Appendix

This appendix provides additional details associated with the proofs of Lemma 5.2, Lemma 5.3, and Lemma 7.1 in the paper "The Euler approximation in state constrained optimal control."

Proof of Lemma 5.2. For completeness, we provide here the analysis associated with the term $\nu_{i+1}(\nabla g(x_i)')$ in the second component of $\mathcal{T}$. Let us apply (44) with the following identifications: Let $y(t)$ denote the continuous, piecewise linear function of t for which $y(t_i) = x_i$ for each i and set $\zeta(t) = \nabla g(y(t))$ to obtain

$$\nu_{i+1}(\nabla g(x_i)') = \frac{\nu_{i+1}}{h}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\nabla g(y(t))dt. \tag{109}$$

Based on this identity, the i-th element of the associated terms in the second component of $\pi^1 - \pi^2$ can be expressed:

$$\frac{1}{h}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\Big(\nu^1_{i+1}\nabla g(y^1(t)) - \nu^2_{i+1}\nabla g(y^2(t)) \\ -\nu^*_{i+1}[(y^1(t) - y^2(t))^{\mathsf{T}}\nabla^2 g(x^*(t))] - (\nu^1_{i+1} - \nu^2_{i+1})\nabla g(x^*(t))\Big)dt, \tag{110}$$

where y^1 and y^2 are the continuous, piecewise linear interpolants associated with x^1 and x^2 respectively. Here a product of the form $y^{\mathsf{T}}\nabla^2 g(x(t))$ stands for a matrix whose j-th row is $y^{\mathsf{T}}\nabla^2 g_j(x(t))$. For any given $\eta > 0$, we now derive a series of estimates, which when combined, show that there exists $r > 0$ such that the sequence whose i-th element is given by (110), has L^2 norm bounded by $\eta\|w_1 - w_2\|_{\mathcal{X}}$ for all $w_1, w_2 \in \mathcal{B}_r(w^*)$.

To start, we decompose (110) into the sum of the following three terms:

$$\textbf{(Term 1)}_i : \ \frac{1}{h}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\left((\nu^1_{i+1} - \nu^2_{i+1})[\nabla g(y^1(t)) - \nabla g(x^*(t))]\right) dt.$$

$$\textbf{(Term 2)}_i : \ \frac{1}{h}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\left((y^1(t) - y^2(t))\nabla^2 g(x^*(t))^{\mathsf{T}}(\nu^2_{i+1} - \nu^*_{i+1})\right) dt.$$

$$\textbf{(Term 3)}_i : \ \frac{1}{h}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\Big(\nu^2_{i+1}[\nabla g(y^1(t)) - \nabla g(y^2(t))] \\ -\nu^2_{i+1}[(y^1(t) - y^2(t))^{\mathsf{T}}\nabla^2 g(x^*(t))]\Big) \ dt.$$

Each of these terms is now analyzed.

Term 1. Taylor's theorem with integral remainder implies that

$$\nabla g_j(y^1(t)) - \nabla g_j(x^*(t)) = (y^1(t) - x^*(t))^{\mathsf{T}}\int_0^1 \nabla^2 g_j((1-s)y^1(t) + sx^*(t))\, ds. \tag{111}$$

Differentiating with respect to t and utilizing (33) gives

$$\left|\frac{d}{dt}\left(\nabla g_j(y^1(t)) - \nabla g_j(x^*(t))\right)\right| \le c\|y^1 - x^*\|_{W^{1,\infty}} \le c(\|x^1 - x^*\|_{W^{1,\infty}} + h). \tag{112}$$

It follows that

$$\|\text{Term 1}\|_{L^2} \le c(\|x^1 - x^*\|_{W^{1,\infty}} + h)\|\nu^1 - \nu^2\|_{L^2}.$$

Hence, by (43) there exists $\bar{h}$ and $r > 0$ such that

$$\|\text{Term 1}\|_{L^2} \le \eta\|\nu^1 - \nu^2\|_{L^2}$$

for all $h \le \bar{h}$ and $x \in \text{Lip}_\xi^1$ with $\|x - x^*\|_{H^1} \le r$.

