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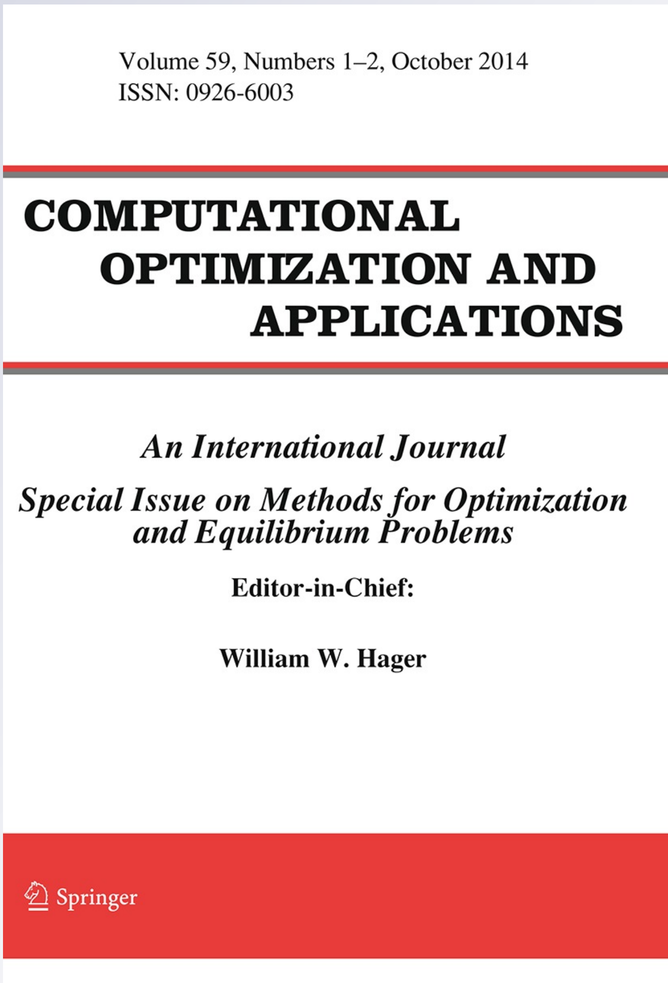

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An affine scaling method for optimization problems with polyhedral constraints

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Abstract Recently an affine scaling, interior point algorithm ASL was developed for box constrained optimization problems with a single linear constraint (Gonzalez-Lima et al., SIAM J. Optim. 21:361–390, 2011). This note extends the algorithm to handle more general polyhedral constraints. With a line search, the resulting algorithm ASP maintains the global and R-linear convergence properties of ASL. In addition, it is shown that the unit step version of the algorithm (without line search) is locally R-linearly convergent at a nondegenerate local minimizer where the second-order sufficient optimality conditions hold. For a quadratic objective function, a sublinear convergence property is obtained without assuming either nondegeneracy or the second-order sufficient optimality conditions.

Keywords Interior point · Affine scaling · Cyclic Barzilai-Borwein methods · CBB · Global convergence · Local convergence · Polyhedral constraints · Box and linear constraints

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1 Introduction

In this paper we develop an interior point algorithm for a polyhedral constrained optimization problem:

$$\min \left\{ f(\mathbf{x}) : \mathbf{x} \in \mathcal{P} \right\}, \tag{1.1}$$

where f is a real-valued, continuously differentiable function and

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \right\}. \tag{1.2}$$

Here $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{l} < \mathbf{u}$ and possibly, $l_i = -\infty$ or $u_i = \infty$. To simplify the exposition, we will focus on the special case

$$\min \left\{ f(\mathbf{x}) : \mathbf{x} \in \Omega \right\}, \quad \Omega = \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \right\}. \tag{1.3}$$

We assume that there exists $\mathbf{x} \in \Omega$ with $\mathbf{x} > \mathbf{0}$, and, without loss of generality, the rows of $\mathbf{A}$ are linearly independent.

Affine scaling methods were first introduced by Dikin [5] for linear programming: $\min \{\mathbf{c}^\mathsf{T}\mathbf{x} : \mathbf{x} \in \Omega\}$. In this context, given an iterate $\mathbf{x}_k$ in the relative interior of the feasible set, the search direction $\mathbf{d}_k$ is the solution of

$$\min \left\{ \mathbf{c}^\mathsf{T}\mathbf{d} + \frac{1}{2}\mathbf{d}^\mathsf{T}\mathbf{X}_k^{-2}\mathbf{d} : \mathbf{A}\mathbf{d} = \mathbf{0} \right\}, \tag{1.4}$$

where $\mathbf{X}_k = \mathbf{diag}(\mathbf{x}_k)$ is the diagonal matrix with $\mathbf{x}_k$ on the diagonal. Hence, we have

$$\mathbf{d}_k = -\mathbf{X}_k^2\left[\mathbf{I} - \mathbf{A}^\mathsf{T}\left(\mathbf{A}\mathbf{X}_k^2\mathbf{A}^\mathsf{T}\right)^{-1}\mathbf{A}\mathbf{X}_k^2\right]\mathbf{c}. \tag{1.5}$$

Given $\beta \in (0, 1)$ near 1, the iterates are expressed

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \beta\alpha_k\mathbf{d}_k, \quad \alpha_k = \max\{\alpha : \mathbf{x}_k + \alpha\mathbf{d}_k \geq \mathbf{0}\}.$$

Dikin's LP affine scaling method was generalized and further analyzed by Saigal in [12] by considering the scaling matrix $\mathbf{X}_k^{-2\gamma}$, where $\gamma > 0$ is a parameter, while the extension to nonlinear objective functions is given in [3, 6, 10, 13, 15]. In [14] global convergence is established for a general nonlinear objective f with scaling matrix $\mathbf{X}_k^{-2\gamma}$. In this case, the search direction $\mathbf{d}_k$ is the solution of

$$\min \left\{ \nabla f(\mathbf{x}_k)\mathbf{d} + \frac{1}{2}\mathbf{d}^\mathsf{T}\mathbf{X}_k^{-2\gamma}\mathbf{d} : \mathbf{A}\mathbf{d} = \mathbf{0} \right\}. \tag{1.6}$$

Analogous to (1.5), $\mathbf{d}_k$ is given by

$$\mathbf{d}_k = -\mathbf{X}_k^{2\gamma}\left[\mathbf{I} - \mathbf{A}^\mathsf{T}\left(\mathbf{A}\mathbf{X}_k^{2\gamma}\mathbf{A}^\mathsf{T}\right)^{-1}\mathbf{A}\mathbf{X}_k^{2\gamma}\right]\mathbf{g}_k, \tag{1.7}$$

where $\mathbf{g}_k = \nabla f(\mathbf{x}_k)^\mathsf{T}$. In [14] the stepsize was determined by an Armijo line search criterion. Under surprisingly weak conditions, more specialized global and local convergence results were obtained in [14] when f was a quadratic objective function.

In this paper, we extend the affine scaling interior point method of [7] from a single linear constraint to a system of linear constraints. We call this algorithm ASP (affine scaling for polyhedral constraints). Analogous to Dikin's affine scaling method, the search direction in ASP is the solution of a quadratic program with a diagonal scaling matrix:

$$\min \left\{ \mathbf{g}_k^\top \mathbf{d} + \frac{1}{2}\mathbf{d}^\top \boldsymbol{\Sigma}_k^{-1} \mathbf{d} : \mathbf{A}\mathbf{d} = \mathbf{0} \right\}. \tag{1.8}$$

The scaling matrix $\boldsymbol{\Sigma}_k$, given in the next section, is obtained from a quasi-Newton approximation to the first order optimality conditions of (1.3). By the structure of $\boldsymbol{\Sigma}_k$, $\mathbf{x}_k + \mathbf{d}_k$ lies in the interior of Ω. As in [7] or [9], the algorithm can be implemented using a nonmonotone line search. We show that under a nondegeneracy assumption, any cluster point of our method is a stationary point of (1.3), while traditional affine scaling methods based on Dikin's method require additional assumptions such as convexity or concavity (for example, see [14, Theorem 1]). Similar to Dikin-type affine scaling methods, we show that when f is a quadratic, the iterates converge sublinearly, without either nondegeneracy or second-order sufficient optimality assumptions, and the asymptotic convergence speed is on the order of $k^{-\sigma}$, for arbitrary $\sigma \in (0, \infty)$, where k is the iteration number. For a general nonlinear objective function, it is shown that the unit step version of the algorithm (without line search) is locally R-linearly convergent at a nondegenerate local minimizer where the second-order sufficient optimality conditions are satisfied. Consequently, our method performs locally like a Barzilai-Borwein gradient method in a "free" subspace of the null space of $\mathbf{A}$, while the traditional affine scaling method with an Armijo stepsize may have more difficulty with ill-conditioned problems, similar to the slow convergence of steepest descent with a Cauchy stepsize when the problem is ill conditioned.

Our paper is organized as follows: In Sect. 2 we present the algorithm and analyze the existence of the iterate. Section 3 gives a global convergence result. Section 4 gives special sublinear convergence results for quadratic objective functions without making assumptions regarding the local minimizer. Section 5 studies linear convergence at a nondegenerate local minimizer satisfying the second-order sufficient optimality conditions.

