Convergence of a Gauss Pseudospectral Method for Optimal Control

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A convergence theory is presented for approximations of continuous-time optimal control problems based on a Gauss pseudospectral discretization. Under assumptions of coercivity and smoothness, the Gauss pseudospectral method has a local minimizer that converges exponentially fast in the sup-norm to a local minimizer of the continuous-time optimal control problem. The convergence theorem is presented and an example is given that illustrates the exponential convergence.

I. Introduction

Over the last decade, pseudospectral methods have risen to prominence in the numerical solution of optimal control problems.[1–22] Pseudospectral methods are a class of *collocation* schemes where the optimal control problem is transcribed to a nonlinear programming problem (NLP) by parameterizing the state and control using global polynomials and collocating the differential-algebraic equations using nodes obtained from a Gaussian quadrature. Some researchers prefer the term *orthogonal collocation*,[23–25] but the terms pseudospectral and orthogonal collocation have the same meaning. The use of global polynomials is most appropriate for problems with smooth solutions. For problems where the solutions are nonsmooth or not well approximated by global polynomials of a reasonably low degree, it is preferable to use a piecewise polynomial approach (see Refs. 24 and 25) where the time interval is partitioned into subintervals and a different polynomial is used over each subinterval. Continuity is preserved by matching the values of the state at the interface between subintervals. The *hp*-pseudospectral scheme in Refs. 26,27 can also be applied to problems with a nonsmooth solution.

The three most commonly used sets of collocation points are *Legendre-Gauss* (LG), *Legendre-Gauss-Radau* (LGR), and *Legendre-Gauss-Lobatto* (LGL) points. These three sets of points are obtained from the roots of a Legendre polynomial and/or linear combinations of a Legendre polynomial and its derivatives. All three sets of points are defined on the domain $[-1, 1]$, but differ significantly in that the LG points include *neither* of the endpoints, the LGR points include *one* of the endpoints, and the LGL points include *both* of the endpoints. In recent years, the two most well documented pseudospectral methods are the *Lobatto pseudospectral method*[4,8] and the *Gauss pseudospectral method*.[12–16]

The objective of this paper is to lay out the convergence analysis for the Gauss pseudospectral method. We remark that in Ref. 28 consistency results are given for the Lobatto pseudospectral

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method applied to a very general constrained control problem. The results that we establish are somewhat different since we focus on a less general unconstrained control problems; on the other hand, we are able to analyze both consistency of the discretization and convergence of a solution of the discrete problem to a solution of the continuous problem. Finally, an example is given to illustrate the theory.

II. Convergence Analysis

The convergence of the Gauss pseudospectral method is analyzed for the following continuous-time optimal control problem:

$$\left.\begin{array}{ll} \text{minimize} & C(\mathbf{x}(1)) \\ \text{subject to} & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad t \in [-1, 1], \\ & \mathbf{x}(-1) = \mathbf{x}_0, \end{array}\right\} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state, $\mathbf{x}_0 \in \mathbb{R}^n$ is the initial condition (assumed to be given), $\mathbf{u}(t) \in \mathbb{R}^m$ is the control, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $C : \mathbb{R}^n \to \mathbb{R}$. We assume that both C and $\mathbf{f}$ are continuously differentiable.

Let t_i, $1 \le i \le N$, be the zeros of the Legendre polynomial of degree N. We also define $t_0 = -1$ and $t_{N+1} = 1$. Let $\mathbf{X}_i$ be an approximation to $\mathbf{x}(t_i)$, $0 \le i \le N+1$, and $\mathbf{U}_i$ be an approximation to $\mathbf{u}(t_i)$, $1 \le i \le N$. According to the Gauss pseudospectral method as described in Refs. 12,13,21, the continuous optimal control problem is discretized to the following nonlinear programming problem:

