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J. Math. Anal. Appl. 341 (2008) 1028–1041

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

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A generalized eigenproblem for the Laplacian which arises in lightning ☆

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Received 5 October 2007

Available online 17 November 2007

Submitted by Goong Chen

Abstract

The following generalized eigenproblem is analyzed: Find $u \in H_0^1(\Omega)$, $u \neq 0$, and $\lambda \in \mathbb{R}$ such that

$$\langle \nabla u, \nabla v \rangle_D = \lambda \langle \nabla u, \nabla v \rangle_\Omega$$

for all $v \in H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, D is a subdomain with closure contained in Ω, and $\langle \cdot, \cdot \rangle_\Omega$ is the inner product

$$\langle \nabla u, \nabla v \rangle_\Omega = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

It is proved that any $f \in H_0^1(\Omega)$ can be expanded in terms of orthogonal eigenfunctions for the generalized eigenproblem. During the analysis, we present a new inner product on $H^{1/2}(\partial D)$ with the following properties: (a) the norm associated with the inner product is equivalent to the usual norm on $H^{1/2}(\partial D)$, and (b) the double layer potential operator is self adjoint with respect to the new inner product and compact as a mapping from $H^{1/2}(\partial D)$ into itself. The analysis identifies four classes of eigenfunctions for the generalized eigenproblem:

1. The function Π which is 1 on D and harmonic on $\Omega \setminus D$; the eigenvalue is 0.
2. Functions in $H_0^1(\Omega)$ with support in $\Omega \setminus D$; the eigenvalue is 0.
3. Functions in $H_0^1(\Omega)$ with support in D; the eigenvalue is 1.
4. Excluding Π, the harmonic extension of the eigenfunctions of a double layer potential on ∂D. The eigenvalues are contained in the open interval $(0, 1)$. The only possible accumulation point is $\lambda = 1/2$.

A positive lower bound for the smallest positive eigenvalue is obtained. These results can be used to evaluate the change in the electric potential due to a lightning discharge.

☆ This material is based upon work supported by the National Science Foundation under Grants 0203270, 0619080, 0620286, 0724750, and 0749396.

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doi:10.1016/j.jmaa.2007.11.007

Keywords: Laplacian; Generalized eigenproblem; Double layer potential; Complete eigenbasis; Lightning

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let D be a connected subdomain with closure contained in Ω and with the complement $D^c = \Omega \setminus D$ connected. This paper focuses on the following generalized eigenproblem: Find $u \in H_0^1(\Omega)$, $u \neq 0$, and $\lambda \in \mathbb{R}$ such that

$$\langle \nabla u, \nabla v \rangle_D = \lambda \langle \nabla u, \nabla v \rangle_\Omega \tag{1.1}$$

for all $v \in H_0^1(\Omega)$, where $\langle \cdot, \cdot \rangle_\Omega$ is the $L^2(\Omega)$ inner product

$$\langle \nabla u, \nabla v \rangle_\Omega = \int_\Omega \nabla u \cdot \nabla v \, dx. \tag{1.2}$$

Throughout this paper, we view $H_0^1(\Omega)$ as a Hilbert space for which the inner product between functions u and $v \in H_0^1(\Omega)$ is given by (1.2). Our main result is the following:

Theorem 1.1. *If $\partial\Omega$ is C^2 and ∂D is $C^{2,\alpha}$, for some $\alpha \in (0,1)$* (*the exponent of Hölder continuity for the second derivative*)*, then any $f \in H_0^1(\Omega)$ has an expansion of the form*

$$f = \sum_{i=1}^{\infty} \phi_i,$$

where the ϕ_i are eigenfunctions of (1.1) *which are orthogonal relative to the inner product* (1.2). *Here the convergence is with respect to the norm of $H_0^1(\Omega)$.*

For all the analysis in this paper up to the proof of Theorem 1.1, we require at most C^2 boundaries for Ω and D. The Hölder continuity of the second derivative is required during the proof of Theorem 1.1 when we utilize a result of Kirsch [12] (also see [6, Theorem 3.6] and [13, Theorem 8.20]) concerning the regularity of a double layer potential operator.

As explained in Section 2, this generalized eigenproblem arises in the modeling of a lightning discharge for a thundercloud [10]. In the process of proving Theorem 1.1, we identify an inner product for which the double layer potential operator is self adjoint and compact in $H^{1/2}(\partial D)$. The inner product amounts to harmonically extending functions from ∂D into Ω and forming the $H_0^1(\Omega)$ inner product. The norm associated with this inner product is equivalent to the usual norm on $H^{1/2}(\partial D)$. In [11] it is shown that the double layer potential operator is symmetrizable by introducing an appropriate (incomplete) inner product on $L^2(\partial D)$. Here, we use a new inner product to obtain an eigenexpansion which converges in the norm of $H^{1/2}(\partial D)$.

Our analysis exhibits four classes of eigenfunctions for (1.1):

1. The function Π which is 1 on D and harmonic on $\Omega \setminus D$; the eigenvalue is 0.
2. Functions in $H_0^1(\Omega)$ with support in $\Omega \setminus D$; the eigenvalue is 0.
3. Functions in $H_0^1(\Omega)$ with support in D; the eigenvalue is 1.
4. Excluding Π, the harmonic extensions of the eigenfunctions of a double layer potential on ∂D. The eigenvalues are contained in the open interval $(0, 1)$. The only possible accumulation point is $\lambda = 1/2$.

The eigenfunction Π is particularly important since the electric potential in D right after the lightning discharge is obtained by projecting the pre-discharge potential on Π.

The paper is organized as follows. Section 2 shows how the generalized eigenproblem arises in the modeling of lightning. Section 3 gives the ordinary eigenproblem associated with the generalized eigenproblem (1.1). Section 4 derives eigenfunctions of types 1, 2, and 3, while Section 5 reformulates the eigenproblem for the remaining eigenfunctions in terms of extensions of the eigenfunctions of a double layer potential for ∂D. Section 6 obtains a lower bound for the smallest positive eigenvalue, and it proves Theorem 1.1 by exploiting a complete inner product on $H^{1/2}(\partial\Omega)$ for which the double layer potential operator is self adjoint. Section 7 examines eigenfunctions in one dimension. Final conclusions appear in Section 8.

Notation. Throughout this paper, we use the following notation. The complement of D is $D^c = \Omega \setminus D$. $H^1(\Omega)$ is the usual Sobolev space consisting of functions whose value and derivative are square integrable on Ω, with the usual norm given by

$$\|u\|^2_{H^1(\Omega)} = \langle \nabla u, \nabla u\rangle_\Omega + \langle u, u\rangle_\Omega.$$

$C_0^\infty(\Omega)$ is the collection of infinitely differentiable functions with compact support in Ω and $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. C^k denotes the set of k-times continuously differentiable functions, while $C^{k,\alpha}$ is the subset of C^k whose kth derivative is Hölder continuous with exponent α. $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the outward normal to D.