Notation Throughout the analysis, c denotes a generic constant which has different values in different equations. We let $\mathbf{a}_i \in \mathbb{R}^m$ denote the i-th column of $\mathbf{A}$ and $\mathbf{e}_i$ be the i-th column of the n by n identity matrix. If $F \subset \{1, \ldots, n\}$, then $\mathbf{A}_F$ is the submatrix formed by the columns $\mathbf{a}_i$, $i \in F$. For any scalar t, $t^+ = \max\{0, t\}$, while for any vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v}^+$ is the vector whose i-th component is v_i^+. The positive span of a set $\mathcal{S}$, denoted $\text{span}_+(\mathcal{S})$, is the set of linear combinations of vectors in $\mathcal{S}$ with nonnegative coefficients. $\nabla f(\mathbf{x})$ denotes the gradient of f, a row vector. The gradient of $f(\mathbf{x})$, arranged as a column vector, is $\mathbf{g}(\mathbf{x})$. The subscript k often represents the iteration number in an algorithm, and $\mathbf{g}_k$ stands for $\mathbf{g}(\mathbf{x}_k)$. If $\mathbf{x}^*$ is an optimal solution of (1.3), then $\mathbf{g}^*$ denotes $\mathbf{g}(\mathbf{x}^*)$. We let x_{ki} denote the i-th component of the iterate $\mathbf{x}_k$. The Hadamard (or component-wise) product $\mathbf{x} \circ \mathbf{y}$ of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is the vector in $\mathbb{R}^n$ defined by $(\mathbf{x} \circ \mathbf{y})_i = x_i y_i$. Given $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{diag}(\mathbf{x})$ is an n by n diagonal matrix with i-th diagonal element x_i. $\|\cdot\|$ is the Euclidean norm and $\|\cdot\|_p$ is the p-norm. $|\mathcal{S}|$ is the number of elements in the set $\mathcal{S}$, and $\mathcal{S}^c$ is the complement of $\mathcal{S}$.

2 The ASP method

Our ASP algorithm starts at a point $\mathbf{x}_1$ in the relative interior of the feasible set, and generates a sequence $\mathbf{x}_k$, $k \geq 2$, by the following rule:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + s_k \mathbf{d}_k \tag{2.1}$$

where $s_k \in (0, 1]$ is a positive stepsize and the i-th component of $\mathbf{d}_k$ is given by

$$d_{ki} = -\left(\frac{1}{\lambda_k + (g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k)^+/x_{ki}}\right)(g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k); \tag{2.2}$$

here λ_k is a positive scalar and $\boldsymbol{\mu}_k \in \mathbb{R}^m$ is chosen so that $\mathbf{A}\mathbf{d}_k = \mathbf{0}$. Equivalently,

$$\mathbf{d}_k = -\boldsymbol{\Sigma}_k \left(\mathbf{g}_k - \mathbf{A}^\mathsf{T}\boldsymbol{\mu}_k\right), \tag{2.3}$$

where $\boldsymbol{\Sigma}_k \in \mathbb{R}^{n\times n}$ is a diagonal matrix and the i-th diagonal element of $\boldsymbol{\Sigma}_k$ is given by

$$\Sigma_{k,ii} = \frac{1}{\lambda_k + (g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k)^+/x_{ki}} = \frac{x_{ki}}{\lambda_k x_{ki} + (g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k)^+}.$$

If $\mathbf{x}_k > \mathbf{0}$, then the denominator of $\Sigma_{k,ii}$ cannot vanish since $\lambda_k > 0$; moreover, from the formula for $\mathbf{d}_k$, we have $\mathbf{x}_{k+1} > \mathbf{0}$. The condition $\mathbf{A}\mathbf{d}_k = \mathbf{0}$ is equivalent to requiring that $\boldsymbol{\mu}_k$ is a root of the equation $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$ where

$$\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{A}\mathbf{d}_k = -\mathbf{A}\ \boldsymbol{\Sigma}(\mathbf{x}_k, \boldsymbol{\mu}, \lambda_k)\ (\mathbf{g}_k - \mathbf{A}^\mathsf{T}\boldsymbol{\mu}), \tag{2.4}$$

and $\boldsymbol{\Sigma}$ is a diagonal matrix with i-th diagonal element

$$\Sigma_{ii}(\mathbf{x}, \boldsymbol{\mu}, \lambda) = \frac{x_i}{\lambda x_i + \left(g_i(\mathbf{x}) - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}\right)^+}. \tag{2.5}$$

Throughout the paper, we assume that there exists a root $\boldsymbol{\mu}_k$ of the equation $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$ at every iteration. When the root exists, it is unique as we show below. Theorem 2.2 below exhibits some cases where the existence of a root can be proved.

To guarantee global convergence for a general nonlinear objective function, the ASP iteration must be combined with a line search. The same nonmonotone line search scheme given in [7] can be used; for reference, we repeat the scheme here:

AFFINE SCALING INTERIOR POINT METHOD FOR POLYHEDRAL CONSTRAINTS (ASP)