$$\left.\begin{array}{ll} \text{minimize} & C(\mathbf{X}_{N+1}) \\ \text{subject to} & \sum_{j=0}^{N} D_{ij}\mathbf{X}_j = \mathbf{f}(\mathbf{X}_i, \mathbf{U}_i), \quad 1 \le i \le N, \\ & \mathbf{X}_{N+1} = \mathbf{X}_0 + \sum_{j=1}^{N} \omega_j \mathbf{f}(\mathbf{X}_j, \mathbf{U}_j), \quad \mathbf{X}_0 = \mathbf{x}_0, \end{array}\right\} \tag{2}$$

where ω_j, $1 \le j \le N$, are the Legendre-Gauss quadrature weights, and $\mathbf{D}$ is the differentiation matrix defined by $D_{ij} = \dot{L}_j(t_i)$ where $L_j(t)$ is the Lagrange polynomial

$$L_j(t) = \prod_{i=0, i\neq j}^{n} \frac{t - t_i}{t_j - t_i}, \quad j = 0, 1, \ldots, N.$$

We utilize the sup-norm for $\mathbf{X} = (\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_{N+1})$ and $\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_N)$ defined by

$$\|\mathbf{X}\|_\infty = \sup_{0 \le i \le N+1} |\mathbf{X}_i| \quad \text{and} \quad \|\mathbf{U}\|_\infty = \sup_{1 \le i \le N} |\mathbf{U}_i|,$$

where $|\cdot|$ is the Euclidean norm. Let L^∞ denote the space of essentially bounded functions and let $W^{1,\infty}$ denote the space of Lipschitz continuous functions. Suppose $(\mathbf{x}^*, \mathbf{u}^*) \in W^{1,\infty}(\mathbb{R}^n) \times L^\infty(\mathbb{R}^m)$ is a local minimizer of the optimal control problem given in (1). We assume that the Pontryagin's minimum principle is satisfied. That is, there exists $\boldsymbol{\lambda}^* \in W^{1,\infty}(\mathbb{R}^n)$ and the following equations are satisfied for $\mathbf{x} = \mathbf{x}^*$, $\mathbf{u} = \mathbf{u}^*$, and $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$:

$$\begin{aligned} &\boldsymbol{\lambda}(1) = \nabla C(\mathbf{x}(1)), \\ &\dot{\boldsymbol{\lambda}}(t) = -\nabla_x H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)), \\ &0 = \nabla_u H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)), \end{aligned}$$

where H is the Hamiltonian defined by $H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = \langle \boldsymbol{\lambda}, \mathbf{f}(\mathbf{x}, \mathbf{u}) \rangle$ and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.

We define vector sequences $\mathbf{X}^*$ and $\mathbf{U}^*$, analogous to $\mathbf{X}$ and $\mathbf{U}$, by $\mathbf{X}_i^* = \mathbf{x}^*(t_i)$, $0 \le i \le N+1$, and $\mathbf{U}_i^* = \mathbf{u}^*(t_i)$, $1 \le i \le N$. We show that under a smoothness and a coercivity assumption, the Gauss pseudospectral discretization (2) has a local minimizer $(\mathbf{X}^N, \mathbf{U}^N)$ which converges to $(\mathbf{X}^*, \mathbf{U}^*)$ exponentially fast in the L^∞ norm. Our convergence theory employs the following assumptions:

(A1) $\mathbf{x}^*$ and $\boldsymbol{\lambda}^* \in C^{k+1}(\mathbb{R}^n)$, for some $k \ge 3$.

(A2) There exists an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ and $\rho > 0$ such that $B_\rho(\mathbf{x}^*(t), \mathbf{u}^*(t)) \subset \Omega$ for every $t \in [-1, 1]$, where $B_\rho(z)$ denotes the ball with center z and radius ρ. The first two derivatives of $\mathbf{f}$ are Lipschitz continuous in Ω, and the first two derivatives of C are Lipschitz continuous in $B_\rho(\mathbf{x}^*(1))$.