2. Origin of the generalized eigenproblem

During lightning, a region of space, the lightning channel, becomes highly conductive, leading to a jump discontinuity in the electric potential throughout the atmosphere. The electric potential change due to lightning is evaluated in [10] using the eigendecomposition developed in this paper. Here we briefly review the connection between the generalized eigenproblem and lightning. By Ampere's law, the electric potential ϕ satisfies an evolution equation of the form

$$\frac{\partial \Delta\phi}{\partial t} = -\nabla\cdot(\sigma\nabla\phi) + \nabla\cdot J, \quad (x,t) \in \Omega\times[0,\infty), \tag{2.1}$$

$$\phi(x,t) = 0, \quad (x,t) \in \partial\Omega\times[0,\infty), \tag{2.2}$$

$$\phi(x,0) = \phi_0(x), \quad x \in \Omega, \tag{2.3}$$

where $\sigma \in L^\infty(\Omega)$ is the conductivity divided by the permittivity of the atmosphere, $J \in L^2(\Omega)$ is due to transport by wind of charged ice and water particles in the thundercloud, and $\phi_0 \in H_0^1(\Omega)$. Equation (2.1) is interpreted in a weak sense.

When the electric field in a thundercloud reaches the "breakdown threshold," the lightning channel D turns into a plasma where conductivity is large. The lightning domain D (see Fig. 1) is essentially a network of connected, thin open tubes. Since the conductivity of the ionized region is extremely high, the effect of lightning in the partial differential equation (2.1) is to replace σ by $\sigma + \tau\Psi$ where Ψ is the characteristic function of D and τ is large.

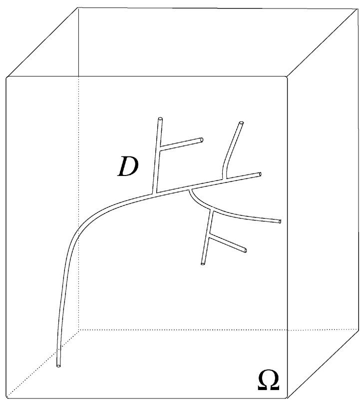

Fig. 1. A sketch of D and Ω for a lightning discharge.

If the lightning occurs at time $t = 0$, then in the moments after the lightning, the electric potential is governed by the equation

$$\frac{\partial \Delta\phi}{\partial t} = -\nabla \cdot (\sigma \nabla\phi) - \tau\nabla \cdot (\Psi \nabla\phi) + \nabla \cdot J \quad \text{in } \Omega \times [0, \infty), \tag{2.4}$$

subject to the boundary conditions (2.2) and (2.3).

If $\phi_\tau(x, t)$ denotes the solution to (2.4) at position $x \in \Omega$ and time t, and if the lightning is infinitely fast and the conductivity of the channel is infinitely large, then the potential right after the lightning is given by

$$\phi^+(x) = \lim_{t\to 0^+} \lim_{\tau\to\infty} \phi_\tau(x, t). \tag{2.5}$$

We show [10] that ϕ^+ can be expressed in the following way:

Theorem 2.1. *If $\partial\Omega$ is C^2 and ∂D is $C^{2,\alpha}$, for some $\alpha \in (0, 1)$, then the electric potential ϕ^+ immediately after the lightning discharge is given by*

$$\phi^+(x) = \begin{cases} \phi_L & \text{if } x \in D, \\ \phi_0(x) + \xi(x) & \text{if } x \in D^c, \end{cases} \tag{2.6}$$

where

$$\phi_L = \frac{\langle \nabla\phi_0, \nabla\Pi\rangle_\Omega}{\langle \nabla\Pi, \nabla\Pi\rangle_\Omega}, \tag{2.7}$$

and where Π and ξ are harmonic functions in D^c with boundary conditions as specified below:

$$\Delta\Pi = 0 \quad \text{in } D^c, \qquad \Pi = 0 \quad \text{on } \partial\Omega, \qquad \Pi = 1 \quad \text{in } D, \tag{2.8}$$

$$\Delta\xi = 0 \quad \text{in } D^c, \qquad \xi = 0 \quad \text{on } \partial\Omega, \qquad \xi = \phi_L - \phi_0 \quad \text{on } \partial D. \tag{2.9}$$

As explained in [10], when D touches $\partial\Omega$, as it would during a cloud-to-ground flash, $\phi_L = 0$ and Π can be eliminated.

Thus in the lightning channel D, the electric potential is the projection of the pre-flash potential ϕ_0 along the type 1 eigenfunction Π. Outside the lightning channel, the change in the electric potential is a linear combination of type 4 eigenfunctions. Hence, the change in the electric potential is harmonic outside of D, and the boundary conditions can be expressed in term of the pre-flash potential ϕ_0 and the post-flash potential ϕ_L along the lightning channel. We prove Theorem 2.1 by expanding the solution to (2.4) in terms of the eigenfunctions of the generalized eigenproblem (1.1) and analyzing limits as τ tends to ∞ and t tends to 0^+. When ϕ is an eigenfunction, the τ term in (2.4), which operates on the subdomain D, is transformed to a term on the entire domain Ω by (1.1).

3. The ordinary eigenproblem

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square, symmetric n by n matrices with $\mathbf{B}$ positive definite. Consider the following generalized eigenproblem: Find $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, and $\lambda \in \mathbb{R}$ such that

$$\mathbf{Ax} = \lambda\mathbf{Bx}. \tag{3.1}$$

By defining $\mathbf{y} = \mathbf{B}^{1/2}\mathbf{x}$, where $\mathbf{B}^{1/2}$ is the symmetric square root of $\mathbf{B}$, we obtain the following ordinary, symmetric eigenproblem: Find $\mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that

$$\left(\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\right)\mathbf{y} = \lambda\mathbf{y}.$$

Hence, finding the eigenvalues of the generalized eigenproblem (3.1) is equivalent to finding the ordinary eigenvalues of $\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}$. We now derive an ordinary, self adjoint eigenproblem associated with (1.1) in which the Laplacian operator plays the role of the matrix $\mathbf{B}$.

