^k$ *satisfy* (17). *If, for some* $\varepsilon > 0$, *the Independence and Controllability Assumptions hold for* $A = A_\varepsilon$ *and the Coercivity Assumption holds for* $A = J_\varepsilon$, *then, for* x_0, u_0, λ_0, ν_0, *and* β_0 *in the neighborhoods of* x_*, u_*, λ_*, ν_*, *and* β_*, *respectively, the SQP iterates are uniquely defined, and, for every* k, *we have*

$$\begin{aligned}
&\|x_{k+1} - x_*\|_{W^{1,\infty}} + \|u_{k+1} - u_*\|_{L^\infty} + \|\lambda_{k+1} - \lambda_*\|_{W^{1,\infty}} \\
&\qquad + \|\nu_{k+1} - \nu_*\|_{L^\infty} + |\beta_{k+1} - \beta_*| \\
&\quad \le C(\|x_k - x_*\|_{L^\infty} + \|u_k - u_*\|_{L^\infty} + \|\lambda_k - \lambda_*\|_{L^\infty} \\
&\qquad + \|\nu_k - \nu_*\|_{L^\infty} + |\beta_k - \beta_*|)^2. \qquad (49)
\end{aligned}$$

Proof. Given a state $\bar{x}$, a control $\bar{u}$, and multipliers $\bar{\lambda}$, $\bar{\nu}$, and $\bar{\beta}$, consider the following linear-quadratic program:

$$\begin{aligned}
&\text{minimize} \quad \langle \nabla \overline{\Psi}, x - \bar{x} \rangle + \langle \nabla_x \overline{\varphi}, x - \bar{x} \rangle + \langle \nabla_u \overline{\varphi}, u - \bar{u} \rangle + \tfrac{1}{2} \overline{B}(x - \bar{x}, u - \bar{u}) \\
&\text{subject to} \quad \overline{L}(x - \bar{x}, u - \bar{u}) = \bar{f} - \dot{\bar{x}}, \quad \overline{E}(x - \bar{x}) = -\overline{\Phi}, \\
&\qquad\qquad\quad \bar{g} + \nabla g(\bar{u})(u - \bar{u}) \le 0. \qquad (50)
\end{aligned}$$

Analogous to the notation in (48), $\overline{B}$, $\overline{L}$, and $\overline{E}$ are defined like B, L, and E except that all the functions are evaluated at $\bar{x}$, $\bar{u}$, $\bar{\lambda}$, $\bar{\nu}$, and $\bar{\beta}$ instead of at x_*, u_*, λ_*, ν_*, and β_*, respectively. Similarly, the functions $\overline{\varphi}$, $\overline{\Psi}$, $\bar{f}$, $\overline{\Phi}$, and $\bar{g}$ stand for $\varphi(\bar{x}, \bar{u})$, $\Psi(\bar{x}(0), \bar{x}(1))$, $f(\bar{x}, \bar{u})$, $\Phi(\bar{x}(0), \bar{x}(1))$, and $g(\bar{u})$, respectively. We apply Corollary 2 to the program (50) where the parameter p is identified with variables containing bars:

$$p = (\bar{x}, \bar{u}, \bar{\lambda}, \bar{\nu}, \bar{\beta}). \qquad (51)$$

It can be verified that the hypotheses of Corollary 2 hold at

$$p_* = (x_*, u_*, \lambda_*, \nu_*, \beta_*) = \omega_{p_*}.$$

The inequality (49) is obtained by substituting the following values into (46):

$$p = p_*, \qquad q = (x_k, u_k, \lambda_k, \nu_k, \beta_k), \quad \text{and}$$
$$\omega_q = (x_{k+1}, u_{k+1}, \lambda_{k+1}, \nu_{k+1}, \beta_{k+1}).$$

With these substitutions, the left-hand side of (46) is equivalent to the left-hand side of (49). In evaluating the right-hand side of (46), the only subtle point is to recall that the Hamiltonian associated with the operator T in (46) is the Hamiltonian for the linear-quadratic problem, not the Hamiltonian of the original nonlinear problem (27)—we are applying (46) to (50) not to (27). To illustrate evaluation of the right-hand side of (46), let us consider the third component of $T_p(\omega_p) - T_q(\omega_p)$. For the general value of p in (51), the third component of T_p evaluated at x, u, and λ can be expressed as

$$\dot{\lambda} + \nabla_x H(\bar{x}, \bar{u}, \lambda) + \nabla_{xx}^2 H(\bar{x}, \bar{u}, \bar{\lambda})(x - \bar{x}) + \nabla_{xu}^2 H(\bar{x}, \bar{u}, \bar{\lambda})(u - \bar{u}).$$