(A3) For some $\alpha > 0$, the smallest eigenvalue of the matrices below are greater than α:

$$\mathbf{V} = \nabla_{xx} C(\mathbf{x}^*(1)) \quad \text{and} \quad \begin{pmatrix} \mathbf{Q}(t) & \mathbf{S}(t) \\ \mathbf{S}^\mathsf{T}(t) & \mathbf{R}(t) \end{pmatrix} \text{ for all } t \in [-1, 1],$$

where

$$\begin{aligned} \mathbf{Q}(t) &= \nabla_{xx} H(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)), \quad \mathbf{S}(t) = \nabla_{xu} H(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)), \\ \mathbf{R}(t) &= \nabla_{uu} H(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)). \end{aligned}$$

(A4) $|\mathbf{A}(t)| \le .25$ for all $t \in [-1, 1]$ where $\mathbf{A}(t) = \nabla_x \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t))$.

Theorem 1. *Suppose* (1) *has a local minimizer* $\mathbf{x}^* \in W^{1,\infty}(\mathbb{R}^n)$ *and* $\mathbf{u}^* \in L^\infty(\mathbb{R}^m)$. *If* (A1)–(A4) *hold, then* (2) *has a local minimizer* $(\mathbf{X}^N, \mathbf{U}^N)$ *satisfying*

$$\|\mathbf{X}^N - \mathbf{X}^*\|_\infty + \|\mathbf{U}^N - \mathbf{U}^*\|_\infty \le \frac{c}{N^{k-\frac{5}{2}}},$$

where N is the number of Gauss points and c is a constant independent of N.

III. Overview of the Analysis

The analysis is based on the following special case of Proposition 3.1 in Ref. 29:

Proposition 2. *Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ be a linear normed space with the norms in both spaces denoted $\|\cdot\|$. Let $\mathcal{T}\colon \mathcal{X} \longmapsto \mathcal{Y}$ with $\mathcal{T}$ continuously Fréchet differentiable in $B_r(\boldsymbol{\theta}^*)$ for some $\boldsymbol{\theta}^* \in \mathcal{X}$ and $r > 0$. Suppose that the following conditions hold:*

(P1) $\|\nabla\mathcal{T}(\boldsymbol{\theta}) - \nabla\mathcal{T}(\boldsymbol{\theta}^*)\| \le \epsilon$ *for all* $\boldsymbol{\theta} \in B_r(\boldsymbol{\theta}^*)$ *and for some* $\epsilon > 0$.

(P2) $\nabla\mathcal{T}(\boldsymbol{\theta}^*)$ *is invertible.*

If $\epsilon\mu < 1$ where $\mu = \|\nabla\mathcal{T}(\boldsymbol{\theta}^*)^{-1}\|$ and $\|\mathcal{T}(\boldsymbol{\theta}^*)\| \le (1-\mu\epsilon)r/\mu$, then there exists a unique $\boldsymbol{\theta} \in B_r(\boldsymbol{\theta}^*)$ such that $\mathcal{T}(\boldsymbol{\theta}) = \mathbf{0}$. Moreover, we have the estimate

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\| \le \frac{\mu}{1-\mu\epsilon}\|\mathcal{T}(\boldsymbol{\theta}^*)\|.$$

We apply Proposition 2 to the KKT conditions associated with the discretized control problem. Hence, the map $\mathcal{T}$ corresponds to the first-order optimality conditions associated with (2) and $\boldsymbol{\theta}$ is the discrete triple $(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})$ where $\boldsymbol{\Lambda}$ is a discrete approximation to the costate variable. As explained in Ref. 20, $\boldsymbol{\Lambda}$ can be expressed as a linear combination of the KKT multipliers associated

with nonlinear programming problem (2). The first-order optimality conditions for (2) can be written in the following form:

$$
\begin{aligned}
\mathbf{X}_0 &= \mathbf{x}_0, && \text{(initial state)}\\
\textstyle\sum_{j=0}^{N} D_{ij}\mathbf{X}_j &= \mathbf{f}(\mathbf{X}_i,\mathbf{U}_i), \quad 1\le i\le N, && \text{(state collocation)}\\
\mathbf{X}_{N+1} &= \mathbf{X}_0 + \textstyle\sum_{i=1}^{N}\omega_i\mathbf{f}(\mathbf{X}_i,\mathbf{U}_i), && \text{(terminal state)}\\
\mathbf{\Lambda}_{N+1} &= \nabla_X C(\mathbf{X}_{N+1}), && \text{(terminal adjoint)}\\
\textstyle\sum_{j=1}^{N+1} D^{\dagger}_{ij}\mathbf{\Lambda}_j &= -\nabla_x\langle\mathbf{\Lambda}_i,\mathbf{f}(\mathbf{X}_i,\mathbf{U}_i)\rangle, \quad 1\le i\le N, && \text{(adjoint collocation)}\\
\mathbf{0} &= \nabla_u\langle\mathbf{\Lambda}_i,\mathbf{f}(\mathbf{X}_i,\mathbf{U}_i)\rangle, \quad 1\le i\le N, && \text{(control minimum)}
\end{aligned}
$$