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Given λ0 > 0, δ and η ∈ (0,1), and an integer M ≥ 0.
Choose x1 > 0 with Ax1 = b and set k = 1.
  Step 1.  Choose λk ≥ λ0 and find μk such that Adk = 0, where dk
           is defined in (2.2).
  Step 2.  If dk = 0, stop.
  Step 3.  Choose sk = η^j with j ≥ 0 the smallest integer such that
           f(xk + sk dk) ≤ fk^R + δ sk ∇f(xk) dk,
           where fk^R = max{f(x_{k−j}) : 0 ≤ j ≤ min(k − 1, M)}.
  Step 4.  Set x_{k+1} = xk + sk dk.
  Step 5.  Set k = k + 1 and go to step 1.
```

The parameter f_k^R is the reference function value. For an Armijo line search, $M = 0$ and $f_k^R = f(\mathbf{x}_k)$, while for a GLL line search [8], f_k^R is the maximum function value occurring in the previous $M + 1$ iterations when $k \geq M + 1$. In our numerical experiments with the case $m = 1$, we took $M = 8$, $\eta = 0.5$, $\delta = 0.0001$, and $\lambda_0 = 10^{-30}$.

As explained in [7], this algorithm is an approximate Newton step applied to the optimality conditions for (1.3). Let $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be the Lagrangian defined by

$$L(\mathbf{x}, \boldsymbol{\mu}) = f(\mathbf{x}) + \boldsymbol{\mu}^{\mathsf{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}).$$

The first-order optimality conditions (KKT conditions) for (1.3) can be expressed:

$$\mathbf{X}^1(\mathbf{x}, \boldsymbol{\mu}) \circ \nabla_x L(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{0}, \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}, \tag{2.6}$$

where

$$X_i^1(\mathbf{x}, \boldsymbol{\mu}) = \begin{cases} 1 & \text{if } g_i(\mathbf{x}) - \mathbf{a}_i^{\mathsf{T}}\boldsymbol{\mu} \leq 0, \\ x_i & \text{otherwise.} \end{cases} \tag{2.7}$$

If the first equation in the KKT conditions (2.6) is linearized around $\mathbf{x}_k$ and the Hessian $\nabla^2 f(\mathbf{x}_k)$ is approximated by $\lambda_k \mathbf{I}$, then we obtain the ASP iteration (2.1)–(2.2). In practice, λ_k is often given by a quasi-Newton condition introduced later in (5.2).

The ASP algorithm is defined when the equation $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$ has a solution. In the case that $\mathbf{A}$ is 1 by n, we show in [7] that there is always a unique solution. We now consider the existence and uniqueness of a solution to $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$ in the more general case where $\mathbf{A}$ is m by n with $m \geq 1$.

Lemma 2.1 *If the equation $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$ has a solution, then it is unique.*

Proof Let $\mathbf{t}$ be defined by

$$\mathbf{t} = \mathbf{g}_k - \mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}.$$

Observe that

$$\left(\boldsymbol{\Sigma}_k\left(\mathbf{g}_k - \mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}\right)\right)_i = (\boldsymbol{\Sigma}_k \mathbf{t})_i = \frac{x_{ki} t_i}{\lambda_k x_{ki} + t_i^+}. \tag{2.8}$$

Although $\mathbf{t}^+$ is not differentiable at $\mathbf{t} = \mathbf{0}$, the fraction (2.8) is differentiable since the point of nondifferentiability in the denominator is precisely the point where the numerator vanishes. It is easily seen that

$$\frac{d(\boldsymbol{\Sigma}_k \mathbf{t})_i}{dt_i} = \frac{\lambda_k x_{ki}^2}{(\lambda_k x_{ki} + t_i^+)^2}.$$

By the chain rule, it follows that

$$\nabla \mathbf{r}_k(\boldsymbol{\mu}) = \lambda_k \mathbf{A} \boldsymbol{\Sigma}_k^2 \mathbf{A}^{\mathsf{T}}. \tag{2.9}$$

Since the rows of $\mathbf{A}$ are linearly independent, $\lambda_k \geq \lambda_0 > 0$, and $\boldsymbol{\Sigma}_k$ is a diagonal matrix with positive diagonal when $\mathbf{x}_k > \mathbf{0}$, the Jacobian in (2.9) is positive definite. Suppose that $\mathbf{r}_k(\boldsymbol{\mu}_1) = \mathbf{r}_k(\boldsymbol{\mu}_2) = \mathbf{0}$. By Taylor's theorem, we have

$$\mathbf{0} = \mathbf{r}_k(\boldsymbol{\mu}_1) - \mathbf{r}_k(\boldsymbol{\mu}_2) = \left(\int_0^1 \nabla \mathbf{r}_k\big((1-s)\boldsymbol{\mu}_2 + s\boldsymbol{\mu}_1\big) ds \right)(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2).$$

Since the sum of positive definite matrices is positive definite, the integral above is positive definite, which implies that $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$. Hence, any solution of $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$ is unique. □

If the function $\mathbf{r}_k(\boldsymbol{\mu})^\mathsf{T}\mathbf{r}_k(\boldsymbol{\mu})$ attains a minimum at $\boldsymbol{\mu} = \boldsymbol{\nu}$, then by the first-order optimality conditions, we have

$$\nabla \mathbf{r}_k(\boldsymbol{\nu})\mathbf{r}_k(\boldsymbol{\nu}) = \mathbf{0}.$$

Since $\nabla \mathbf{r}_k(\boldsymbol{\nu})$ is positive definite, it follows that $\mathbf{r}_k(\boldsymbol{\nu}) = \mathbf{0}$. Hence, any minimizer of $\mathbf{r}_k(\boldsymbol{\mu})^\mathsf{T}\mathbf{r}_k(\boldsymbol{\mu})$ yields the unique solution of $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$. However, showing that $\mathbf{r}_k(\boldsymbol{\mu})^\mathsf{T}\mathbf{r}_k(\boldsymbol{\mu})$ attains a minimum does not seem easy in general. We now give both local and global results concerning the existence of a solution to $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$.

Theorem 2.2 *For any* $\mathbf{x}_k > \mathbf{0}$, $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$ *has a unique solution in any of the following situations*:

(I) *The matrix* $\mathbf{A}$ *has the following form*:

$$\mathbf{A} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{p}_m \end{bmatrix},$$

where the $\mathbf{p}_i$ *are nonzero row vectors.*

(II) *span*$_+\{\mathbf{a}_i : i = 1, \dots, n\} = \mathbb{R}^m$.

(III) (1.3) *has a local minimizer* $\mathbf{x}^*$ *with properties* (a) *and* (b) *below and* $\mathbf{x}_k$ *is sufficiently close to* $\mathbf{x}^*$.

(a) $\mathbf{A}_F$ *has rank* m *where* $F = \{i : x_i^* > 0\}$. *Moreover, if* $\boldsymbol{\mu}^*$ *is the KKT multiplier associated with* $\mathbf{x}^*$, *then*

$$g_i(\mathbf{x}^*) - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}^* > 0 \quad \textit{when } x_i^* = 0. \tag{2.10}$$

(b) f *is twice continuously differentiable near* $\mathbf{x}^*$ *and* $\lambda_k \in [\lambda_{\min}, \lambda_{\max}] \subset (0, \infty)$.

Proof In case (I), the equation $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$ uncouples into m independent equations for each of the components of $\boldsymbol{\mu}$. By the [7, Proposition 2.2], each of these m equations has a unique (scalar) solution. These unique scalar solutions form the components of the unique solution vector $\boldsymbol{\mu}_k$.

In case (III), let us define the function $\mathbf{r}$ as follows:

$$\mathbf{r}(\mathbf{x}, \boldsymbol{\mu}, \lambda) = \mathbf{A}\boldsymbol{\Sigma}(\mathbf{x}, \boldsymbol{\mu}, \lambda)\big(\mathbf{g}(\mathbf{x}) - \mathbf{A}^\mathsf{T}\boldsymbol{\mu}\big),$$

where the diagonal matrix $\boldsymbol{\Sigma}$ is given in (2.5). Since $g_i(\mathbf{x}^*) - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}^* > 0$ when $x_i^* = 0$ and $g_i(\mathbf{x}^*) - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}^* = 0$ when $x_i^* > 0$, it follows from the KKT conditions that $\mathbf{r}(\mathbf{x}^*, \boldsymbol{\mu}^*, \lambda) = \mathbf{0}$ for all $\lambda > 0$. The Jacobian of $\mathbf{r}(\mathbf{x}^*, \boldsymbol{\mu}, \lambda)$ with respect to $\boldsymbol{\mu}$ can be computed in the same way that we computed the Jacobian of $\mathbf{r}_k$ in (2.9), the only difference is that $\Sigma_{ii}(\mathbf{x}^*, \boldsymbol{\mu}, \lambda) = 0$ if $x_i^* = 0$; the final result is

$$\nabla_\mu \mathbf{r}\big(\mathbf{x}^*, \boldsymbol{\mu}, \lambda\big) = \lambda \mathbf{A}_F \boldsymbol{\Sigma}_{FF}\big(\mathbf{x}^*, \boldsymbol{\mu}, \lambda\big)^2 \mathbf{A}_F^\mathsf{T}, \tag{2.11}$$

where $\boldsymbol{\Sigma}_{FF}$ is the submatrix of $\boldsymbol{\Sigma}$ corresponding to rows and columns associated with $i \in F$. Since the rows of $\mathbf{A}_F$ are linearly independent by (a), $\lambda > 0$, and $\boldsymbol{\Sigma}_{FF}$ is a diagonal matrix with positive diagonal, it follows that the Jacobian (2.11) is invertible. The derivatives of $\mathbf{r}(\mathbf{x}, \boldsymbol{\mu}^*, \lambda)$ with respect to $\mathbf{x}$ and λ are slightly more complex than (2.11), but easily computed; the derivatives involve the derivative of $\mathbf{g}$, or the second derivative of f. Hence, these derivatives are continuous since f is twice continuously differentiable near $\mathbf{x}^*$. By the implicit function theorem, it follows that for $\mathbf{x}_k$ near $\mathbf{x}^*$ and for any $\lambda_k \in [\lambda_{\min}, \lambda_{\max}]$, the equation