Term 2. Letting $G_j(t)$ denote the Hessian matrix $\nabla^2 g_j(x^*(t))$, $\|G\|_{W^{1,\infty}}$ is finite by Smoothness, while (32) implies that

$$\|y^1 - y^2\|_{H^1} \le \|x^1 - x^2\|_{H^1}. \tag{113}$$

Therefore, by (40) and for $r > 0$ sufficiently small,

$$\|\text{Term 2}\|_{L^2} \le c\|\nu^2 - \nu^*\|_{L^\infty}\|x^1 - x^2\|_{H^1} \le \eta\|x^1 - x^2\|_{H^1}$$

for all $\nu^2 \in \text{Lip}_\xi$ with $\|\nu^2 - \nu^*\|_{L^2} \le r$.

Term 3. Let $\bar{G}_j$ be defined by

$$\bar{G}_j = \int_0^1 \nabla^2 g_j((1-s)y^1(t) + sy^2(t))\, ds.$$

This is the same Hessian matrix appearing in (111) except that x^* has been replaced by y^2. With this notation, Term 3 can be written as

$$\frac{1}{h}\left|\int_{t_i}^{t_{i+1}} \frac{d}{dt}\left(\nu_{i+1}^2(y^1(t) - y^2(t))^{\mathsf{T}}(\bar{G}(t) - G(t))\right)\, dt\right| \le$$

$$\frac{(c|\nu_{i+1}^2|)\|\bar{G} - G\|_{W^{1,\infty}}}{\sqrt{h}}\|y^1 - y^2\|_{H^1([t_i,t_{i+1}])}. \tag{114}$$

For any $\eta > 0$, the continuity of the third derivatives of g given in Smoothness implies that

$$\|\bar{G} - G\|_{W^{1,\infty}} \le \eta \tag{115}$$

when $\|x^1 - x^*\|_{W^{1,\infty}} + \|x^2 - x^*\|_{W^{1,\infty}}$ is sufficiently small. Hence, by (43), there exists $r > 0$ such that (115) holds for all x^1 and $x^2 \in \text{Lip}_\xi^1$ with $\|x^1 - x^*\|_{H^1} + \|x^2 - x^*\|_{H^1} \le r$. Utilizing (115) in (114), we see that

$$|(\text{Term 3})_i| \le \frac{c\eta|\nu_i^2|}{\sqrt{h}}\|y^1 - y^2\|_{H^1([t_i,t_{i+1}])}.$$

Hence, taking η small enough and applying (41) and (113), we conclude that for some $r > 0$,

$$\|\text{Term } 3\|_{L^2} \le c\eta \|\nu^2\|_{L^\infty} \|y^1 - y^2\|_{H^1} \le \eta \|x^1 - x^2\|_{H^1}$$

for all $\nu^2 \in \text{Lip}_\xi$ and $x^1, x^2 \in \text{Lip}_\xi^1$ with

$$\|\nu^2 - \nu^*\|_{L^2} + \|x^1 - x^*\|_{H^1} + \|x^2 - x^*\|_{H^1} \le r.$$

In each of the three terms, η can be made arbitrarily small by shrinking $\bar{h}$ and r. Hence, terms 1, 2, and 3 are all consistent with (51) when $\bar{h}$ and r are sufficiently small. □

Proof of Lemma 5.3. For completeness, we provide here the analysis associated with the term $\nu_{i+1}(\nabla g(x_i)')$ in the second component of $\mathcal{T}$. Again, let $q(t)$ denotes the quadratic on $[t_{i-1}, t_{i+1}]$ for which $q(t_j) = x_j$ for $j = i-1$, i, and $i+1$, and let $\nu(\cdot)$ denote the linear function for which $\nu(t_{i+1}) = \nu_{i+1}$ and $\nu(t_{i+2}) = \nu_{i+2}$. Observe that

$$\begin{aligned}
(\nu_{i+1} \nabla g(x_i)')' &= \frac{1}{h}\Big(\nu_{i+1}(\nabla g(x_{i+1}) - \nabla g(x_i))\Big)' \\
&= \frac{1}{h}\Big((\nu_{i+1}\nabla g(x_{i+1}))' - (\nu_{i+1}\nabla g(x_i))'\Big) \\
&= \frac{1}{h^2}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\Big(\nu(t+h)[\nabla g(q(t+h)) - \nabla g(q(t))]\Big)\, dt \\
&= \frac{1}{h^2}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\Big(\nu_+(t)[\nabla g(q_+(t)) - \nabla g(q(t))]\Big)\, dt, \qquad (116)
\end{aligned}$$