where the adjoint differentiation matrix $\mathbf{D}^{\dagger}$ is given by

$$
D^{\dagger}_{ij} = -\frac{\omega_j}{\omega_i}D_{ji}, \quad 1\le i\le N, \quad 1\le j\le N, \quad D^{\dagger}_{i,N+1} = -\sum_{j=1}^{N} D^{\dagger}_{ij}, \quad 1\le i\le N.
$$

Again, see Ref. 20 for the derivation of this formulation of the KKT conditions. The parameter $\boldsymbol{\theta}^*$ in Proposition 2 is identified with the triple $(\mathbf{X}^*,\mathbf{U}^*,\mathbf{\Lambda}^*)$ where $\mathbf{\Lambda}^*$ is defined by $\mathbf{\Lambda}^*_i = \lambda^*(t_i)$, $0\le i\le N+1$.

The application of Proposition 2 to the pseudospectral control problem proceeds as follows. We first show that $\mathcal{T}(\boldsymbol{\theta}^*)$ tends to zero as N tends to infinity. In other words, we take the continuous optimal state, control, and costate, we evaluate them at the collocation points, and we insert them into the first-order optimality conditions for the discrete problem. We show that as the number of collocation points increases, the continuous solution satisfies the discrete first-order conditions more accurately. The condition $\|\mathcal{T}(\boldsymbol{\theta}^*)\| \le (1-\mu\epsilon)r/\mu$ in Proposition 2 is satisfied by taking N sufficiently large, or equivalently, by using enough collocation points. By a theorem of Jackson (for example, see Theorem 1.5 in Ref. 30) and by a bound we obtain for the Lebesgue constant associated with the interpolation points t_i, $0\le i\le N$, we are able to estimate the sup-norm of $\mathcal{T}(\boldsymbol{\theta}^*)$:

$$
\|\mathcal{T}(\boldsymbol{\theta}^*)\|_\infty \le \frac{c}{N^{k-\frac{5}{2}}},
$$

where c is a constant independent of N. Referring to Proposition 2, this bound for $\|\mathcal{T}(\boldsymbol{\theta}^*)\|_\infty$ ultimately translates into a bound for the error in the pseudospectral approximation. A convergence result of this nature is usually referred to as exponential convergence since for a smooth problem, we can take the exponent k large to achieve rapid decay in the error as a function of the polynomial degree N.

The other requirement of Proposition 2 that must be analyzed is the invertibility of the Jacobian matrix $\nabla\mathcal{T}(\boldsymbol{\theta}^*)$. Using an idea from Lemma 1 of Ref. 31, the invertibility is reduced to the analysis of a related discrete linear/quadratic control problem. The analysis is somewhat intricate and employs some results that were established by numerical computation. For example, by numerical computation, we can show that the sup-norm of both $\mathbf{D}^{-1}_{1:N}$ and $(\mathbf{D}^{\dagger}_{1:N})^{-1}$ are bounded by 2, independent of N. Ideally, one would like a theorem which proves this upper bound of 2. But from a practical perspective, N will never be very large; and in this case, we can simply compute the sup-norm of these matrices; the norms are always less than 2 and tend towards 2 as N becomes large. The derivation of the bound for the inverse of $\nabla\mathcal{T}(\boldsymbol{\theta}^*)$ also requires control over the size of $|\mathbf{A}(t_i)|$, $0\le i\le N+1$. The condition (A4) ensures that $\|\nabla\mathcal{T}(\boldsymbol{\theta}^*)^{-1}\|$ is bounded, independent of N.