$$\mathbf{r}(\mathbf{x}_k, \boldsymbol{\mu}, \lambda_k) = \mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$$

has a unique solution $\boldsymbol{\mu}_k$ near $\boldsymbol{\mu}^*$.

Finally, let us consider case (II). In this case, we show that $\|\mathbf{r}_k(\boldsymbol{\mu})\|$ tends to infinity as $\|\boldsymbol{\mu}\|$ tends to infinity. Hence, as noted before the theorem, the minimizer of $\|\mathbf{r}_k(\boldsymbol{\mu})\|^2$ is the unique solution of $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$. The growth of $\|\mathbf{r}_k(\boldsymbol{\mu})\|$ is analyzed as follows: First, consider any $\boldsymbol{\mu} \in \mathbb{R}^m$ with $\|\boldsymbol{\mu}\| = 1$. By assumption, there exists $\boldsymbol{\theta} \in \mathbb{R}^n$ with $\boldsymbol{\theta} \geq \mathbf{0}$ and $\boldsymbol{\mu} = \mathbf{A}\boldsymbol{\theta}$. Moreover, it can be arranged so that the columns of $\mathbf{A}$ associated with $\theta_i > 0$ are linearly independent. Let β be defined as follows:

$$\beta = \max\big\{\big\|\big(\mathbf{A}_I^\mathsf{T}\mathbf{A}_I\big)^{-1}\mathbf{A}_I^\mathsf{T}\big\| \colon \text{ rank } \mathbf{A}_I = |I|,\ I \subset \{1, \ldots, n\}\big\}.$$

Here the rank condition basically means that the vectors $\mathbf{a}_i$, $i \in I$, are linearly independent. Since $\mathbf{A}\boldsymbol{\theta} = \boldsymbol{\mu}$ and the columns of $\mathbf{A}$ corresponding nonzero components of $\boldsymbol{\theta}$ are linearly independent, it follows that

$$\|\boldsymbol{\theta}\|_1 \leq n\|\boldsymbol{\theta}\| \leq n\beta\|\boldsymbol{\mu}\| = n\beta.$$

Define $j = \arg\max\{\boldsymbol{\mu}^\mathsf{T}\mathbf{a}_i : 1 \leq i \leq n\}$. We have

$$1 = \boldsymbol{\mu}^\mathsf{T}\boldsymbol{\mu} = \sum_{i=1}^n \theta_i\big(\boldsymbol{\mu}^\mathsf{T}\mathbf{a}_i\big) \leq \big(\boldsymbol{\mu}^\mathsf{T}\mathbf{a}_j\big)\|\boldsymbol{\theta}\|_1 \leq n\beta\big(\boldsymbol{\mu}^\mathsf{T}\mathbf{a}_j\big). \tag{2.12}$$

Let us define the sets

$$\mathcal{L}_-(t) = \big\{i : g_{ki} - t\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu} \leq 0\big\} \quad \text{and} \quad \mathcal{L}_+(t) = \big\{i : g_{ki} - t\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu} > 0\big\}.$$

With these definitions,

$$\mathbf{r}_k(t\boldsymbol{\mu}) = \sum_{i\in\mathcal{L}_-(t)} \mathbf{a}_i \left(\frac{g_{ki} - t\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}}{\lambda_k}\right) + \sum_{i\in\mathcal{L}_+(t)} \mathbf{a}_i \left(\frac{x_{ki}(g_{ki} - t\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu})}{\lambda_k x_{ki} + g_{ki} - t\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}}\right).$$

Since the coefficient of $\mathbf{a}_i$ in the $\mathcal{L}_+(t)$ sum is bounded by x_{ki}, it follows that

$$\left\| \sum_{i\in\mathcal{L}_+(t)} \mathbf{a}_i \left(\frac{x_{ki}(g_{ki} - t\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu})}{\lambda_k x_{ki} + g_{ki} - t\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}}\right)\right\| \le \|\mathbf{x}_k\|_\infty \sum_{i=1}^{n} \|\mathbf{a}_i\|. \tag{2.13}$$

By (2.12), for any $\boldsymbol{\mu}$ with $\|\boldsymbol{\mu}\| = 1$, there exists j such that $\boldsymbol{\mu}^\mathsf{T}\mathbf{a}_j \ge 1/(n\beta)$. Hence, if t satisfies the inequality

$$t \ge n\beta\|\mathbf{g}_k\|_\infty$$

then $j \in \mathcal{L}_-(t)$. Consequently, by (2.12) we have

$$\left\| \sum_{i\in\mathcal{L}_-(t)} \mathbf{a}_i\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu} \right\| \ge \boldsymbol{\mu}^\mathsf{T}\left(\sum_{i\in\mathcal{L}_-(t)} \mathbf{a}_i\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}\right) = \sum_{i\in\mathcal{L}_-(t)} \left(\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}\right)^2 \ge \left(\mathbf{a}_j^\mathsf{T}\boldsymbol{\mu}\right)^2 \ge 1/(n\beta)^2.$$

From this lower bound and (2.13), it follows that $\|\mathbf{r}_k(t\boldsymbol{\mu})\|$ tends to infinity, asymptotically at least as fast as $t/[\lambda_k(n\beta)^2]$. Equivalently, if $\boldsymbol{\mu} \in \mathbb{R}^m$ is an arbitrary vector, not necessarily a unit vector, then $\|\mathbf{r}_k(\boldsymbol{\mu})\|$ tends to infinity, asymptotically at least as fast as $\|\boldsymbol{\mu}\|/[\lambda_k(n\beta)^2]$. □

In case (I) of Theorem 2.2, the equation $\mathbf{r}_k(\boldsymbol{\mu})$ uncouples into m independent equations for the components of $\boldsymbol{\mu}_k$. Each of these independent equations can be solved using the techniques developed in [7]. In general, since the Jacobian of $\mathbf{r}_k(\boldsymbol{\mu})$ in (2.9) is positive definite, Newton's method might be applied to compute the root $\boldsymbol{\mu}_k$ of $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$. When f is quadratic and $\mathbf{A}$ has the form given in case (I), the resulting problem is called a multi-Standard quadratic optimization problem (multi-StQP), which has important applications in computer imaging and pattern recognition (see [2]).

The linear independence condition in (IIIa) is often referred to as "primal nondegeneracy" [14]. The condition (2.10) amounts to nondegeneracy for the dual multiplier associated with the bound constraint.

3 Global convergence

In this section, we give a global convergence result for ASP. Several properties established in [7] for the search direction $\mathbf{d}_k$ remain valid when $m > 1$, simply by changing the scalar a_i to the vector $\mathbf{a}_i$. Hence, we simply state these results that directly extend to a system of linear constraints.

Proposition 3.1 *If $\mathbf{x}_k$ lies in the relative interior of the feasible set Ω for (1.3) and $\mathbf{A}^\mathsf{T}\mathbf{d}_k = \mathbf{0}$, then $\mathbf{x}_{k+1} = \mathbf{x}_k + s_k\mathbf{d}_k$ lies in the relative interior of Ω for all $s_k \in [0, 1]$.*

Proposition 3.2 *The search direction* $\mathbf{d}_k$ *given by* (2.2) *satisfies*

$$\mathbf{g}_k^\mathsf{T}\mathbf{d}_k = -\left\| \boldsymbol{\Sigma}_k^{-1/2}\mathbf{d}_k \right\|^2 \leq -\lambda_k \|\mathbf{d}_k\|^2.$$

Proposition 3.3 *If* f *is continuously differentiable,* $\lambda_k \in [\lambda_{\min}, \lambda_{\max}] \subset (0, \infty)$ *for all* k, *and the iterates* $(\mathbf{x}_k, \boldsymbol{\mu}_k)$ *are uniformly bounded with* $\mathbf{x}_k$ *in the relative interior of* Ω, *then*

$$\lim_{k\to\infty} \mathbf{d}_k = \mathbf{0} \quad \textit{if and only if} \quad \lim_{k\to\infty} \mathbf{X}^1(\mathbf{x}_k, \boldsymbol{\mu}_k) \circ \nabla_x L(\mathbf{x}_k, \boldsymbol{\mu}_k) = \mathbf{0},$$

where $\mathbf{X}^1$ *are defined in* (2.7).

Theorem 3.4 *If* $\lambda_k \in [\lambda_{\min}, \lambda_{\max}] \subset (0, \infty)$ *for all* k, *the level set of* f *is bounded at* $\mathbf{x}_1$, *and* f *is Lipschitz continuously differentiable, then ASP either terminates in a finite number of iterations at a KKT point, or*

$$\lim_{k\to\infty} \mathbf{d}_k = \mathbf{0}.$$

We use Theorem 3.4 to show that every convergent subsequence of the ASP iterates approaches a stationary point when a nondegeneracy condition holds.

Theorem 3.5 *Suppose the hypotheses of Theorem* 3.4 *are satisfied and a subsequence of the iterates* $\mathbf{x}_k$ *generated by ASP approaches a limit* $\mathbf{x}^*$. *If* $\mathbf{A}_F$ *has rank* m *where* $F = \{i : x_i^* > 0\}$, *then there exists* $\boldsymbol{\mu}^* \in \mathbb{R}^m$ *such that* $(\mathbf{x}^*, \boldsymbol{\mu}^*)$ *satisfy the KKT conditions* (2.6).

Proof For convenience and without loss of generality, we assume that the entire sequence $\mathbf{x}_k$ converges to $\mathbf{x}^*$. If $\liminf_{k\to\infty} g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k < 0$, then there is a scalar $\alpha < 0$ and a subsequence $\mathcal{K} \subset \{1, 2, \ldots\}$ such that $g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k \leq \alpha < 0$ for all $k \in \mathcal{K}$. Hence, by (2.2) and for all $k \in \mathcal{K}$, we have

$$d_{ki} = -\left(\frac{g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k}{\lambda_k} \right) \geq \frac{\alpha}{\lambda_{\max}} > 0,$$