Let $(-\Delta)^{-\frac{1}{2}} : L^2(\Omega) \to H_0^1(\Omega)$ be the inverse of the square root of the Laplacian (see [2]). We claim that the ordinary eigenproblem associated with (1.1) is the following: Find $U \in L^2(\Omega)$ such that

$$\left\langle \nabla(-\Delta)^{-\frac{1}{2}} U, \nabla(-\Delta)^{-\frac{1}{2}} V\right\rangle_D = \lambda\langle U, V\rangle_\Omega \tag{3.2}$$

for all $V \in L^2(\Omega)$. The operator on the left side is the analogue of the matrix $\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}$. To see the connection between (3.2) and (1.1), let $u = (-\Delta)^{-1/2}U$ and $v = (-\Delta)^{-1/2}V$ denote the functions in $H_0^1(\Omega)$ corresponding to U and $V \in L^2(\Omega)$. In terms of u and v, (3.2) can be stated in the following way: Find $u \in H_0^1(\Omega)$ such that

$$\langle \nabla u, \nabla v\rangle_D = \lambda\big\langle(-\Delta)^{\frac{1}{2}}u, (-\Delta)^{\frac{1}{2}}v\big\rangle_\Omega \tag{3.3}$$

for all $v \in H_0^1(\Omega)$. If $u \in C_0^\infty(\Omega)$, then

$$\big\langle(-\Delta)^{\frac{1}{2}}u, (-\Delta)^{\frac{1}{2}}v\big\rangle_\Omega = -\langle \Delta u, v\rangle_\Omega = \langle \nabla u, \nabla v\rangle_\Omega. \tag{3.4}$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ and the operators $(-\Delta)^{\frac{1}{2}}$ and ∇ are both bounded in $H_0^1(\Omega)$, the identity (3.4) is valid for all $u \in H_0^1(\Omega)$. Hence, (3.3) reduces to (1.1).

4. Eigenfunctions of types 1, 2, and 3

The eigenfunctions of types 1, 2, and 3 are now derived. By (1.1), we have

$$\lambda = \frac{\langle \nabla u, \nabla u\rangle_D}{\langle \nabla u, \nabla u\rangle_\Omega} = \frac{\langle \nabla u, \nabla u\rangle_D}{\langle \nabla u, \nabla u\rangle_D + \langle \nabla u, \nabla u\rangle_{D^c}}, \tag{4.1}$$

which implies that $0 \leqslant \lambda \leqslant 1$. Let $H_0^1(D) \subset H_0^1(\Omega)$ denote the subspace consisting of functions with support in D. Similarly, let $H_0^1(D^c) \subset H_0^1(\Omega)$ denote the subspace consisting of functions with support in D^c.

Proposition 4.1. *$\lambda = 1$ and $u \in H_0^1(\Omega)$ is an eigenpair of* (1.1) *if and only if the support of u is contained in D. If $u \in H_0^1(D^c)$, then u is an eigenfunction of* (1.1) *corresponding to the eigenvalue* 0. *The only other eigenfunction of* (1.1) *corresponding to the eigenvalue* 0, *which is orthogonal to $H_0^1(D^c)$, is the solution $\Pi \in H_0^1(\Omega)$ of*

$$\langle \nabla \Pi, \nabla v\rangle_\Omega = 0 \quad \textit{for all } v \in H_0^1\big(D^c\big), \qquad \Pi = 1 \quad \textit{on } D. \tag{4.2}$$

Proof. If $\lambda = 1$ and $u \in H_0^1(\Omega)$ is an eigenpair of (1.1), then by (4.1), we have

$$\langle \nabla u, \nabla u\rangle_{D^c} = 0.$$

Hence, $\nabla u = 0$ in D^c, which implies that u is constant in D^c since D^c is connected. Since $u \in H_0^1(\Omega)$, $u = 0$ in D^c. Conversely, if $u = 0$ in D^c, then by (1.1), u is an eigenfunction corresponding to the eigenvalue 1. If $u = 0$ in D, then u is an eigenfunction corresponding to the eigenvalue 0. The solution Π of (4.2) is an eigenfunction of (1.1) corresponding to the eigenvalue 0 since $\nabla \Pi = 0$ in D.

Let $w \in H_0^1(\Omega)$ be any eigenfunction of (1.1) corresponding to the eigenvalue 0 which is orthogonal to $H_0^1(D^c)$. By (1.1), we have $\langle \nabla w, \nabla w\rangle_D = 0$, which implies that $\nabla w = 0$ in D, or w is constant in D since D is connected. Without loss of generality, let us assume that $w = 1$ in D. Since w is orthogonal to the functions $v \in H_0^1(D^c)$, we have

$$\langle \nabla w, \nabla v\rangle_\Omega = 0 \quad \text{for all } v \in H_0^1\big(D^c\big).$$

Combining this with (4.2) gives

$$\big\langle \nabla(w - \Pi), \nabla v\big\rangle_\Omega = 0 \quad \text{for all } v \in H_0^1\big(D^c\big).$$

Since $\partial D^c = \partial\Omega \cup \partial D$ and since $w - \Pi$ vanishes on both $\partial\Omega$ and ∂D, it follows that $w = \Pi$. □

5. Reformulation of eigenproblem in $\mathcal{H}$ using double-layer potential

Proposition 4.1 describes eigenfunctions of types 1, 2, and 3. In this section, we focus on type 4 eigenfunctions. Let $\mathcal{H}$ be the space which consists of all $u \in H_0^1(\Omega)$ satisfying the conditions

$$\langle \nabla u, \nabla v\rangle_\Omega = 0 \quad \text{for all } v \in H_0^1(D) \quad \text{and} \tag{5.1}$$

$$\langle \nabla u, \nabla w\rangle_\Omega = 0 \quad \text{for all } w \in H_0^1\big(D^c\big). \tag{5.2}$$

$\mathcal{H}$ is a subspace of $H_0^1(\Omega)$ consisting of functions harmonic in D and D^c ($\Delta u = 0$ in D and $\Delta u = 0$ in D^c). Note that $\Pi \in \mathcal{H}$. Since $H_0^1(D)$ and $H_0^1(D^c)$ are orthogonal with respect to the $H_0^1(\Omega)$ inner product, and since $\mathcal{H}$ is the orthogonal complement of $H_0^1(D) \oplus H_0^1(D^c)$ in $H_0^1(\Omega)$, we have the orthogonal decomposition

$$H_0^1(\Omega) = \mathcal{H} \oplus H_0^1(D) \oplus H_0^1\big(D^c\big).$$

The following series of lemmas reformulates the generalized eigenvalue problem (1.1) on $\mathcal{H}$ in terms of an integral operator.

Lemma 5.1. *$u \in \mathcal{H}$ is a solution of the generalized eigenproblem* (1.1) *if and only if*

$$\frac{\partial u}{\partial n}^{-} = -\lambda \left[\frac{\partial u}{\partial n}\right] \quad \text{on } \partial D, \tag{5.3}$$

where

$$\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u}{\partial n}^{+} - \frac{\partial u}{\partial n}^{-} \in H^{-1/2}(\partial D).$$

Here n is the outward unit normal to D and the $-$ and $+$ refer to the limits from the interior and exterior of D, respectively.