where the + subscript denotes translation by h; for example, $q_+(t) = q(t+h)$. The terms in

$$((\mathcal{T} - \mathcal{L})(w) - \pi^*)' = ((\mathcal{T} - \mathcal{L})(w) - (\mathcal{T} - \mathcal{L})(w^*))'$$

corresponding to the term (116) are obtained by subtracting from it the linearization around x^* and ν^* to obtain the following expression:

$$\begin{aligned}
&\frac{1}{h^2}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\Big(\nu_+[\nabla g(q_+) - \nabla g(q)] - \nu_+^*[\nabla g(x_+^*) - \nabla g(x^*)] \\
&- \nu_+^*[(q_+ - x_+^*)^\mathsf{T} G_+ - (q - x^*)^\mathsf{T} G] - (\nu_+ - \nu_+^*)[K_+ - K]\Big)\, dt. \qquad (117)
\end{aligned}$$

The first line of the expression (117) is simply (116) minus the same expression with (q, ν) replaced by (x^*, ν^*). The second line is the linearization, with respect to first the state variable, and then the multiplier. As in Section 6, we derive a series of inequalities which together show that for all $x \in \text{Lip}_\xi^1$ and $\nu \in \text{Lip}_\xi$ with $\|x - x^*\|_{H^1} \le r$ and $\|\nu - \nu^*\|_{L^2} \le r$, the expression (117) can be made arbitrarily small by taking r sufficiently small.

To start, we decompose (117) into the sum of following three terms:

$$\textbf{(Term 1)}_i: \ \frac{1}{h^2}\int_{t_i}^{t_{i+1}} \frac{d}{dt}\Big((\nu_+ - \nu_+^*)[\nabla g(q_+) - \nabla g(q) - (K_+ - K)]\Big)\, dt.$$

$$\textbf{(Term 2)}_i: \ \frac{1}{h^2}\int_{t_i}^{t_{i+1}} \Big(\dot\nu_+^*[\nabla g(q_+) - \nabla g(q)] - \dot\nu_+^*[\nabla g(x_+^*) - \nabla g(x^*)] \\ -\dot\nu_+^*[(q_+ - x_+^*)^\mathsf{T} G_+ - (q - x^*)^\mathsf{T} G]\Big)\, dt\ .$$

$$\textbf{(Term 3)}_i: \ \frac{1}{h^2}\int_{t_i}^{t_{i+1}} \Big(\nu_+^*[\nabla \dot g(q_+) - \nabla \dot g(q)] - \nu_+^*[\nabla \dot g(x_+^*) - \nabla \dot g(x^*)] \\ -\frac{d}{dt}\nu_+^*[(q_+ - x_+^*)^\mathsf{T} G_+ - (q - x^*)^\mathsf{T} G]\Big)\, dt\ .$$

Each of these terms is now analyzed.

Term 1. First, note that

$$\begin{aligned} |[\nabla g(q_+) - \nabla g(q) - (K_+ - K)](t)| &= |[(q_+ - x_+^*)^\mathsf{T}\bar G_+ - (q - x^*)^\mathsf{T}\bar G](t)| \\ &= \left|\int_t^{t+h} \frac{d}{ds}[(q(s) - x^*(s))^\mathsf{T}\bar G(s)]\, ds\right| \\ &\le ch\|q - x^*\|_{W^{1,\infty}([t_i,t_{i+2}])} \end{aligned} \tag{118}$$

for all $t \in [t_i, t_{i+1}]$, where

$$\bar G_j = \int_0^1 \nabla^2 g_j((1-\tau)q(t) + \tau x^*(t))d\tau. \tag{119}$$

Since $\ddot q = x_i''$ and $x \in \mathrm{Lip}_\xi^1$, it follows that $\dot q$ is Lipschitz continuous with Lipschitz constant bounded by ξ. By (47), q is Lipschitz continuous with Lipschitz constant at most $\gamma + \|x^*\|_{W^{1,\infty}}$. Hence, we have