which contradicts Theorem 3.4. It follows that

$$\liminf_{k\to\infty} g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k \geq 0, \quad \text{for } i = 1, \ldots, n. \tag{3.1}$$

Define the set

$$\mathcal{F} = \left\{ i \in \{1, \ldots, n\} : \limsup_{k\to\infty} g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k < +\infty \right\}. \tag{3.2}$$

If $i \notin \mathcal{F}$, then by the definition of d_{ki}, we see that d_{ki} converges to $-x_i^*$. Since d_{ki} tends to zero by Theorem 3.4, it follows that $x_i^* = 0$ when $i \notin \mathcal{F}$. Hence, $F = \{i : x_i^* > 0\} \subset \mathcal{F}$. If $i \in \mathcal{F}$, then from the boundedness of $g_{ki} - \mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k$, we conclude that $\mathbf{a}_i^\mathsf{T}\boldsymbol{\mu}_k$ is bounded. Hence, $\mathbf{p}_k = \mathbf{A}_\mathcal{F}^\mathsf{T}\boldsymbol{\mu}_k$ is bounded and a subsequence approaches a limit $\mathbf{p}^*$. Again, for convenience and without loss of generality, we assume that the entire sequence $\mathbf{p}_k$ approaches $\mathbf{p}^*$. Since $F \subset \mathcal{F}$, $\mathbf{A}_{\mathcal{F}}$ has rank m and

$$\boldsymbol{\mu}_k = \left(\mathbf{A}_{\mathcal{F}}\mathbf{A}_{\mathcal{F}}^{\mathsf{T}}\right)^{-1}\mathbf{A}_{\mathcal{F}}\mathbf{p}_k.$$

Since $\mathbf{p}_k$ converges to $\mathbf{p}^*$, it follows that $\boldsymbol{\mu}_k$ converges to

$$\boldsymbol{\mu}^* = \left(\mathbf{A}_{\mathcal{F}}\mathbf{A}_{\mathcal{F}}^{\mathsf{T}}\right)^{-1}\mathbf{A}_{\mathcal{F}}\mathbf{p}^*.$$

By (3.1),

$$\mathbf{g}^* - \mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}^* \geq \mathbf{0}. \tag{3.3}$$

Since $x_i^* = 0$ when $i \notin \mathcal{F}$, we conclude that $i \in \mathcal{F}$ when $x_i^* > 0$. Moreover, when $x_i^* > 0$, we must have $g_i^* - \mathbf{a}_i^{\mathsf{T}}\boldsymbol{\mu}^* = 0$ since $\mathbf{d}_k$ tends to $\mathbf{0}$. Hence, the complementary slackness condition $(\mathbf{g}^* - \mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}^*)^{\mathsf{T}}\mathbf{x}^* = 0$ is satisfied. Since $\mathbf{x}^* \geq \mathbf{0}$ and (3.3) holds, the KKT conditions are satisfied by $(\mathbf{x}^*, \boldsymbol{\mu}^*)$. □

In this section, we have not introduced any convexity assumptions for f. In [7] we establish a global convergence result for ASL when f is strongly convex over the feasible set, and the global minimizer is unique. The same global convergence property holds for ASP.

4 Convergence for quadratic programs

We now establish a sublinear convergence rate for ASP when f is quadratic. The proof utilizes ideas found in Theorem 2 of [14] for the analysis of a different affine scaling algorithm. The following result is used in the analysis:

Lemma 4.1 *Suppose that* $x, t, w, \epsilon \in \mathbb{R}$ *with* $x > 0$, $\epsilon > 0$, $\lambda \in [\lambda_{\min}, \lambda_{\max}] \subset (0, \infty)$, *and*

$$w := t\sqrt{\frac{x}{\lambda x + t^+}}.$$

If

$$\frac{x}{\lambda x + t^+} \leq |w|^{2\epsilon} \quad \text{and} \quad \lambda_{\max}|w|^{2\epsilon} \leq 1/2, \tag{4.1}$$

then

$$t > 0, \quad w > 0, \quad xt \leq 2w^2, \quad \text{and} \quad x \leq 2w^{1+\epsilon}. \tag{4.2}$$

Proof If $t \leq 0$, then the first inequality in (4.1) implies that $\lambda_{\max}|w|^{2\epsilon} \geq \lambda|w|^{2\epsilon} \geq 1$, which contradicts the second inequality in (4.1). Hence, $t > 0$ which implies that $w > 0$. Also, by (4.1), we have

$$x\left(1 - \lambda w^{2\epsilon}\right) \leq t w^{2\epsilon}.$$

Since $\lambda w^{2\epsilon} \le \lambda_{\max} w^{2\epsilon} \le 1/2$, it follows that

$$x \le 2t w^{2\epsilon}. \tag{4.3}$$

The definition of w can be rearranged to obtain

$$\sqrt{t} = w\sqrt{\frac{1+\lambda(x/t)}{x}}. \tag{4.4}$$

If $\lambda w^{2\epsilon} \le \lambda_{\max} w^{2\epsilon} \le 1/2$, then (4.3) implies that $\lambda x/t \le 1$, and (4.4) gives

$$\sqrt{t} \le w\sqrt{\frac{2}{x}}.$$

Squaring this inequality yields the second relation in (4.2), and combining with (4.3) gives

$$x \le 2t w^{2\epsilon} \le 4w^{2+2\epsilon}/x.$$

Rearranging this gives the third inequality in (4.2). □

Theorem 4.2 *Suppose f is quadratic*:

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x} + \mathbf{q}^{\mathsf{T}}\mathbf{x}$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is symmetric and $\mathbf{q} \in \mathbb{R}^n$, and that the infimum of f over the feasible set Ω is finite. If $M = 0$ in ASP (or equivalently, $f_k^R = f(\mathbf{x}_k)$) and $\lambda_k \in [\lambda_{\min}, \lambda_{\max}] \subset (0, \infty)$, then either the algorithm terminates in a finite number of iterations at a stationary point, or the convergence has the following property:

(a) *$f(\mathbf{x}_k)$ approaches a limit f^* and for each $\eta \in (0,\infty)$, there exists a constant c such that*

$$0 \le f\left(\mathbf{x}^k\right) - f^* \le ck^{-\eta}. \tag{4.5}$$

(b) *The iterates $\mathbf{x}_k$ approach a limit $\mathbf{x}^*$ and for each $\eta \in (0,\infty)$, there exists a constant c such that*

$$\left\|\mathbf{x}_k - \mathbf{x}^*\right\| \le ck^{-\eta}.$$

In case (b), the KKT conditions hold at $\mathbf{x}^*$ if the rank condition of Theorem 3.5 is satisfied.

Proof We assume that ASP does not stop in finite number of iterations. By [7, Proposition 3.4], the ASP stepsize s_k is bounded from below, uniformly in k, by a positive scalar. Hence, by the line search criterion in Step 3 of ASP and by Proposition 3.2, we have

$$\begin{aligned} f(\mathbf{x}_{k+1}) &\le f(\mathbf{x}_k) + \delta s_k \mathbf{g}_k^{\mathsf{T}}\mathbf{d}_k = f(\mathbf{x}_k) - \delta s_k \left\|\mathbf{\Sigma}_k^{-1/2}\mathbf{d}_k\right\|^2 \\ &\le f(\mathbf{x}_k) - c\|\mathbf{w}_k\|^2, \end{aligned} \tag{4.6}$$

where $\mathbf{w}_k = \boldsymbol{\Sigma}_k^{-1/2}\mathbf{d}_k$; throughout the proof (and the paper), c is a generic positive constant (independent of k). Since the infimum of f over the feasible set Ω is finite, (1.3) has a solution, the monotone decreasing sequence $f(\mathbf{x}_k)$ approaches a limit f^* and $\mathbf{w}_k$ approaches $\mathbf{0}$.

Let $\boldsymbol{\sigma}_k$ denote the diagonal of $\boldsymbol{\Sigma}_k$, define

$$\mathbf{t}_k = \mathbf{g}_k - \mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}_k = \mathbf{Q}\mathbf{x}_k + \mathbf{q} - \mathbf{A}^{\mathsf{T}}\boldsymbol{\mu}_k, \tag{4.7}$$

and choose any $\epsilon \in (0,1)$. The parameter $\epsilon \in (0,1)$ is related to the parameter $\eta \in (0,\infty)$ in (4.5) by $\eta = \epsilon^{-1} - 1$. By the definition of the search direction $\mathbf{d}_k$, we have $d_{ki} = -\sigma_{ki} t_{ki}$ and by the definition of $\mathbf{w}_k$, we have $w_{ki} = d_{ki}/\sqrt{\sigma}_{ki}$. We combine these relations to obtain

$$\sigma_{ki} = w_{ki}^2/t_{ki}^2. \tag{4.8}$$

For any $\mathcal{J} \subset \{1,\ldots,n\}$, we define the set

$$\mathcal{K}_{\mathcal{J}} = \left\{k \ge 1 : \sigma_{kj} \le |w_{kj}|^{2\epsilon} \quad \text{for all } j \in \mathcal{J} \text{ and } \sigma_{kj} > |w_{kj}|^{2\epsilon} \text{ for all } j \in \mathcal{J}^c\right\}.$$

Since there are a finite number of $\mathcal{J} \subset \{1,\ldots,n\}$, there exists a subset $\mathcal{J}$ (possibly empty) for which $\mathcal{K}_{\mathcal{J}}$ has an infinite number of elements.

Now consider any $\mathcal{J}$ such that $\mathcal{K}_{\mathcal{J}}$ has an infinite number of elements. If $j \in \mathcal{J}^c$, then by (4.8), we have $w_{kj}^2/t_{kj}^2 = \sigma_{kj} > |w_{kj}|^{2\epsilon}$ or $|t_{kj}| < |w_{kj}|^{1-\epsilon}$. Consequently,

$$|t_{kj}| < |w_{kj}|^{1-\epsilon} \quad \text{for all } j \in \mathcal{J}^c \text{ and } k \in \mathcal{K}_{\mathcal{J}}. \tag{4.9}$$

For any $j \in \mathcal{J}$ and $k \in \mathcal{K}_{\mathcal{J}}$, we have $\sigma_{kj} \le |w_{kj}|^{2\epsilon}$, or equivalently,

$$\frac{x_{kj}}{\lambda_k x_{kj} + t_{kj}^+} \le |w_{kj}|^{2\epsilon}, \quad \text{where } w_{kj} = t_{kj}\sqrt{\sigma_{kj}} = t_{kj}\sqrt{\frac{x_{kj}}{\lambda_k x_{kj} + t_{kj}^+}}.$$

Since $\mathbf{w}_k$ tends to $\mathbf{0}$, $\lambda_{\max}\|\mathbf{w}_k\|_1^{2\epsilon} \le 1/2$ for k sufficiently large. It follows from Lemma 4.1 that

$$t_{kj} > 0, \quad w_{kj} > 0, \quad x_{kj}t_{kj} \le 2w_{kj}^2, \quad \text{and} \quad x_{kj} \le 2w_{kj}^{1+\epsilon} < w_{kj}^{1-\epsilon}, \tag{4.10}$$

for all $j \in \mathcal{J}$ and $k \in \mathcal{K}_{\mathcal{J}}$. Hence, by (4.9) and (4.10), we have