Proof. First we show a generalized eigenpair also satisfies (5.3). By (1.1) we have

$$\langle \nabla u, \nabla v\rangle_D = \lambda \langle \nabla u, \nabla v\rangle_\Omega = \lambda\big(\langle \nabla u, \nabla v\rangle_{D^c} + \langle \nabla u, \nabla v\rangle_D\big) \tag{5.4}$$

for any $v \in H_0^1(\Omega)$. Integrating by parts and utilizing the fact that u is harmonic in both D and D^c gives

$$\int_{\partial D} v \frac{\partial u}{\partial n}^{-} d\gamma = -\lambda \int_{\partial D} v \frac{\partial u}{\partial n}^{+} d\gamma + \lambda \int_{\partial D} v \frac{\partial u}{\partial n}^{-} d\gamma, \tag{5.5}$$

where γ denotes the boundary measure on ∂D. Hence, we have

$$\int_{\partial D} v \left[\frac{\partial u}{\partial n}^{-} + \lambda\left(\frac{\partial u}{\partial n}^{+} - \frac{\partial u}{\partial n}^{-}\right)\right] d\gamma = 0$$

for any $v \in H_0^1(\Omega)$. Since any $v \in H^{1/2}(\partial D)$ has an $H_0^1(\Omega)$ extension, (5.3) holds.

Conversely, suppose that u satisfies (5.3). As in (5.4)–(5.5), we have

$$\lambda \langle \nabla v, \nabla u\rangle_\Omega = -\lambda \int_{\partial D} v \frac{\partial u}{\partial n}^{+} d\gamma + \lambda \int_{\partial D} v \frac{\partial u}{\partial n}^{-} d\gamma.$$

Applying (5.3) gives

$$\lambda \langle \nabla v, \nabla u\rangle_\Omega = \int_{\partial D} v \frac{\partial u}{\partial n}^{-} = \langle \nabla v, \nabla u\rangle_D$$

since u is harmonic in D. Hence, u satisfies (1.1). $\square$

Now let us introduce the Green's function on Ω:

$$\Delta_y G(x, y) = \delta_x(y) \quad \text{in } \Omega, \qquad G(x, y) = 0 \quad \text{for } y \in \partial\Omega, \tag{5.6}$$

where δ_x is the Dirac delta function located at x. The piecewise harmonic functions $u \in \mathcal{H}$ can be described in terms of the jump on ∂D of the normal derivative.

Lemma 5.2. *Suppose that ∂D and $\partial\Omega$ are C^2. If $u \in \mathcal{H}$, $x \in \Omega$, and $x \notin \partial D$, then*

$$u(x) = \int_{\partial D} \left[\frac{\partial u}{\partial n}\right](y) G(x, y)\, d\gamma_y. \tag{5.7}$$

Proof. This follows from (5.6). Multiply both sides by $u(y)$, integrate over $y \in \Omega$, and integrate by parts. The terms involving the trace of u on ∂D cancel since the exterior and interior traces match for functions in $\mathcal{H}$. $\square$

The following lemma is the well-known limit of a double layer potential. See for example [4, Theorem 6.5.1] or [8, Theorem 3.22] for the case of free space potentials. We restate this limit here for completeness.

Lemma 5.3. *Suppose $\phi \in H^{1/2}(\partial D)$ and both ∂D and $\partial\Omega$ are C^2. For $x \in \Omega$, $x \notin \partial D$, let $v(x)$ be defined by*

$$v(x) = \int_{\partial D} \phi(y) \frac{\partial G}{\partial n_y}(x, y)\, d\gamma_y.$$

The trace v^+ of v onto ∂D from the exterior of D and the trace v^- of v onto ∂D from the interior D are given by

$$v^\pm(x) = \mp\frac{1}{2}\phi(x) + \int_{\partial D} \phi(y) \frac{\partial G}{\partial n_y}(x, y)\, d\gamma_y. \tag{5.8}$$

Proof. We express G as the sum of the free space Green's function F (fundamental solution) for the Laplacian and a harmonic function: $G = F + H$ where

$$F(x, y) = \begin{cases} \frac{|x-y|^{2-n}}{(2-n)\omega_n}, & n > 2, \\ \frac{1}{2\pi}\log|x-y|, & n = 2, \end{cases} \tag{5.9}$$

and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. By [4, Theorem 6.5.1],

$$\lim_{\substack{x \to \partial D \\ x \in D^c}} \int_{\partial D} \phi(y) \frac{\partial F}{\partial n_y}(x, y)\, d\gamma_y = -\frac{1}{2}\phi(x) + \int_{\partial D} \phi(y) \frac{\partial F}{\partial n_y}(x, y)\, d\gamma_y. \tag{5.10}$$

For $x \in \Omega$, the harmonic function H has C^2 boundary values given by $H(x, y) = -F(x, y)$ for $y \in \partial\Omega$. By the smoothness of H and (5.10), we obtain (5.8). $\square$

Using Lemmas 5.1–5.3, we reformulate the generalized eigenproblem (1.1) on $\mathcal{H}$ in terms of a boundary integral operator. By the trace theorem [1, Theorem 7.53], any $u \in \mathcal{H} \subset H_0^1(\Omega)$ has a trace on ∂D in $H^{1/2}(\partial D)$. Conversely, $u \in H^{1/2}(\partial D)$ has a unique harmonic extension into both D and D^c with $u = 0$ on $\partial\Omega$. Hence, there is a one-to-one correspondence between elements of $\mathcal{H}$ and elements of $H^{1/2}(\partial D)$.

Define

$$T : L^2(\partial D) \to L^2(\partial D)$$

by

$$T\phi(x) = \int_{\partial D} \phi(y) K(x, y)\, d\gamma_y, \quad K(x, y) := \frac{\partial G}{\partial n_y}(x, y). \tag{5.11}$$

By [8, Proposition 3.17], K is a continuous kernel of order $n-2$ on ∂D. It follows from [8, Proposition 3.12] that T is a compact operator from $L^2(\partial D)$ to itself.