$$\left|\frac{d}{dt}(\nabla g(q(t+h)) - \nabla g(q(t)))\right| \le ch \quad \text{and} \quad |\dot K(t+h) - \dot K(t)| \le ch, \tag{120}$$

for all $t \in [t_i, t_{i+1}]$. Combining (118) and (120) gives

$$\begin{aligned} |(\text{Term 1})_i| &\le c(\|q - x^*\|_{W^{1,\infty}} + \|\nu - \nu^*\|_{L^\infty}) \\ &\le c(h + \|x - x^*\|_{W^{1,\infty}} + \|\nu - \nu^*\|_{L^\infty}), \end{aligned} \tag{121}$$

where the last inequality is based on (46). By (41) and (43), the right hand side of (121) can be made arbitrarily small, for all $x \in \mathrm{Lip}_\xi^1$ and $\nu \in \mathrm{Lip}_\xi$ with $\|x - x^*\|_{H^1} \le r$ and $\|\nu - \nu^*\|_{L^2} \le r$, by taking r and h sufficiently small.

Term 2. Since

$$\dot\nu^*(t)(\nabla g(q(t)) - \nabla g(x^*(t))) = \dot\nu^*(t)[(q(t) - x^*(t))^\mathsf{T}\bar G(t)],$$

where $\bar{G}$ is defined in (119). Term 2 takes the following form

$$\frac{\dot{\nu}_+^*}{h^2}\int_{t_i}^{t_{i+1}}\Big((q_+ - x_+^*)^\mathsf{T}(\bar{G}_+ - G_+) - (q-x^*)^\mathsf{T}(\bar{G}-G)\Big)dt$$
$$= \frac{\dot{\nu}_+^*}{h^2}\int_{t_i}^{t_{i+1}}\int_t^{t+h}\frac{d}{ds}\Big[(q(s)-x^*(s))^\mathsf{T}(\bar{G}(s)-G(s))\Big]dsdt.$$

Due to (47) and Smoothness, for any given $\gamma > 0$, there exists $r > 0$ such that

$$\|\dot{\bar{G}}_j - \dot{G}_j\|_{L^\infty} \le \gamma \quad \text{and} \quad \|\bar{G}_j - G_j\|_{L^\infty} \le c\|q-x^*\|_{L^\infty}$$

for all $x \in \mathrm{Lip}_\xi^1$ with $\|x - x^*\|_{H^1} \le r$. It follows that

$$|(\text{Term 2})_i| \le c\|q-x^*\|_{L^\infty}(\gamma + c\|q-x^*\|_{W^{1,\infty}}),$$

which by (46) can be made arbitrarily small for all $x \in \mathrm{Lip}_\xi^1$ with $\|x-x^*\|_{H^1} \le r$, by taking r sufficiently small.

Term 3. We decompose Term 3 further into two terms, the first being

$$\frac{\nu_+^*}{h^2}\int_{t_i}^{t_{i+1}}\Big((q_+ - x_+^*)^\mathsf{T}\dot{G}_+ - (q-x^*)^\mathsf{T}\dot{G}\Big)dt$$
$$= \frac{\nu_+^*}{h^2}\int_{t_i}^{t_{i+1}}\int_t^{t+h}\frac{d}{ds}[(q(s)-x^*(s))^\mathsf{T}\dot{G}(s)]\,dsdt.$$

By Smoothness, $\dot{G}$ is Lipschitz continuous, so we have

$$\|(q-x^*)^\mathsf{T}\dot{G}\|_{W^{1,\infty}} \le c\|q-x^*\|_{W^{1,\infty}},$$

which implies that

$$\left|\frac{\nu_+^*}{h^2}\int_{t_i}^{t_{i+1}}\int_t^{t+h}\frac{d}{ds}\Big[(q(s)-x^*(s))^\mathsf{T}\dot{G}(s)\Big]dsdt\right| \le c\|q-x^*\|_{W^{1,\infty}}. \tag{122}$$

By (47), this can be made arbitrarily small, for all $x \in \mathrm{Lip}_\xi^1$ with $\|x-x^*\|_{H^1} \le r$, by taking h and $r > 0$ sufficiently small.