Although the convergence theory has a gap when condition (A4) is not satisfied, there is a way to address this problem. Instead of using a global pseudospectral scheme, we first partition the

original interval $[-1,+1]$ into subinterval and use a different pseudospectral polynomial on each subinterval. If h is the width of the subinterval, then the condition (A4) is replaced by a condition of the form $|\mathbf{A}(t)| \leq 1/(2h)$ for all $t \in [-1,1]$. By taking enough subintervals, or equivalently, by taking h sufficiently small, the condition $|\mathbf{A}(t)| \leq 1/(2h)$ will be satisfied. This idea of introducing an initial mesh and using a different pseudospectral polynomial in each interval is what we call an hp-pseudospectral scheme. It is developed in Refs. 26, 27.

IV. Example

We now consider an example that demonstrates empirically the convergence of the Gauss pseudospectral method. While the convergence analysis of Section III was provided for an initial-value problem, the Gauss pseudospectral method can (and has been) applied to more general control problems. The example considered here is the *brachistochrone problem*, which can be formulated as a control problem with both initial and terminal constraints. Since an analytic solution is known, it is possible to evaluate the error in the Gauss pseudospectral approximation. As we will now show, the error in the discrete state, control, and costate approaches zero exponentially fast, similar to the convergence result for the unconstrained control problem.

The *brachistochrone problem* is a minimum time problem that can be stated as follows:

$$\text{minimize } t_f \tag{3}$$

subject to the dynamic constraints

$$\left.\begin{array}{rcr} \dot{x} &=& v\sin u, \\ \dot{y} &=& -v\cos u, \\ \dot{v} &=& g\cos u, \end{array}\right\} \tag{4}$$

and the boundary conditions

$$\left.\begin{array}{rclcrcl} x(0) &=& x_0 &,& x(t_f) &=& x_f, \\ y(0) &=& y_0 &,& y(t_f) &=& y_f, \\ v(0) &=& v_0 &,& v(t_f) &=& \text{Free.} \end{array}\right\} \tag{5}$$

The optimal solution to this problem is

$$\begin{aligned} x^*(t) &= R(2u^*(t) - \sin 2u^*(t)), \quad R = -\frac{y_f}{1-\cos a_0}, \\ y^*(t) &= -R(1-\cos 2u^*(t)), \\ v^*(t) &= \sqrt{-2gy^*(t)}, \quad g = 10, \\ \lambda_x^*(t) &= -\frac{\omega}{g}, \quad \omega = \frac{\sqrt{-2gy_f}}{4R\sin a_0/2}, \\ \lambda_y^*(t) &= \frac{\omega}{g}\cot \omega t_f^*, \\ \lambda_v^*(t) &= \frac{\lambda_x^*(t)}{\omega}\cos u^*(t) + \frac{\lambda_y^*(t)}{\omega}\sin u^*(t), \\ u^*(t) &= \omega t_f^*, \quad t_f^* = \frac{-2y_f}{(a_0/2)/(\sin(a_0/2)\sqrt{-2gy_f}}, \end{aligned}$$

where a_0 is the solution to the algebraic equation

$$\frac{a_0 - \sin a_0}{1-\cos a_0} + \frac{x_f}{y_f} = 0.$$

The optimal control problem given in Eqs. (3)–(5) was solved using the Gauss pseudospectral method with the boundary conditions $x(0) = y(0) = v(0) = 0$ and $x(t_f) = -y(t_f) = 2$ for $5 \leq N \leq 8$ using the NLP solver SNOPT with optimality tolerance 10^{-6} and feasibility tolerance 10^{-10}, and with the exact solution of the continuous control problem as the initial guess. The error is in the sup-norm at the approximation points (collocation points for the control and collocation points *plus* noncollocated endpoints for the state and costate). Figs. 1–3 show the base ten logarithm of the sup-norm errors for the state, control, and costate, respectively, for the Gauss pseudospectral method. It is seen that the state, control, and costate errors decrease exponentially fast for $N \leq 8$ with the the error roughly proportional to 10^{-N}.