Proposition 5.4. *If both ∂D and $\partial\Omega$ are C^2, then $(u, \lambda) \in \mathcal{H} \times \mathbb{R}$ is a generalized eigenpair for* (1.1) *if and only if the corresponding $u \in H^{1/2}(\partial D)$ is an eigenfunction of T with associated eigenvalue* $1/2 - \lambda$; *that is,*

$$Tu = (1/2 - \lambda)u. \tag{5.12}$$

Proof. First, let us assume that $(u, \lambda) \in \mathcal{H} \times \mathbb{R}$ is a generalized eigenpair for (1.1). By Lemmas 5.1 and 5.2, we have

$$\lambda u(x) = -\int_{\partial D} \frac{\partial u}{\partial n}^-(y) G(x, y)\, d\gamma_y$$

for $x \in \Omega$ and $x \notin \partial D$. We integrate by parts to obtain

$$\lambda u(x) = -\int_D \nabla u(y)\nabla_y G(x,y)\,dy = -\int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y + \int_D u(y)\Delta_y G(x,y)\,dy.$$

If $x \in D^c$, then the second term above disappears, and we have

$$\lambda u(x) = -\int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y,$$

an equation for a double layer potential. We let $x \in D^c$ approach ∂D. According to Lemma 5.3,

$$\lambda u(x) = \frac{1}{2}u(x) - \int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y,$$

which is equivalent to (5.12).

Conversely, suppose that $u \in H^{1/2}(\partial D)$ satisfies (5.12). We identify u with its harmonic extension in $\mathcal{H}$, and we define $w(x)$ by

$$w(x) = \begin{cases} -\int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y & \text{for } x \in D^c, \\ -\int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y + u(x) & \text{for } x \in D. \end{cases} \tag{5.13}$$

In either D^c and D, w is harmonic. By Lemma 5.3, we have

$$w^+(x) = \frac{1}{2}u(x) - \int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y$$

and

$$w^-(x) = u(x) - \frac{1}{2}u(x) - \int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y = \frac{1}{2}u(x) - \int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y.$$

Utilizing (5.12) yields

$$w^+ = w^- = (1/2 - T)u = \lambda u \quad \text{on } \partial D. \tag{5.14}$$

Observe that w vanishes on $\partial\Omega$ due to the symmetry of $G(x,y)$ [8, Lemma 2.33]; that is, since $G(x,y) = 0$ when $y \in \partial\Omega$, we have by symmetry $G(x,y) = 0$ when $x \in \partial\Omega$. Hence, the normal derivative in (5.13) vanishes when $x \in \partial\Omega$. Since w is harmonic in each subdomain and it is equal to λu on both ∂D (see (5.14)) and $\partial\Omega$ (they both vanish), it follows that $w = \lambda u$ in Ω. We replace w with λu in (5.13) to obtain

$$\lambda u(x) = \begin{cases} -\int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y & \text{for } x \in D^c, \\ -\int_{\partial D} u(y)\frac{\partial G}{\partial n_y}(x,y)\,d\gamma_y + u(x) & \text{for } x \in D. \end{cases} \tag{5.15}$$

Integrating both right-hand sides in (5.15) by parts into D and using Eq. (5.6) for G, we have for any $x \notin \partial D$

$$\lambda u(x) = -\int_{\partial D} \frac{\partial u}{\partial n}^-(y)G(x,y)\,d\gamma_y. \tag{5.16}$$

By Lemma 5.2,

$$\lambda u(x) = \lambda \int_{\partial D} \left[\frac{\partial u}{\partial n}\right](y)G(x,y)\,d\gamma_y. \tag{5.17}$$

Subtracting (5.17) from (5.16) gives

$$s(x) := \int_{\partial D} \phi(y)G(x,y)\,d\gamma_y = 0 \quad \text{for any } x \notin \partial D,$$

where

$$\phi(y) = -\frac{\partial u}{\partial n}^{-}(y) - \lambda\left[\frac{\partial u}{\partial n}\right](y).$$

Since $s = 0$ almost everywhere in Ω, ϕ lies in the null space of the single layer potential operator. Since the null space is the zero function, it follows that $\phi = 0$ or

$$\frac{\partial u}{\partial n}^{-} = -\lambda\left[\frac{\partial u}{\partial n}\right].$$

Lemma 5.1 completes the proof.

It is proved in [4, Theorem 6.8.2] that the null space of the single layer potential operator associated with the fundamental solution for the Laplacian is the zero function when Ω is simply connected and $\Omega \subset \mathbb{R}^3$ with smooth boundary. For completeness, we provide a proof of this result in our slightly different setting. Given a smooth function r defined on ∂D, let r also denote any smooth extension into Ω with compact support in Ω. By the symmetry of G, we have

$$r(y) = \int_{\Omega} \left[\Delta_x r(x)\right] G(x, y)\, dx.$$

Forming the $L^2(\Omega)$ inner product between s (which vanishes almost everywhere) and Δr yields

$$0 = \langle s, \Delta r\rangle_{\Omega} = \int_{\Omega}\int_{\partial D} \phi(y)\left[\Delta_x r(x)\right] G(x, y)\, d\gamma_y\, dx = \langle \phi, r\rangle_{\partial D}.$$

Since r was an arbitrary smooth function defined on ∂D, it follows that $\phi = 0$. $\square$

Corollary 5.5. *If both ∂D and $\partial\Omega$ are C^2, then the eigenvalues of the double layer potential operator T in* (5.11) *are real and contained in the half-open interval* $(-1/2, 1/2]$. *The only possible accumulation point for the spectrum is* 0.

Proof. The eigenvalues of the generalized eigenproblem (1.1) are all real due to symmetry of the inner product. By Proposition 5.4, the eigenvalues of T are all real. As noted before Proposition 4.1, the eigenvalues of (1.1) are contained on the interval [0, 1]. Moreover, by Proposition 4.1, those eigenfunctions corresponding to the eigenvalue 1 are supported in D. The trace of this eigenfunction on ∂D is 0. The only element in $\mathcal{H}$ with vanishing trace on ∂D is the zero function. Consequently, there is no eigenfunction in $\mathcal{H}$ corresponding to the eigenvalue 1. There is one eigenfunction in $\mathcal{H}$ corresponding to the eigenvalue 0, namely the function Π of Proposition 4.1. Except for the eigenvalue 0, all the remaining eigenvalues for the generalized eigenproblem lie in the open interval (0, 1). Since the eigenvalues of T are 1/2 minus the corresponding eigenvalue of (1.1) in [0, 1), the proof is complete. Since T is compact on $L^2(\Omega)$ [8, Proposition 3.12], the only possible accumulation point for the spectrum is 0. $\square$

A lower bound for the separation between the largest and second largest eigenvalues of T is obtained from Proposition 6.1.