For $p = p_*$, $x = x_*$, $u = u_*$, and $\lambda = \lambda_*$, the third component of $T_p(\omega_p)$ is 0, while the third component of $T_q(\omega_p)$ reduces to

$$\dot{\lambda}_* + \nabla_x H(x_k, u_k, \lambda_*) + \nabla_{xx}^2 H(x_k, u_k, \lambda_k)(x_* - x_k)$$
$$+ \nabla_{xu}^2 H(x_k, u_k, \lambda_k)(u_* - u_k). \tag{52}$$

Since H is an affine function of λ, we have

$$\nabla_{xx}^2 H(x_k, u_k, \lambda_k) = \nabla_{xx}^2\big(f(x_k, u_k)^T(\lambda_k - \lambda_*)\big) + \nabla_{xx}^2 H(x_k, u_k, \lambda_*)$$

and

$$\nabla_{xu}^2 H(x_k, u_k, \lambda_k) = \nabla_{xu}^2\big(f(x_k, u_k)^T(\lambda_k - \lambda_*)\big) + \nabla_{xu}^2 H(x_k, u_k, \lambda_*).$$

After making these substitutions in (52) along with the substitution $\dot{\lambda}_* = -\nabla_x H(x_*, u_*, \lambda_*)$, and after expanding in a Taylor series about x_k and u_k, we obtain an expression that is bounded by the right-hand side of (49) for an appropriate choice of C. The treatment of the other components of $T_p(\omega_p) - T_q(\omega_p)$ is similar. □

7. Penalty/Multiplier Approximations

In this section we analyze the following penalty/multiplier approximation to (27) in which the differential equation is penalized by a parameter $p > 0$ while the

Lagrange multiplier associated with the differential equation is approximated by some $\bar{\lambda} \in L^\infty$:

$$\begin{aligned}
&\text{minimize} \quad C_p(x,u) \\
&\text{subject to} \quad g(u(t)) \le 0 \quad \text{a.e. } t \in [0,1], \\
&\qquad\qquad \Phi(x(0),x(1)) = 0, \quad x \in W^{1,\infty}, \quad u \in L^\infty
\end{aligned} \tag{53}$$

where

$$\begin{aligned}
C_p(x,u) &= \Psi(x(0),x(1)) + \int_0^1 \varphi(x(t),u(t))\,dt \\
&\quad + \frac{1}{2p}\langle f(x,u) - \dot{x}, f(x,u) - \dot{x}\rangle + \langle \bar{\lambda}, f(x,u) - \dot{x}\rangle .
\end{aligned}$$

Suppose that the hypotheses of Theorem 3 are in effect. Hence, (x_*, u_*) is a local minimizer for (27), associated multipliers $\lambda_* \in W^{1,\infty}$, $\nu_* \in L^\infty$, and $\beta_* \in R^k$ satisfying (17) exist, and, for some $\varepsilon > 0$, the Independence and Controllability Assumptions hold for $A = A_\varepsilon$, and the Coercivity Assumption holds with $A = J_\varepsilon$. Let us consider the following perturbation of (17):

$$\begin{cases}
\dot{x} = f(x,u) + p(\bar{\lambda} - \lambda), \qquad \Phi(x(0),x(1)) = 0, \\
\dot{\lambda}^T = -\nabla_x H(x,u,\lambda), \\
\qquad (-\lambda(0)^T, \lambda(1)^T) = \beta^T \nabla\Phi(x(0),x(1)) + \nabla\Psi(x(0),x(1)), \\
\nabla_u \hat{H}(x,u,\lambda,\nu) = 0, \\
g(u) \in N_U(\nu),
\end{cases} \tag{54}$$