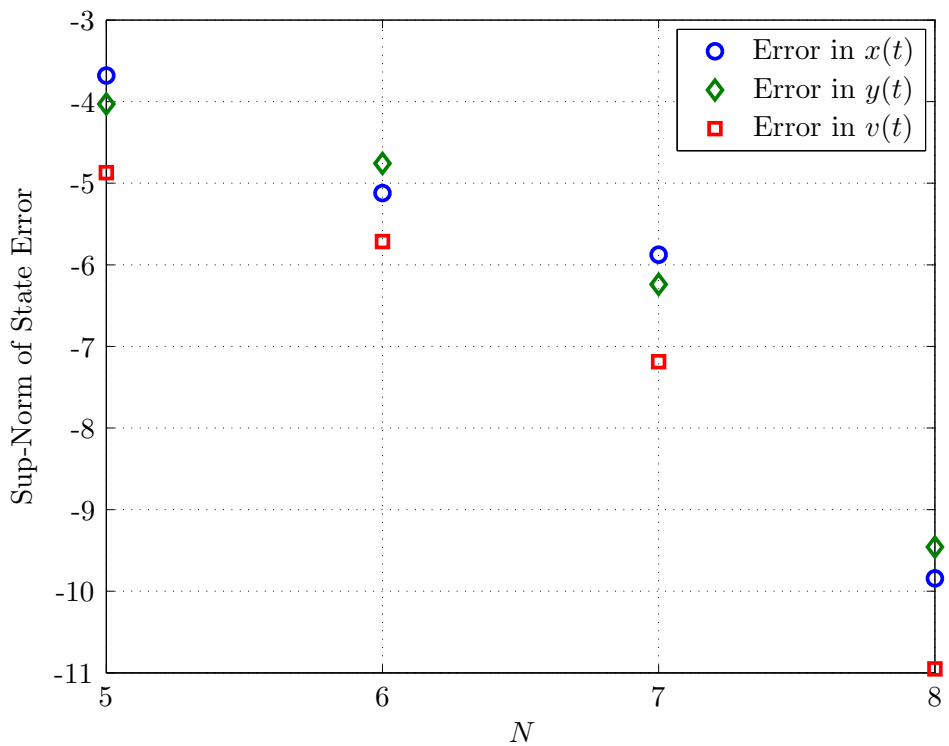

Figure 1. Base Ten Logarithm of sup-Norm State Error as a Function of N for Brachistochrone Problem Using Gauss Pseudospectral Method.

V. Conclusions

A convergence theory has been presented for the approximation of a continuous-time optimal control problems using the Gauss pseudospectral method. The main convergence theorem has been given along with an outline of its proof. Under coercivity and smoothness assumptions, there exists a solution to the Gauss pseudospectral discrete problem (2) and in the sup-norm, the error in the discrete solution is $O(N^{k-2.5})$ where N is the degree of the pseudospectral polynomials and k is the number of continuous derivatives in the solution. This is often described as exponential convergence since for a problem with a smooth solution, we can take k large in the error bound. After stating the convergence theorem and discussing its proof, an example was given that illustrates the exponential convergence.