6. Eigenvalue separation and completeness of eigenfunctions

Due to Proposition 4.1, the generalized eigenproblem (1.1) restricted to $\mathcal{H}$ has a simple eigenvalue $\lambda = 0$ corresponding to the eigenfunction $\Pi \in \mathcal{H}$ while the remaining eigenvalues are positive. By Proposition 5.4, the only possible accumulation point for the spectrum is $\lambda = 1/2$. Hence, there is an interval $(0, \rho)$, $\rho > 0$, where the generalized eigenproblem has no eigenvalues. We now give an explicit positive lower bound for ρ in terms of three embedding constants:

(E1) Let u_a denote the constant function on Ω whose value is the average of $u \in H^1(\Omega)$ over D:

$$u_a = \frac{1}{\text{measure}(D)} \int_D u(x)\, dx.$$

By [7, Theorem 1, p. 275], there exists a constant $\theta_1 > 0$ such that

$$\|\nabla u\|^2_{L^2(D)} \geqslant \theta_1 \|u - u_a\|^2_{H^1(D)}$$

for all $u \in H^1(\Omega)$.

(E2) By [1, Theorem 7.53], there exists a constant $\theta_2 > 0$ such that

$$\|u\|^2_{H^1(D)} \geqslant \theta_2 \|u\|^2_{H^{1/2}(\partial D)}$$

for all $u \in H^1(\Omega)$.

(E3) There exists a constant $\theta_3 > 0$ such that

$$\|u\|^2_{H^{1/2}(\partial D)} \geqslant \theta_3 \|u\|^2_{H^1(D^c)} \tag{6.1}$$

for all $u \in H^1_0(\Omega)$ which are harmonic in D^c (in other words, (5.2) holds). This is a standard result for harmonic extensions of $H^{1/2}$ functions. That is, by [9, Theorem 1.5.1.3], there exists a bounded linear map $\mathcal{T} : H^{1/2}(\partial D) \to H^1_0(\Omega)$ with the property that $\mathcal{T}(g)$ has trace g on ∂D. Consequently, if g is the trace of a harmonic function $u \in H^1(D^c)$, then we have

$$\big\langle \nabla u, \nabla\big(u - \mathcal{T}(g)\big)\big\rangle_{D^c} = 0,$$

which implies that $\|\nabla u\|_{L^2(D^c)} \leqslant \|\nabla \mathcal{T}(g)\|_{L^2(D^c)} \leqslant (\|\mathcal{T}\|)\|g\|_{H^{1/2}(\partial D)}$. Since u vanishes on $\partial\Omega$, (6.1) holds.

Proposition 6.1. *If both ∂D and $\partial\Omega$ are Lipschitz, then the generalized eigenproblem* (1.1) *has no eigenvalues in the interval* $(0, \rho)$ *where*

$$\rho = \min\{1, \theta_2\theta_3\}\theta_1/2.$$

Proof. Let μ be the smallest positive eigenvalue for the generalized eigenproblem (1.1), and let u be an associated eigenfunction with normalization $\langle \nabla u, \nabla u\rangle_\Omega = 1$. If $\Pi \in \mathcal{H}$ is the eigenfunction described in (4.2), then we have

$$\begin{aligned}
\mu &= \|\nabla u\|^2_{L^2(D)} \\
&\geqslant \theta_1 \|u - u_a\|^2_{H^1(D)} && (6.2)\\
&= \theta_1 \|u - \Pi u_a\|^2_{H^1(D)} && (6.3)\\
&\geqslant \theta_1\theta_2 \|u - \Pi u_a\|^2_{H^{\frac{1}{2}}(\partial D)} && (6.4)\\
&\geqslant \theta_1\theta_2\theta_3 \|u - \Pi u_a\|^2_{H^1(D^c)}. && (6.5)
\end{aligned}$$

Above, (6.3) is due to the fact that $\Pi = 1$ on D, while (6.2), (6.4), and (6.5) come from (E1)–(E3), respectively.

Suppose that the proposition does not hold, in which case $\mu < \theta_1/2$ and $\mu < \theta_1\theta_2\theta_3/2$. By (6.3) and (6.5), we have

$$\|u - \Pi u_a\|^2_{H^1(D)} < 1/2 \quad \text{and} \quad \|u - \Pi u_a\|^2_{H^1(D^c)} < 1/2.$$

Combining these gives

$$\|u - \Pi u_a\|^2_{H^1(\Omega)} < 1. \tag{6.6}$$

On the other hand, u and Π are orthogonal since these eigenfunctions correspond to distinct eigenvalues. Since $u_a\Pi$ is a multiple of Π which is orthogonal to u, it follows that

$$1 \leqslant \big\|\nabla(u - \Pi u_a)\big\|^2_{L^2(\Omega)} \leqslant \big\|\nabla(u - \Pi u_a)\big\|^2_{H^1(\Omega)}. \tag{6.7}$$

Comparing (6.6) and (6.7), we have a contradiction. Hence, either $\mu \geqslant \theta_1/2$ or $\mu \geqslant \theta_1\theta_2\theta_3/2$. □

We continue to develop properties for the eigenfunctions of the generalized eigenproblem (1.1) by exploiting the connection, given in Proposition 5.4, between the eigenfunctions of the generalized eigenproblem (1.1) and those of the double layer potential T in (5.11). As noted before Proposition 5.4, there is a one-to-one correspondence between elements of $\mathcal{H}$ and elements of $H^{1/2}(\partial D)$. If $u \in H^{1/2}(\partial D)$, then the corresponding $E(u) \in \mathcal{H}$ is the harmonic extension of $u \in H^{1/2}(\partial D)$ into Ω which vanishes on $\partial\Omega$. For any $u, v \in H^{1/2}(\partial D)$, we define the inner product

$$(u, v) = \langle \nabla E(u), \nabla E(v) \rangle_{\Omega}. \tag{6.8}$$

In other words, harmonically extend u and v in Ω and form the $H_0^1(\Omega)$ inner product of the extended functions. We now show that T is self adjoint and compact relative to this new inner product.

Lemma 6.2. *The following properties are satisfied*:

(T1) *If $\partial\Omega$ and ∂D are Lipschitz, then the norm $(\cdot,\cdot)^{1/2}$ is equivalent to the usual norm for $H^{1/2}(\partial D)$. That is, there exist positive constants c_1 and c_2 such that*

$$c_1(v, v) \leqslant \|v\|^2_{H^{1/2}(\partial D)} \leqslant c_2(v, v)$$

for all $v \in H^{1/2}(\partial D)$.

(T2) *If $\partial\Omega$ and ∂D are C^2, then the double layer potential operator T in* (5.11) *is self adjoint relative to the inner product* (6.8).