$$x_{kj} \le w_{kj}^{1-\epsilon} \quad \text{for all } j \in \mathcal{J} \quad \text{and} \quad 0 < t_{kj} \le w_{kj}^{1-\epsilon} \quad \text{for all } j \in \mathcal{J}^c. \tag{4.11}$$

Since $\mathbf{w}_k$ tends to $\mathbf{0}$, it follows form (4.11) that x_{kj} for $j \in \mathcal{J}$ and t_{kj} for $j \in \mathcal{J}^c$ both tend to 0 as $k \in \mathcal{K}_{\mathcal{J}}$ tends to infinity.

Consider the following linear system in $(\mathbf{x}, \boldsymbol{\mu})$:

$$\begin{cases} x_j = x_{kj} & \text{for } j \in \mathcal{J}, \\ (\mathbf{Q}\mathbf{x} + \mathbf{q} - \mathbf{A}^{\mathsf{T}}\boldsymbol{\mu})_j = t_{kj} & \text{for } j \in \mathcal{J}^c, \\ \mathbf{x} \ge \mathbf{0}, & \mathbf{A}\mathbf{x} = \mathbf{b}. \end{cases} \tag{4.12}$$

By the definition of $\mathbf{t}_k$, $(\mathbf{x}, \boldsymbol{\mu}) = (\mathbf{x}_k, \boldsymbol{\mu}_k)$ is a solution of (4.12). We just showed that x_{kj} for $j \in \mathcal{J}$ and t_{kj} for $j \in \mathcal{J}^c$ both tend to 0 as $k \in \mathcal{K}_{\mathcal{J}}$ tends to infinity. Hence, the terms x_{kj} for $j \in \mathcal{J}$ and t_{kj} for $j \in \mathcal{J}^c$ in (4.12) can be considered small perturbations in the system (4.12) which tend to 0. By Robinson's continuity property [11, Proposition 1] for polyhedral multifunctions, we know that the limiting system

$$\begin{cases} x_j = 0 & \text{for } j \in \mathcal{J}, \\ (\mathbf{Q}\mathbf{x} + \mathbf{q} - \mathbf{A}^{\mathsf{T}}\boldsymbol{\mu})_j = 0 & \text{for } j \in \mathcal{J}^c, \\ \mathbf{x} \geq \mathbf{0}, & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{cases} \tag{4.13}$$

is feasible, and there exists c such that for each $k \in \mathcal{K}_{\mathcal{J}}$, we can find a solution $(\bar{\mathbf{x}}_k, \bar{\boldsymbol{\mu}}_k)$ of (4.13) that is close to $(\mathbf{x}_k, \boldsymbol{\mu}_k)$ in the following sense:

$$\left\| (\mathbf{x}_k - \bar{\mathbf{x}}_k, \boldsymbol{\mu}_k - \bar{\boldsymbol{\mu}}_k) \right\| \leq c \|\mathbf{w}_k\|_1^{1-\epsilon}. \tag{4.14}$$

Here $\|\mathbf{w}_k\|_1^{1-\epsilon}$ is a bound for the perturbation terms obtained from (4.11). Since there are a finite number of subsets $\mathcal{J}$ of $\{1, \ldots, n\}$, we can choose c large enough so that (4.14) holds for all $\mathcal{J}$ where $\mathcal{K}_{\mathcal{J}}$ has an infinite number of elements.

In [14] it is shown that all solutions $(\mathbf{x}, \boldsymbol{\mu})$ of (4.13) for a given choice of $\mathcal{J}$ yield precisely the same objective function value $f(\mathbf{x})$. Let $f(\mathcal{J})$ denote the objective function value associated with any solution of (4.13). We expand the quadratic objective function in a Taylor series around $\bar{\mathbf{x}}_k$ and use the identity $\mathbf{A}(\mathbf{x}_k - \bar{\mathbf{x}}_k) = \mathbf{0}$ and the bound (4.14) to obtain

$$\begin{aligned} \left| f(\mathbf{x}_k) - f(\mathcal{J}) \right| &= \left| \frac{1}{2}(\mathbf{x}_k - \bar{\mathbf{x}}_k)\mathbf{Q}(\mathbf{x}_k - \bar{\mathbf{x}}_k) + (\mathbf{Q}\bar{\mathbf{x}}_k + \mathbf{q})^{\mathsf{T}}(\mathbf{x}_k - \bar{\mathbf{x}}_k) \right| \\ &\leq c\|\mathbf{w}_k\|_1^{2(1-\epsilon)} + \left| (\mathbf{Q}\bar{\mathbf{x}}_k + \mathbf{q} - \mathbf{A}^{\mathsf{T}}\bar{\boldsymbol{\mu}}_k)^{\mathsf{T}}(\mathbf{x}_k - \bar{\mathbf{x}}_k) \right| \end{aligned} \tag{4.15}$$

By (4.13), $(\mathbf{Q}\bar{\mathbf{x}}_k + \mathbf{q} - \mathbf{A}^{\mathsf{T}}\bar{\boldsymbol{\mu}}_k)_j = 0$ for $j \in \mathcal{J}^c$ and $\bar{x}_{kj} = 0$ for $j \in \mathcal{J}$; it follows that

$$\begin{aligned} \left| \left(\mathbf{Q}\bar{\mathbf{x}}_k + \mathbf{q} - \mathbf{A}^{\mathsf{T}}\bar{\boldsymbol{\mu}}_k\right)^{\mathsf{T}}(\mathbf{x}_k - \bar{\mathbf{x}}_k) \right| &= \left| \sum_{j \in \mathcal{J}} \left(\mathbf{Q}\bar{\mathbf{x}}_k + \mathbf{q} - \mathbf{A}^{\mathsf{T}}\bar{\boldsymbol{\mu}}_k\right)_j (x_{kj} - \bar{x}_{kj}) \right| \\ &= \left| \sum_{j \in \mathcal{J}} \left(\mathbf{Q}(\bar{\mathbf{x}}_k - \mathbf{x}_k) - \mathbf{A}^{\mathsf{T}}(\bar{\boldsymbol{\mu}}_k - \boldsymbol{\mu}_k)\right)_j x_{kj} + x_{kj} t_{kj} \right| \\ &\leq c\|\mathbf{w}_k\|_1^{2(1-\epsilon)} + 2\|\mathbf{w}_k\|_1^2 \leq c\|\mathbf{w}_k\|_1^{2(1-\epsilon)}. \end{aligned} \tag{4.16}$$

Here the second equality uses the definition of $\mathbf{t}_k$ in (4.7) and the first inequality is from (4.11), (4.14), and (4.10). Since $f(\mathbf{x}_k)$ approaches f^* monotonically and $\mathbf{w}_k$ tends to $\mathbf{0}$, it follows that $f(\mathcal{J}) = f^*$ whenever $\mathcal{K}_{\mathcal{J}}$ has an infinite number of elements. It follows from (4.15), (4.16), and (4.6) that

$$f(\mathbf{x}_k) - f^* \leq c\|\mathbf{w}_k\|_1^{2(1-\epsilon)} \leq \beta\left(f(\mathbf{x}_k) - f(\mathbf{x}_{k+1})\right)^{(1-\epsilon)}, \tag{4.17}$$

where β is a specific generic constant that will be used below.

The inequality (4.17) is rearranged to give

$$\Delta_{k+1} \le \Delta_k - (\Delta_k/\beta)^\tau,$$

where $\Delta_k = f(\mathbf{x}_k) - f^*$ and $\tau = 1/(1-\epsilon) > 1$ since $\epsilon \in (0,1)$. Since $\Delta_k > 0$, this is equivalent to

$$1 \le \frac{\beta^\tau}{\Delta_k^\tau}(\Delta_k - \Delta_{k+1}). \tag{4.18}$$

Since the Δ_k decrease monotonically, it follows that

$$\frac{\Delta_k - \Delta_{k+1}}{\Delta_k^\tau} \le \int_{\Delta_{k+1}}^{\Delta_k} s^{-\tau} ds = \frac{1}{1-\tau}\big(\Delta_k^{1-\tau} - \Delta_{k+1}^{1-\tau}\big). \tag{4.19}$$

Combine (4.18) and (4.19) to obtain

$$1 \le \frac{\beta^\tau}{1-\tau}\big(\Delta_k^{1-\tau} - \Delta_{k+1}^{1-\tau}\big). \tag{4.20}$$

Let k_1 be chosen large enough that for any $k \ge k_1$, the associated set $\mathcal{K}_{\mathcal{J}}$ containing k has an infinite number of elements. We sum (4.20) from k_1 up to $k-1$ to obtain

$$k - k_1 \le \frac{\beta^\tau}{1-\tau}\big(\Delta_{k_1}^{1-\tau} - \Delta_k^{1-\tau}\big),$$

which is rearranged into

$$\Delta_k^{\tau-1} \le \frac{\gamma_1}{k+\gamma_2}, \quad \gamma_1 = \frac{\beta^\tau}{\tau-1}, \ \gamma_2 = \Delta_{k_1}^{1-\tau}\beta^\tau/(\tau-1) - k_1. \tag{4.21}$$

Since $\tau > 1$, $\Delta_{k_1}^{1-\tau}$ tends to infinity as k_1 grows. Choose k_1 large enough that $\gamma_2 > 0$. In this case, $1/(k+\gamma_2) \le 1/k$, and (4.21) implies that

$$\Delta_k \le \frac{c}{k^{1/(\tau-1)}} = ck^{-\epsilon^{-1}+1} = ck^{-\eta} \quad \text{where } \eta = \epsilon^{-1} - 1,$$

which establishes (a).

Now consider (b) and suppose that $\epsilon \in (0, .5)$. Since $\boldsymbol{\Sigma}_k$ is diagonal, we have $\|\boldsymbol{\Sigma}_k\| \le 1/\lambda_{\min}$. It follows that

$$\|\mathbf{x}_{k+1} - \mathbf{x}_k\| = s_k\|\mathbf{d}_k\| = s_k\big\|\boldsymbol{\Sigma}_k^{1/2}\mathbf{w}_k\big\| \le s_k\|\mathbf{w}_k\|/\sqrt{\lambda_{\min}}. \tag{4.22}$$

By Proposition 3.2,

$$\|\mathbf{w}_k\| = \frac{\|\mathbf{w}_k\|^2}{\|\mathbf{w}_k\|} \le \frac{\sqrt{n}\|\mathbf{w}_k\|^2}{\|\mathbf{w}_k\|_1} = \frac{\sqrt{n}|\mathbf{g}_k^{\mathsf{T}}\mathbf{d}_k|}{\|\mathbf{w}_k\|_1}. \tag{4.23}$$

By (4.6),

$$s_k\big|\mathbf{g}_k^{\mathsf{T}}\mathbf{d}_k\big| \le \big(f(\mathbf{x}_k) - f(\mathbf{x}_{k+1})\big)/\delta = (\Delta_k - \Delta_{k+1})/\delta, \tag{4.24}$$

and by (4.17),

$$\|\mathbf{w}_k\|_1 \geq (\Delta_k/c)^{\tau/2} \quad \text{or} \quad 1/\|\mathbf{w}_k\|_1 \leq (\Delta_k/c)^{-\tau/2}. \tag{4.25}$$

Combining relations (4.22)–(4.25) gives

$$\begin{aligned}\|\mathbf{x}_{k+1} - \mathbf{x}_k\| &\leq c(\Delta_k - \Delta_{k+1})\Delta_k^{-\tau/2} \leq c\int_{\Delta_{k+1}}^{\Delta_k} s^{-\tau/2} ds \\ &= \frac{c}{1-\tau/2}\left(\Delta_k^{1-\tau/2} - \Delta_{k+1}^{1-\tau/2}\right).\end{aligned}$$

Recall that $\epsilon \in (0, .5)$ which implies that $1 < \tau = 1/(1-\epsilon) < 2$ and $0 < 1-\tau/2 < .5$. We sum from $k = k_1$ to k_2 to obtain

$$\sum_{k=k_1}^{k_2} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| \leq \frac{c}{1-\tau/2}\left(\Delta_{k_1}^{1-\tau/2} - \Delta_{k_2+1}^{1-\tau/2}\right) \leq \frac{c}{1-\tau/2}\Delta_{k_1}^{1-\tau/2}.$$

Since Δ_{k_1} tends to 0 as k_1 tends to ∞ and $1 - \tau/2 > 0$, $\{\mathbf{x}_k\}$ is a Cauchy's sequence with limit denoted $\mathbf{x}^*$. By the triangle inequality,

$$\|\mathbf{x}_{k_2+1} - \mathbf{x}_{k_1}\| \leq \sum_{k=k_1}^{k_2} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| \leq \frac{c}{1-\tau/2}\Delta_{k_1}^{1-\tau/2}.$$

Let $k_2 \to \infty$ and use (4.5) to establish (b). □