Putting $p = 0$ and $q = p$ in (46), it follows from Corollary 2 that, for p near 0, there is a solution $x = x_p$, $u = u_p$, $\lambda = \lambda_p$, $\nu = \nu_p$, and $\beta = \beta_p$ of (54) satisfying an estimate of the form

$$\begin{aligned}
&\|x_p - x_*\|_{W^{1,\infty}} + \|u_p - u_*\|_{L^\infty} + \|\lambda_p - \lambda_*\|_{W^{1,\infty}} + \|\nu_p - \nu_*\|_{L^\infty} + |\beta_p - \beta_*| \\
&\quad \le Cp\|\bar{\lambda} - \lambda_*\|_{L^\infty}.
\end{aligned} \tag{55}$$

We now use Theorem 1 to show that (x_p, u_p) is a local minimizer of (53). Employing the same approach used in the proof of Corollary 1, we first consider the following control problem:

$$\begin{aligned}
&\text{minimize} \quad C_p(x,u) \\
&\text{subject to} \quad \dot{x}(t) = f(x(t),u(t)) \quad \text{a.e. } t \in [0,1], \\
&\qquad\qquad g^{I_\varepsilon}(u(t)) \le 0 \quad \text{and} \quad g^{J_\varepsilon}(u(t)) \le 0 \quad \text{a.e. } t \in [0,1], \\
&\qquad\qquad \Phi(x(0),x(1)) = 0, \quad x \in W^{1,\infty}, \quad u \in L^\infty.
\end{aligned} \tag{56}$$

The constraint $h(z) \in K_h$ of (1) is identified with the constraint $g^{J_\varepsilon}(u(t)) \leq 0$ of (56), while the constraint $g(z) \in K_g$ of (1) is identified with all the other constraints of (56). By (54), the first-order conditions (2) hold. By (55), $\nu_p(t) \geq \varepsilon/2$ for every $t \in J_\varepsilon$ and p near 0, which implies that condition (6) of Section 1 holds. From the proof of Theorem 4, the analogues of the Independence, Controllability, and Coercivity Assumptions obtained by replacing the * by p hold when p is near 0. Lemma 1 implies that (8) holds. By the Controllability Assumption and Lemma 5 of [7], scalars α and $q > 0$ exist such that

$$B_p(x,u) + p^{-1}\langle L_p(x,u), L_p(x,u)\rangle \geq \alpha[\langle x,x\rangle + \langle u,u\rangle]$$

$$\text{for every} \quad x \in H^1 \quad \text{and} \quad u \in L^2 \quad \text{satisfying} \quad E_p(x) = 0$$

$$\text{and} \quad \nabla g_p^{J_\varepsilon} u = 0,$$

and for every positive $p \leq q$. Here the p subscripts mean that in the definition of each operator, every * is replaced by p. By Theorem 1, (x_p, u_p) is a strict local minimizer of (56). Since (53) has more constraints than (56), yet (x_p, u_p) is feasible in (53) for p near 0, we conclude that (x_p, u_p) is a local minimizer of (53) for p near 0. These observations are summarized in

Theorem 6. *Suppose that a local minimizer* (x_*, u_*) *of* (27) *exists and that associated multipliers* $\lambda_* \in W^{1,\infty}$, $\nu_* \in L^\infty$, *and* $\beta_* \in R^k$ *satisfying* (17) *exist. If, for some* $\varepsilon > 0$, *the Independence and Controllability Assumptions hold for* $A = A_\varepsilon$ *and the Coercivity Assumption holds for* $A = J_\varepsilon$, *then, for p in a neighborhood of* 0, *a solution* $x = x_p$, $u = u_p$, $\lambda = \lambda_p$, $\nu = \nu_p$, *and* $\beta = \beta_p$ *of* (54) *exists with* (x_p, u_p) *a local minimizer of* (53). *Moreover, the estimate* (55) *holds where C is independent of p in a neighborhood of* 0.

Appendix. Measurability

In this appendix we show that the function $\bar{u}$ in (21) is measurable. Suppose that A is a map from $[0,1]$ to the subsets of $\{1,\ldots,l\}$ with the property that the sets

$$A^j = A^{-1}(j) = \{t \in [0,1]: j \in A(t)\}, \qquad j = 1,2,\ldots,l,$$

are measurable. Let S_i, $i = 1,2,3,\ldots,2^l$, denote the 2^l distinct subsets of $\{1,\ldots,l\}$, and define