(T3) *If $\partial\Omega$ is C^2 and ∂D is $C^{2,\alpha}$, then T is a compact operator from $H^{1/2}(\partial D)$ into $H^{1/2}(\partial D)$.*

Proof. We begin by showing that the norm of $H^{1/2}(\partial D)$ and the norm $(\cdot,\cdot)^{1/2}$ are equivalent. First, recall [7, p. 265] that there exists a constant $\theta_4 > 0$ such that

$$\|\nabla v\|^2_{L^2(\Omega)} \geqslant \theta_4 \|v\|^2_{H^1(\Omega)}$$

for each $v \in \mathcal{H}$ which vanishes on $\partial\Omega$. Combining this with (E2) gives the lower bound

$$(v, v) = \langle \nabla E(v), \nabla E(v) \rangle_{\Omega} \geqslant \theta_4 \|E(v)\|^2_{H^1(\Omega)} \geqslant \theta_4 \|E(v)\|^2_{H^1(D)} \geqslant \theta_2\theta_4 \|v\|^2_{H^{1/2}(\partial D)}. \tag{6.9}$$

An upper bound for (v, v) is obtained from (E3):

$$(v, v) = \langle \nabla E(v), \nabla E(v) \rangle_{\Omega} \leqslant \|E(v)\|^2_{H^1(\Omega)} \leqslant \left(\theta_3^{-1} + \bar{\theta}_3^{-1}\right) \|v\|^2_{H^{1/2}(\partial D)}. \tag{6.10}$$

Here $\bar{\theta}_3 > 0$ is analogous to θ_3 in (6.1) except that it relates D to ∂D:

$$\|v\|^2_{H^{1/2}(\partial D)} \geqslant \bar{\theta}_3 \|E(v)\|^2_{H^1(D)}.$$

Relations (6.9) and (6.10) yield (T1).

To show that T is self adjoint relative to the inner product (6.8), we must verify the identity

$$(Tu, v) = \langle \nabla E(Tu), \nabla E(v) \rangle_{\Omega} = \langle \nabla E(u), \nabla E(Tv) \rangle_{\Omega} = (u, Tv) \tag{6.11}$$

for all u and $v \in H^{1/2}(\partial D)$. We first observe that the extension of Tu has the form

$$E(Tu)(x) = \begin{cases} \frac{1}{2}E(u)(x) + \int_{\partial D} u(y) \frac{\partial G(x,y)}{\partial n_y}\, d\gamma_y & \text{for } x \in D^c, \\ -\frac{1}{2}E(u)(x) + \int_{\partial D} u(y) \frac{\partial G(x,y)}{\partial n_y}\, d\gamma_y & \text{for } x \in D. \end{cases} \tag{6.12}$$

By Lemma 5.3, the trace of the right side of (6.12) is Tu from either side of ∂D. Moreover, the right side is harmonic and it vanishes on $\partial\Omega$ since $E(u)$ vanishes on $\partial\Omega$ and $G(x, y) = 0$, independent of $y \in \Omega$, when $x \in \partial\Omega$. Since the right side is harmonic and satisfies the boundary conditions associated with $E(Tu)$, it must equal $E(Tu)$.

Integrating by parts and utilizing (6.12), we obtain

$$\begin{aligned} (Tu, v) &= \langle \nabla E(Tu), \nabla v \rangle_{\Omega} = \langle \nabla E(Tu), \nabla E(v) \rangle_{D} + \langle \nabla E(Tu), \nabla E(v) \rangle_{D^c} \\ &= -\frac{1}{2} \int_{\partial D} \left(\frac{\partial E(u)^-}{\partial n} + \frac{\partial E(u)^+}{\partial n} \right) E(v)\, d\gamma. \end{aligned} \tag{6.13}$$

The term in $E(Tu)$ associated with the Green's function cancels since the normal derivative of a double layer potential operator is continuous across ∂D (for example, see [6, Theorem 3.1], [5, Theorem 2.21], [13, Theorem 6.13]).

For any p and $q \in \mathcal{H}$, we have the identities

$$\langle \nabla p, \nabla q \rangle_D = \int_{\partial D} q \frac{\partial p}{\partial n}^- \, d\gamma = \int_{\partial D} p \frac{\partial q}{\partial n}^- \, d\gamma,$$

and

$$\langle \nabla p, \nabla q \rangle_{D^c} = -\int_{\partial D} q \frac{\partial p}{\partial n}^+ \, d\gamma = -\int_{\partial D} p \frac{\partial q}{\partial n}^+ \, d\gamma.$$

Hence, the normal derivatives in (6.13) can be moved from the u terms to v to obtain

$$(Tu, v) = -\frac{1}{2} \int_{\partial D} \left(\frac{\partial E(v)^-}{\partial n} + \frac{\partial E(v)^+}{\partial n} \right) E(u) \, d\gamma = (u, Tv),$$

which establishes (T2).

We now show that T is compact on $H^{1/2}(\partial D)$. Consider the corresponding free space double layer potential operator T_F defined by

$$T_F \phi(x) = \int_{\partial D} \phi(y) \frac{\partial F}{\partial n_y}(x, y) \, d\gamma_y,$$

where F is the free space Green's function defined in (5.9). For $n = 2$, T_F is compact by [13, Theorem 8.20]. For $n \geqslant 3$, Theorem 4.2 in [12] gives the boundedness of T_F as a map from $L^2(\partial D)$ to $H^1(\partial D)$. This result extends to our operator T as follows. The difference, $T - T_F$, is an integral operator on ∂D with kernel

$$\frac{\partial H}{\partial n_y}(x, y) = \frac{\partial G}{\partial n_y}(x, y) - \frac{\partial F}{\partial n_y}(x, y).$$

For $x \in \partial D$, H has no singularity since it is harmonic with smooth boundary data (see the proof of Lemma 5.3). Consequently, $T - T_F$ is bounded from $L^2(\partial D)$ to $H^1(\partial D)$. Since both T_F and $T - T_F$ are bounded from $L^2(\partial D)$ to $H^1(\partial D)$, we conclude that T is bounded from $L^2(\partial D)$ to $H^1(\partial D)$. This implies that T is compact on $H^{1/2}(\partial D)$ since H^1 embeds compactly in $H^{1/2}$; that is, by [9, Theorem 1.4.3.2] H^s embeds compactly in H^t when $s > t \geqslant 0$. Hence, T is compact on $H^{1/2}(\partial D)$. $\square$

We now prove our main result, Theorem 1.1. As pointed out earlier, we have the orthogonal decomposition

$$H_0^1(\Omega) = \mathcal{H} \oplus H_0^1(D) \oplus H_0^1(D^c).$$

By Proposition 4.1, any complete orthonormal basis for $H_0^1(D)$ is an eigenfunction basis corresponding to the eigenvalue 1. Likewise, any complete orthonormal basis for $H_0^1(D^c)$ is a basis whose elements are eigenfunctions of the generalized eigenproblem corresponding to the eigenvalue 0. To complete the proof, we need to show that any $f \in \mathcal{H}$ lies in the span of the remaining eigenfunctions for (1.1).