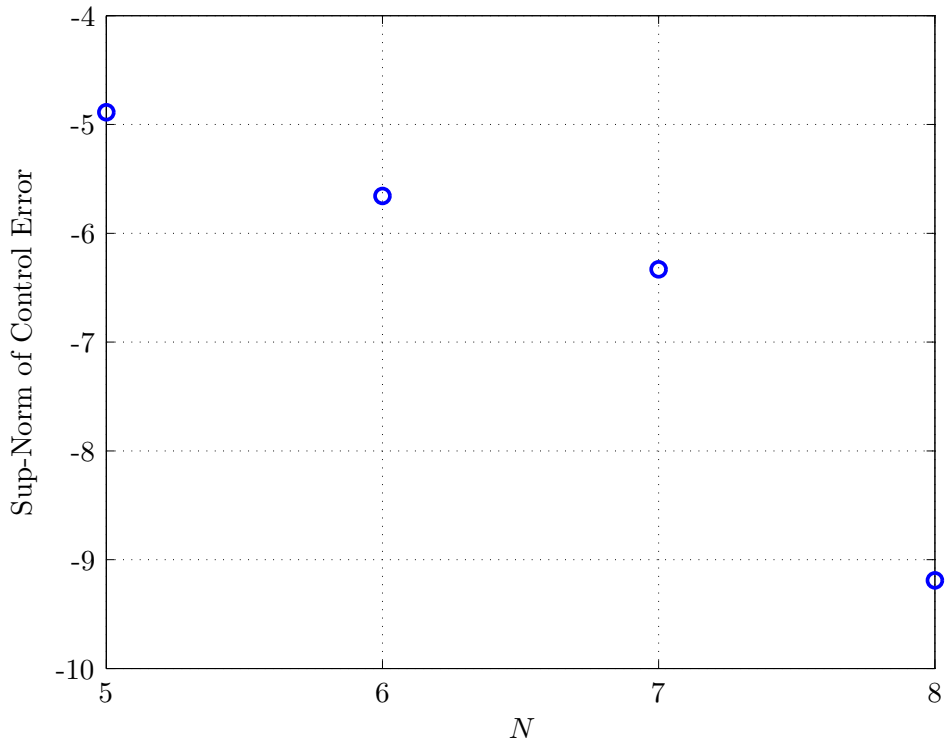

Figure 2. Base Ten Logarithm of sup-Norm Control Error as a Function of N for Brachistochrone Problem Using Gauss Pseudospectral Method.

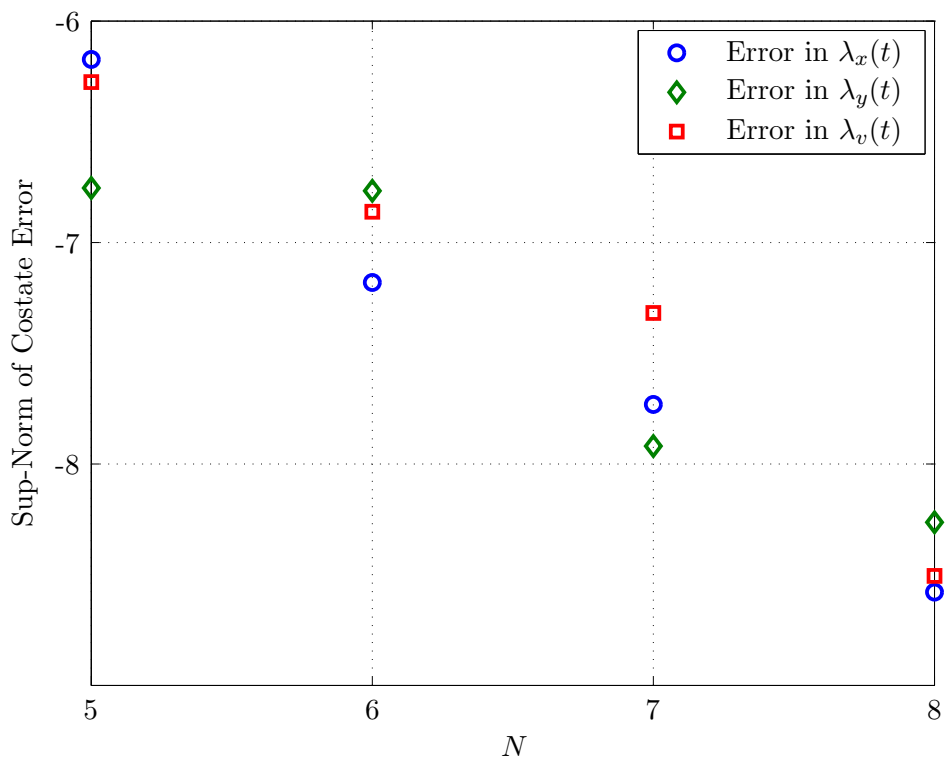

Figure 3. Base Ten Logarithm of sup-Norm Costate Error as a Function of N for Brachistochrone Problem Using Gauss Pseudospectral Method.

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