$$T^i = \{t \in [0,1]: S_i = A(t)\} = \bigcap_{j \in S_i} A^j.$$

The T^i are measurable since the A^j are measurable. Note that T^i and T^j are disjoint for $i \neq j$, and the union of the T^i is the interval $[0,1]$. If the function $\bar{u}$ in (21) is measurable on T^i for each i, then it is measurable on $[0,1]$. Since the set A is invariant on T^i, $\bar{u}$ is measurable on T^i if each of the matrices in (21) is measurable. Hence, showing that $\bar{u}$ is measurable on T^i reduces to the problem of showing that the inverse of a symmetric matrix is measurable when the elements of the matrix are measurable and the smallest eigenvalue is bounded away from zero (the smallest eigenvalue is bounded away from zero by (18)). Recall that the inverse of a matrix can be expressed in terms of the determinant of submatrices divided by the determinant of the entire matrix; moreover, the determinant of a matrix can be expressed both as a sum of a product of elements and as a product of its eigenvalues. These observations, combined with the fact that a continuous function of a measurable function is measurable, implies that the inverse matrix is measurable.

References

1. Alt W (1989) Stability of solutions for a class of nonlinear cone constrained optimization problems, Part 1: Basic theory. Numer Funct Anal Optim 10:1053–1064
2. Alt W (1990) Stability of solutions to control constrained nonlinear optimal control problems. Appl Math Optim 21:53–68
3. Alt W (1991) Parametric optimization with applications to optimal control and sequential quadratic programming. Bayreuth Math Schr 35:1–37
4. Dontchev AL, Hager WW (1993) Lipschitzian stability in nonlinear control and optimization. SIAM J Control Optim 31:569–603
5. Dontchev AL, Hager WW (1994) Implicit functions, Lipschitz maps, and stability in optimization. Math Oper Res 19:753–768
6. Dunn JC, Tian T (1992) Variants of the Kuhn–Tucker sufficient conditions in cones of nonnegative functions. SIAM J Control Optim 30:1361–1384
7. Hager WW (1985) Approximations to the multiplier method. SIAM J Numer Anal 22:16–46
8. Hager WW (1990) Multiplier methods for nonlinear optimal control. SIAM J Numer Anal 27:1061–1080
9. Hager WW, Ianculescu G (1984) Dual approximations in optimal control. SIAM J Control Optim 22:423–465
10. Ito K, Kunisch K (1992) Sensitivity analysis of solutions to optimization problems in Hilbert spaces with applications to optimal control and estimation. J Differential Equations 99:1–40
11. Malanowski K (1988) On stability of solutions to constrained optimal control problems for systems with control appearing linearly. Arch Automat Telemech 33:483–497
12. Malanowski K (1992) Second-order conditions and constraint qualifications in stability and sensitivity analysis of solutions to optimizations problems in Hilbert spaces. Appl Math Optim 25:51–79
13. Malanowski K (preprint) Two norm approach in stability and sensitivity analysis of optimization and optimal control problems
14. Maurer H (1981) First- and second-order sufficient optimality conditions in mathematical programming and optimal control. Math Programming Stud 14:163–177
15. Maurer H, Zowe J (1979) First- and second-order necessary and sufficient optimality conditions for infinite-dimensional programming problems. Math Programming 16:98–110
16. Orrell K, Zeidan V (1988) Another Jacobi sufficient criterion for optimal control with smooth constraints. J Optim Theory Appl 58:283–300
17. Reid WT (1972) Riccati Differential Equations. Academic Press, New York
18. Robinson SM (1976) Stability theory for systems of inequalities, Part II: Differentiable nonlinear systems. SIAM J Numer Anal 13:497–513
19. Robinson SM (1980) Strongly regular generalized equations. Math Oper Res 5:43–62
20. Rudin W (1966) Real and Complex Analysis. McGraw-Hill, New York
21. Ursescu C (1975) Multifunctions with closed convex graph. Czechoslovak Math J 25:438–441

22. Yang B (1991) Numerical Methods for Nonlinear Optimal Control Problems with Equality Control Constraints. PhD dissertation, Department of Mathematics, Colorado State University, Fort Collins, CO
23. Zeidan V (1984) Extended Jacobi sufficiency criterion for optimal control. SIAM J Control Optim 22:294–301
24. Zeidan V (1984) First- and second-order sufficient conditions for optimal control and the calculus of variations. Appl Math Optim 11:209–226
25. Zeidan V (1993) Sufficient conditions for variational problems with variable endpoints: coupled points. Appl Math Optim 27:191–209

Accepted 4 October 1993