5 Local linear convergence

In our earlier work [7], we proved an R-linear convergence result for ASL with a line search. The same convergence result applies to ASP with suitable adjustments in the proof such as replacing scalars like a_i^2 by matrices $\mathbf{a}_i\mathbf{a}_i^{\mathsf{T}}$. In this section, we study the convergence rate of ASP with a unit step (without a line search):

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k, \quad d_{ki} = -\left(\frac{1}{\lambda_k + (g_{ki} - \mathbf{a}_i^{\mathsf{T}}\boldsymbol{\mu}_k)^+/x_{ki}}\right)\left(g_{ki} - \mathbf{a}_i^{\mathsf{T}}\boldsymbol{\mu}_k\right), \tag{5.1}$$

where $\boldsymbol{\mu}_k$ is chosen so that $\mathbf{A}\mathbf{d}_k = \mathbf{0}$ and where λ_k is given by the BB formula [1]:

$$\lambda_k = \lambda_k^{BB} := \arg\min_{\lambda \geq \lambda_0} \|\lambda\mathbf{s}_{k-1} - \mathbf{y}_{k-1}\|_2 = \max\left\{\lambda_0, \frac{\mathbf{s}_{k-1}^{\mathsf{T}}\mathbf{y}_{k-1}}{\mathbf{s}_{k-1}^{\mathsf{T}}\mathbf{s}_{k-1}}\right\}, \tag{5.2}$$

for $k \geq 2$ where $\lambda_0 > 0$, $\mathbf{s}_{k-1} = \mathbf{x}_k - \mathbf{x}_{k-1}$, and $\mathbf{y}_{k-1} = \mathbf{g}_k - \mathbf{g}_{k-1}$. In this case, the iterates are locally R-linearly convergent in a neighborhood of a local minimizer $\mathbf{x}^*$ where both assumption (III) of Theorem 2.2 and the second-order sufficient optimality conditions hold: There exists $\sigma > 0$ such that

$$\mathbf{d}^{\mathsf{T}}\nabla^2 f\left(\mathbf{x}^*\right)\mathbf{d} > \sigma\|\mathbf{d}\|^2, \tag{5.3}$$

for all $\mathbf{d}$ satisfying $\mathbf{A}\mathbf{d} = \mathbf{0}$ and $d_i = 0$ when $i \in \mathcal{A} = \{i : x_i^* = 0\}$.

The proof of R-linear convergence has two parts: In the first part, we show that the components of x_{ki} for $i \in \mathcal{A}$ approach 0 quadratically fast. In the second part, we compare the components of $\mathbf{x}_k$ associated with $F = \{i : x_i^* > 0\}$ to the iterates of the BB method applied to an unconstrained optimization problem. In [4] we give a linear convergence result for the BB method applied to an unconstrained optimization problem. By showing that the ASP iterates are sufficiently close to the unconstrained BB iterates, we are able to deduce R-linear convergence for ASP. This also implies that the ASP method with a unit step behaves locally like an unconstrained BB method. This second part of the analysis is somewhat tedious, and closely parallels the analysis given in [4] and [9]; hence, the second part of the analysis will be summarized. Here we focus on the first part of the analysis, the quadratic convergence of the components of x_{ki} for $i \in \mathcal{A}$. Finally, we remark that although we focus on the BB formula (5.2), the results and analysis also apply to the cyclic BB formula given in [4].

Lemma 5.1 *If condition* (IIIa) *of Theorem* 2.2 *is satisfied, then the matrix* $\bar{\mathbf{A}}$ *obtained by augmenting* $\mathbf{A}$ *with additional rows* $\mathbf{e}_i^{\mathsf{T}}$, $i \in \mathcal{A}$, *has linearly independent rows.*

Proof By assumption (IIIa), $\mathbf{A}_F$ has linearly independent rows. Since the vectors $\mathbf{e}_i^{\mathsf{T}}$, $i \in \mathcal{A}$, are linearly independent and their nonzeros are in the columns of $\mathbf{A}$ associated with the complement of F, it follows that the rows of $\bar{\mathbf{A}}$ are linearly independent. □

Proposition 5.2 *If* (1.3) *has a solution* $\mathbf{x}^*$ *and conditions* (IIIa) *and* (IIIb) *of Theorem* 2.2 *are satisfied, then there exists a neighborhood* $\mathcal{N}$ *of* $\mathbf{x}^*$ *and a constant c with the property that for each* $\mathbf{x}_k \in \mathcal{N} \cap \Omega$, *the equation* $\mathbf{r}_k(\boldsymbol{\mu}) = \mathbf{0}$ *has a unique solution* $\boldsymbol{\mu}_k$ *and*

$$\|\boldsymbol{\mu}_k - \boldsymbol{\mu}^*\| \le c\|\mathbf{x}_k - \mathbf{x}^*\|. \tag{5.4}$$

Proof This follows from the implicit function theorem as applied in the proof of Theorem 2.2, part III. In Theorem 2.2 we only claimed the existence of the solution $\boldsymbol{\mu}_k$. Since f is twice continuously differentiable near $\mathbf{x}^*$, it follows that $\boldsymbol{\mu}_k$ is a continuously differentiable function of $\mathbf{x}_k$ and λ_k. This implies the Lipschitz property (5.4). □

We now analyze the components of $\mathbf{x}_k$ corresponding to the active set $\mathcal{A}$. Given any $\mathbf{x} \in \mathbb{R}^n$, let $\hat{\mathbf{x}}$ denote the vector obtained by replacing with 0 the components associated with active indicates. That is,

$$\hat{x}_i = \begin{cases} x_i & \text{if } i \in \mathcal{A}^c, \\ 0 & \text{if } i \in \mathcal{A}. \end{cases}$$

Thus $\mathbf{x} - \hat{\mathbf{x}}$ is the vector with components

$$x_i - \hat{x}_i = \begin{cases} 0 & \text{if } i \in \mathcal{A}^c, \\ x_i & \text{if } i \in \mathcal{A}. \end{cases}$$

Proposition 5.3 *If* (1.3) *has a solution* $\mathbf{x}^*$ *and conditions* (IIIa) *and* (IIIb) *of Theorem* 2.2 *are satisfied, then there exists a neighborhood* $\mathcal{N}$ *of* $\mathbf{x}^*$ *and a constant* c *with the property that for all* $\mathbf{x}_k \in \mathcal{N} \cap \Omega$, *we have*

$$\|\mathbf{d}_k\| \le c\|\mathbf{x}_k - \mathbf{x}^*\|, \tag{5.5}$$

$$\|\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}\| \le c\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^2, \tag{5.6}$$

$$\|\mathbf{P}(\mathbf{x}_{k+1} - \mathbf{x}^*)\| \le c\|\mathbf{P}(\mathbf{x}_k - \mathbf{x}^*)\|^2, \tag{5.7}$$

where $\mathbf{P} = \bar{\mathbf{A}}^\mathsf{T}(\bar{\mathbf{A}}\bar{\mathbf{A}}^\mathsf{T})^{-1}\bar{\mathbf{A}}$.

Proof Define $\mathbf{h}_k = \mathbf{g}(\mathbf{x}_k) - \mathbf{A}^\mathsf{T}\boldsymbol{\mu}_k$ and $\mathbf{h}^* = \mathbf{g}(\mathbf{x}^*) - \mathbf{A}^\mathsf{T}\boldsymbol{\mu}^*$. By (IIIa), $h_i^* > 0$ for $i \in \mathcal{A}$. Hence, by (5.4) it follows that for $\mathbf{x}_k$ near $\mathbf{x}^*$, we have $h_{ki} > 0$ for $i \in \mathcal{A}$. By (5.1), it follows that $0 \le |d_{ki}| \le x_{ki}$ for $i \in \mathcal{A}$ and

$$\sum_{i\in\mathcal{A}} d_{ki}^2 \le \sum_{i\in\mathcal{A}} x_{ki}^2 \le \|\mathbf{x}_k - \mathbf{x}^*\|^2. \tag{5.8}$$

By complementary slackness, $h_i^* = 0$ for $i \in F$. Therefore, for $i \in F$,

$$\begin{aligned} |d_{ki}| &\le |h_{ki}|/\lambda_{\min} = |h_{ki} - h_i^*|/\lambda_{\min} \\ &\le \left(|g_i(\mathbf{x}_k) - g_i(\mathbf{x}^*)| + |\mathbf{a}_i^\mathsf{T}(\boldsymbol{\mu}_k - \boldsymbol{\mu}^*)|\right)/\lambda_{\min} \\ &\le c\|\mathbf{x}_k - \mathbf{x}^*\|, \end{aligned} \tag{5.9}$$

since f is twice continuously differentiable and (5.4) holds. Combine (5.8) and (5.9) to obtain $\|\mathbf{d}_k\|^2 \le c\|\mathbf{x}_k - \mathbf{x}^*\|^2$ which establishes (5.5).

Choose $\epsilon > 0$ and $\mathcal{N}$ small enough that $h_{ki} > \epsilon$ for all $i \in \mathcal{A}$. Again, by (5.1), we have

$$\begin{aligned} \|\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}\|^2 = \sum_{i\in\mathcal{A}} x_{(k+1)i}^2 &= \sum_{i\in\mathcal{A}} \left(x_{ki} - \frac{x_{ki}}{1 + \lambda_k x_{ki}/h_{ki}}\right)^2 \\ &\le \sum_{i\in\mathcal{A}} \left(\frac{\lambda_k x_{ki}^2}{h_{ki}}\right)^2 \le (\lambda_{\max}/\epsilon)^2 \sum_{i\in\mathcal{A}} x_{ki}^4 \\ &\le (\lambda_{\max}/\epsilon)^2 \left(\sum_{i\in\mathcal{A}} x_{ki}^2\right)^2 = (\lambda_{\max}/\epsilon)^2 \|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^4. \end{aligned}$$

This establishes (5.6).

Now, by Lemma 5.1, we know that $\bar{\mathbf{A}}\bar{\mathbf{A}}^\mathsf{T}$ is invertible. Hence, by the definition of $\mathbf{P}$, we have

$$\begin{aligned} \|\mathbf{P}(\mathbf{x}_k - \mathbf{x}^*)\|^2 &= \|\bar{\mathbf{A}}^\mathsf{T}(\bar{\mathbf{A}}\bar{\mathbf{A}}^\mathsf{T})^{-1}\bar{\mathbf{A}}(\mathbf{x}_k - \mathbf{x}^*)\|^2 \\ &= [\bar{\mathbf{A}}(\mathbf{x}_k - \mathbf{x}^*)]^\mathsf{T}(\bar{\mathbf{A}}\bar{\mathbf{A}}^\mathsf{T})^{-1}[\bar{\mathbf{A}}(\mathbf{x}_k - \mathbf{x}^*)] \end{aligned} \tag{5.10}$$

$$\geq \frac{\|\bar{\mathbf{A}}(\mathbf{x}_k - \mathbf{x}^*)\|^2}{\|\bar{\mathbf{A}}\bar{\mathbf{A}}^\mathsf{T}\|},$$

which implies that

$$\|\mathbf{x}_k - \hat{\mathbf{x}}_k\| = \|\bar{\mathbf{A}}(\mathbf{x}_k - \mathbf{x}^*)\| \leq \sqrt{\|\bar{\mathbf{A}}\bar{\mathbf{A}}^\mathsf{T}\|}\,\|\mathbf{P}(\mathbf{x}_k - \mathbf{x}^*)\|. \tag{5.11}$$