By Lemma 6.2, T is compact and self adjoint relative to the inner product $(\cdot,\cdot)$ defined in (6.8). Hence, every $f \in H^{1/2}(\partial D)$ has a unique expansion in terms of orthogonal eigenfunctions of T (for example, see [3, Theorem 1.28]). Given $f \in \mathcal{H}$, its restriction to ∂D lies in $H^{1/2}(\partial D)$. Therefore, there exist orthogonal eigenfunctions ϕ_i, $i \geqslant 1$, of T such that

$$f = \sum_{i=1}^{\infty} \phi_i \quad \text{on } \partial D.$$

By the linearity and boundedness of the extension operator, we have

$$f = \sum_{i=1}^{\infty} E(\phi_i) \quad \text{on } \Omega.$$

By Proposition 5.4, $E(\phi_i)$ is an eigenfunction for the generalized eigenproblem (1.1). This completes the proof of Theorem 1.1.

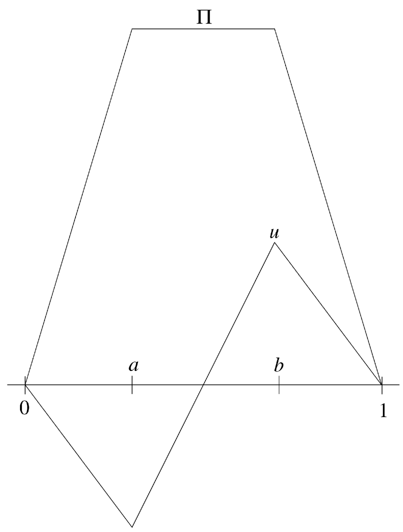

Fig. 2. Eigenfunctions in $\mathcal{H}$ in one dimension.

7. One dimension

Let us consider the generalized eigenproblem (1.1) in one dimension where Ω is the interval $[0, 1]$ and D is a subinterval $[a, b] \subset (0, 1)$. In this case, there are precisely 2 eigenfunctions in $\mathcal{H}$. The functions which are harmonic on both D and D^c are piecewise linear. The eigenfunction Π of Proposition 4.1, corresponding to the eigenvalue 0, is defined by its boundary values $\Pi(0) = \Pi(1) = 0$ and the values $\Pi(x) = 1$ on D. Let s_1, s_2, and s_3 be the slope on the intervals $[0, a]$, $[a, b]$, and $[b, 1]$ respectively of the remaining eigenfunction $u \in \mathcal{H}$. The jump condition of Lemma 5.1 yields

$$s_2 = -\lambda(s_1 - s_2) \quad \text{and} \quad s_2 = -\lambda(s_3 - s_2). \tag{7.1}$$

Hence, $s_1 = s_3$. Let s denote either s_1 or s_3. The boundary conditions $u(0) = u(1) = 0$ imply that

$$0 = \int_0^1 u'(x)\,dx = s_1 a + s_2(b - a) + s_3(1 - b) = s(1 + a - b) + s_2(b - a).$$

This gives

$$s_2 = s\left(\frac{b - a - 1}{b - a}\right).$$

With this substitution in (7.1), we have

$$\lambda = 1 - (b - a).$$

A sketch of these two eigenfunctions appears in Fig. 2.

8. Conclusions

We analyze a generalized eigenproblem (1.1) for the Laplacian. The elements of $H_0^1(D^c)$ are eigenfunctions corresponding to the eigenvalue 0, while the elements of $H_0^1(D)$ are eigenfunctions corresponding to the eigenvalue 1. The remaining eigenfunctions are elements of the piecewise harmonic space $\mathcal{H}$, consisting of functions in $H_0^1(\Omega)$ which are harmonic in both D and D^c. There is a one-to-one correspondence between eigenfunctions of (1.1) in $\mathcal{H}$ and eigenfunctions of the double layer potential T in (5.11). The eigenfunctions of (1.1) are the harmonic extensions of the eigenfunctions of T, and if μ is an eigenvalue of T, then $\lambda = 1/2 - \mu$ is the corresponding eigenvalue of (1.1).

$\Pi \in \mathcal{H}$ (see Proposition 4.1) is the only eigenfunction in $\mathcal{H}$ corresponding to the eigenvalue 0. All the remaining eigenvalues corresponding to eigenfunctions in $\mathcal{H}$ are contained in the open interval $(0, 1)$ and $\lambda = 1/2$ is the only possible accumulation point. Since the eigenvalues of the generalized eigenproblem (1.1) associated with eigenfunctions in $\mathcal{H}$ are contained in the half-open interval $[0, 1)$, the eigenvalues of the double layer potential T in (5.11) are contained in $[-1/2, 1/2)$. Proposition 6.1 gives a lower bound for the positive eigenvalues of the generalized eigenproblem, or equivalently, a lower bound for the gap between the largest and the second largest eigenvalue of T. Based on the fact that the double layer potential T is self adjoint and compact relative to the inner product (6.8), as established in Lemma 6.2, we conclude that any $f \in H_0^1(\Omega)$ can be expressed as a linear combination of orthogonal eigenfunctions for (1.1).

The eigenfunction decomposition developed in this paper can be used to determine the potential change due to lightning as given in Theorem 2.1. Since the eigenfunctions for (1.1) form a basis for $H_0^1(\Omega)$, we can expand the electric potential in terms of the eigenfunctions. Utilizing the structure of the eigenvalues, we are able to evaluate the limit (2.5) of the potential as conductivity σ tends to ∞ in the lightning channel. The key properties of the eigenvalues which enter into the analysis are the following:

(i) The positive eigenvalues are bounded away from 0 (Proposition 6.1). As a result, we show that as σ tends to ∞ in the lightning domain D, the coefficients in the expansion of the potential all decay to zero except for the coefficients associated with zero eigenvalues.
(ii) There is only one eigenfunction with support in the lightning domain D whose eigenvalue is zero, namely Π. Since Π is constant on D, it follows that the potential becomes constant along the lightning channel as σ tends to ∞. The value of the constant is the H^1 projection of the initial potential ϕ_0 along Π. This mathematical result coincides with our physical expectation that the potential in a highly conductive material is constant.
(iii) Outside the lightning channel D, the change in potential is expressed in terms of the eigenfunctions in $\mathcal{H}$. Since these eigenfunctions are all harmonic, we deduce that the change in the potential is harmonic in D^c. Using the known boundary value on ∂D, we determine the potential throughout D^c by solving the Dirichlet problem (2.9).

See [10] for the detailed analysis.

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