The first equality here is due to the fact that the top submatrix in $\bar{\mathbf{A}}$ is $\mathbf{A}$ where $\mathbf{A}(\mathbf{x}_k - \mathbf{x}^*) = \mathbf{0}$, while the bottom submatrix in $\bar{\mathbf{A}}$ contains $\mathbf{e}_i$, $i \in \mathcal{A}$, and $x_i^* = \hat{x}_{ki} = 0$ for $i \in \mathcal{A}$. By (5.10), we have

$$\begin{aligned}
\|\mathbf{P}(\mathbf{x}_{k+1} - \mathbf{x}^*)\| &\leq \sqrt{\|(\bar{\mathbf{A}}\bar{\mathbf{A}}^\mathsf{T})^{-1}\|}\;\|\bar{\mathbf{A}}(\mathbf{x}_{k+1} - \mathbf{x}^*)\| \\
&= \sqrt{\|(\bar{\mathbf{A}}\bar{\mathbf{A}}^\mathsf{T})^{-1}\|}\;\|\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}\| \\
&\leq c\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^2.
\end{aligned}$$

The last inequality here is due to (5.6). Combining this with (5.11) completes the proof. □

In (5.7), Proposition 5.3 basically gives the quadratic convergence of the ASP iterates in the range space of $\bar{\mathbf{A}}$. The second phase of the analysis involves showing linear convergence in the null space of $\bar{\mathbf{A}}$ when the second-order sufficient optimality condition holds. This can be established using a comparison technique along the lines developed in [4] for unconstrained optimization and in [9] for bound constrained optimization. In particular, we compare the ASP iterate to an unconstrained BB iterate defined as follows: Let $\mathbf{N}$ denote a matrix whose columns are an orthonormal basis for the null space of $\bar{\mathbf{A}}$, and define $\mathbf{y}_k = \mathbf{N}^\mathsf{T}\mathbf{x}_k$, $\mathbf{y}^* = \mathbf{N}^\mathsf{T}\mathbf{x}^*$, and

$$\begin{aligned}
\mathbf{z}_{k,0} &= \mathbf{y}_k - \mathbf{y}^*, \\
\mathbf{z}_{k,j+1} &= \mathbf{z}_{k,j} - \lambda_{k,j}^{-1}\mathbf{N}^\mathsf{T}\mathbf{g}(\mathbf{N}\mathbf{z}_{k,j} + \mathbf{x}^*), \quad j \geq 0,
\end{aligned} \tag{5.12}$$

where

$$\lambda_{k,j} = \frac{\mathbf{v}_{k,j}^\mathsf{T}\mathbf{w}_{k,j}}{\mathbf{v}_{k,j}^\mathsf{T}\mathbf{v}_{k,j}}, \quad \mathbf{v}_{k,j} = \mathbf{N}(\mathbf{z}_{k,j} - \mathbf{z}_{k,j-1}), \quad \text{and}$$

$$\mathbf{w}_{k,j} = \mathbf{g}(\mathbf{N}\mathbf{z}_{k,j} + \mathbf{x}^*) - \mathbf{g}(\mathbf{N}\mathbf{z}_{k,j-1} + \mathbf{x}^*).$$

We compare $\mathbf{x}_{k+j}$ to $\mathbf{N}\mathbf{z}_{k,j} + \mathbf{x}^*$.

Note that $\mathbf{z}_{k,j}$ given by (5.12) represents a BB iterate associated with the unconstrained optimization problem

$$\min_{\mathbf{z}\in\mathbb{R}^l} f(\mathbf{N}\mathbf{z} + \mathbf{x}^*), \tag{5.13}$$

where l is the number of columns of $\mathbf{N}$. The Hessian matrix at $\mathbf{z} = \mathbf{0}$ is $\mathbf{Z}^\mathsf{T}\nabla^2 f(\mathbf{x}^*)\mathbf{Z}$ which is positive definite by the second-order sufficient optimality condition (5.3).

Hence, $\mathbf{z} = \mathbf{0}$ is a local minimizer for (5.13). By [4, Theorem 2.3], the $\mathbf{z}_{k,j}$ converge to $\mathbf{0}$ linearly, which yields the following lemma. For more details, see the proof of Proposition 5.3 in [9].

Lemma 5.4 *If f is twice continuously differentiable in a neighborhood of $\mathbf{x}^*$ and the second-order sufficient optimality condition (5.3) is satisfied, then there exist $\delta > 0$ and an integer $J > 0$ such that for all starting points $\mathbf{y}_k = \mathbf{N}^\mathsf{T}\mathbf{x}_k$ with $\mathbf{x}_k \in \mathcal{B}_\delta(\mathbf{x}^*) \cap \Omega$, the BB iterates generated by (5.12) satisfy*

$$\mathbf{z}_{k,j} \in \mathcal{B}_\rho(\mathbf{0}) \quad \textit{and} \quad \mathbf{N}\mathbf{z}_{k,j} + \mathbf{x}^* \geq \mathbf{0} \quad \textit{for } j \geq 0 \quad \textit{and} \tag{5.14}$$

$$\|\mathbf{z}_{k,j}\| \leq \frac{1}{2}\|\mathbf{z}_{k,0}\| \quad \textit{for all } j \geq J. \tag{5.15}$$

The following lemma compares the null space iterates to the ASP iterates:

Lemma 5.5 *Suppose that the assumptions of Lemma (5.4) are satisfied and $\lambda_0 > \sigma/2$ where σ is given in the second-order sufficient optimality condition. Then for any $\Lambda \geq \lambda_0$ and for any positive integer J, there exist positive constants δ and c with the following property: For any $\mathbf{x}_k \in \mathcal{B}_\delta(\mathbf{x}^*)$ satisfying*

$$\left\|\mathbf{P}\big(\mathbf{x}_k - \mathbf{x}^*\big)\right\| \leq \left\|\mathbf{N}\mathbf{N}^\mathsf{T}\big(\mathbf{x}_k - \mathbf{x}^*\big)\right\|^{3/2} \quad \textit{and} \quad \lambda_k \leq \Lambda, \tag{5.16}$$

and for any $\ell \in [0, J]$, if

$$\|\mathbf{z}_{k,j}\| \geq \frac{1}{2}\|\mathbf{z}_{k,0}\| \quad \textit{for all } j \in \big[0, \max\{0, \ell - 1\}\big], \tag{5.17}$$

then we have

$$\left\|\mathbf{x}_{k+j} - \big(\mathbf{N}\mathbf{z}_{k,j} + \mathbf{x}^*\big)\right\| \leq c\left\|\mathbf{x}_k - \mathbf{x}^*\right\|^{3/2} \tag{5.18}$$

for all $j \in [0, \ell]$.

This lemma is analogous to Lemma 2.2 in [4], and the proof is essentially a line-by-line transcription of the proof of Lemma 6.1 in [9]. Since [9] has only bound constraints, not the linear constraint $\mathbf{A}\mathbf{x} = \mathbf{b}$, the projection operations appearing in [9] need to be replaced by the projection $\mathbf{P}$ on the range of $\bar{\mathbf{A}}$ and the projection $\mathbf{P}_N = \mathbf{N}\mathbf{N}^\mathsf{T}$ on the null space of $\bar{\mathbf{A}}$. More precisely, a vector with components $x_{ki} - x_i^*$ for $i \in \mathcal{A}$ appearing in [9] should be replaced by $\mathbf{P}(\mathbf{x}_k - \mathbf{x}^*)$ in this paper. And a vector such as $\hat{\mathbf{x}}_k - \mathbf{x}^*$ appearing in [9] is replaced by $\mathbf{P}_N(\mathbf{x}_k - \mathbf{x}^*)$ in this paper. For example, the following inequality (6.3) in [9]:

$$\max\big\{|x_{ki}| : i \in \mathcal{A}\big\} \leq \left\|\hat{\mathbf{x}}_k - \mathbf{x}^*\right\|^{3/2} \quad \text{and} \quad \lambda_k \leq \Lambda,$$

is transcribed into (5.16) in this paper. As another illustration, the analogue of the comparison result appearing in equation (6.9) of [9] is

$$\left\|\mathbf{x}_k - \big(\mathbf{N}\mathbf{z}_{k,0} + \mathbf{x}^*\big)\right\| = \left\|\mathbf{x}_k - \mathbf{x}^* - \mathbf{N}\big(\mathbf{y}_k - \mathbf{y}^*\big)\right\| = \left\|\mathbf{x}_k - \mathbf{x}^* - \mathbf{N}\mathbf{N}^\mathsf{T}\big(\mathbf{x}_k - \mathbf{x}^*\big)\right\|$$

$$= \left\| \left(\mathbf{I} - \mathbf{N}\mathbf{N}^{\mathsf{T}}\right)\left(\mathbf{x}_k - \mathbf{x}^*\right) \right\| = \left\| \mathbf{P}\left(\mathbf{x}_k - \mathbf{x}^*\right) \right\|$$
$$\leq \left\| \mathbf{P}_N\left(\mathbf{x}_k - \mathbf{x}^*\right) \right\|^{3/2} \leq \left\| \mathbf{x}_k - \mathbf{x}^* \right\|^{3/2},$$

where the next-to-last inequality is (5.16).

Finally, the quadratic convergence of the active constraint components established in Proposition 5.3 together with the comparison result Lemma 5.5 yield the following R-linear convergence result. This result is a line-by-line transcription of Theorem 7.1 in [9].

Theorem 5.6 *Suppose that* (1.3) *has a solution* $\mathbf{x}^*$, *conditions* (IIIa) *and* (IIIb) *of Theorem* 2.2 *are satisfied*, *and the second-order sufficient optimality condition* (5.3) *holds*. *If* λ_0 *is chosen in accordance with Lemma* 5.5, *then there exist positive scalars* δ *and* η, *and a positive scalar* $\gamma < 1$ *with the property that for all starting points* $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{B}_\delta(\mathbf{x}^*)$, $\mathbf{x}_0 \neq \mathbf{x}_1$, *the ASP iterates generated by* (5.1) *satisfy*

$$\left\| \mathbf{x}_k - \mathbf{x}^* \right\| \leq \eta\gamma^k \left\| \mathbf{x}_1 - \mathbf{x}^* \right\|. \tag{5.19}$$

6 Conclusions

The affine scaling interior point method ASL developed in [7] in the context of bound constraints and a single linear constraint was extended to handle a general system of linear constraints. The new algorithm ASP applies to general polyhedral constraints. There is convergence to a stationary point when a nondegeneracy condition holds. Moreover, if a second-order sufficient optimality condition holds, then the convergence is R-linear. When the objective function is quadratic, we obtained convergence on the order of $k^{-\sigma}$ for any $\sigma \in (0, \infty)$ without either the nondegeneracy or second-order sufficient optimality conditions. In [7] it is seen that the affine scaling interior point algorithm can be quite efficient in large-scale ill-posed problems associated with support vector machines.

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