The second part of Term 3 is

$$\frac{\nu_+^*}{h^2}\int_{t_i}^{t_{i+1}}\Big(\dot{q}_+^\mathsf{T}(G_+^I - G_+) - \dot{q}^\mathsf{T}(G^I - G)\Big)\,dt$$
$$= \frac{\nu_+^*}{h^2}\int_{t_i}^{t_{i+1}}\int_t^{t+h}\frac{d}{ds}\Big(\dot{q}(s)^\mathsf{T}[G^I(s)-G(s)]\Big)\,dsdt, \tag{123}$$

where $G_j^I(s) = \nabla^2 g_j(q(s))$. By Smoothness, for any given $\gamma > 0$, we can choose $r > 0$ such that

$$\|\nabla^2 g_j(q) - \nabla^2 g_j(x^*)\|_{W^{1,\infty}} \le \gamma, \quad j = 1, 2, \cdots, k,$$

for all $x \in \mathrm{Lip}_\xi^1$ with $\|x-x^*\|_{H^1} \le r$. Combining this with (47), the bound $|\ddot{q}| \le \xi$, and (122)–(123), we conclude that Term 3 can be made arbitrarily small, for all $x \in \mathrm{Lip}_\xi^1$ with $\|x - x^*\|_{H^1} \le r$, by taking $r > 0$ sufficiently small. ◻

Proof of Lemma 7.1. In [18, Lem. 3.5] we show that there exists $\bar{\beta} > 0$, subsets $J_1, J_2, \cdots, J_l$ of $\{1, 2, \cdots, k\}$, where $J_1 = \emptyset$, corresponding points $0 = \tau_1 < \tau_2 < \cdots < \tau_{l+1} = 1$, and a constant $0 < \eta < \min_q(\tau_{q+1} - \tau_q)$ such that whenever $t \in [\tau_q - \eta, \tau_{q+1} + \eta] \cap [0,1]$ for some $1 \le q \le l$, we have $\mathcal{I}(t) \subset J_q$ and

$$|\sum_{j \in J_q} v_j (K(t)B(t))_j| \ge \bar{\beta} |v_{J_q}|$$

for every choice of v. Since $K_i = K_{i+1} + O(h)$, let us choose $\bar{h} < \eta$ small enough that

$$|\sum_{j \in J_q} v_j (K_{i+1} B_i)_j| \ge .5\bar{\beta} |v_{J_q}|$$

for each $t_i \in [\tau_q - \eta, \tau_{q+1} + \eta] \cap [0,1]$ and $h \le \bar{h}$, and for every choice of v. Our approach is to enforce the following equations

$$(K_i x_i + b_i)_j = 0 \;\text{ for each }\; j \in J_q \setminus J_{q-1},\;\; t_i \in [\tau_q + \eta, \tau_{q+1}], \tag{124}$$

$$(K_i x_i + b_i)_j = 0 \;\text{ for each }\; j \in J_q \cap J_{q-1},\;\; t_i \in [\tau_q, \tau_{q+1}], \tag{125}$$

$q = 2,\ 3,\ \cdots,\ l$, where $L(x,u) + a = 0$, $x_0 = x^0$. Since J_1 is empty, (68) holds trivially on $[\tau_1, \tau_2] = [0, \tau_2]$. Suppose that $q > 1$, and let us consider (68) on the interval $[\tau_q, \tau_{q+1}]$. Since $\mathcal{I}(t_i) \subset J_q$ for $t_i \in [\tau_q, \tau_{q+1}]$, we conclude that any $j \in \mathcal{I}(t_i)$ is contained in either $J_q \cap J_{q-1}$ or $J_q \setminus J_{q-1}$. If $j \in J_q \cap J_{q-1}$, then by (125), (68) holds. If $j \in J_q \setminus J_{q-1}$, then by the construction in Lemma 9.1, $\mathcal{I}(t_i) \subset J_{q-1}$ for all $t_i \in [\tau_q, \tau_q + \eta]$. Hence, if $j \notin J_{q-1}$ then $j \notin \mathcal{I}(t_i) \subset J_{q-1}$, and (68) holds on $[\tau_q, \tau_q + \eta]$ since $j \notin \mathcal{I}(t_i)$. On the other hand, if $j \in J_q \setminus J_{q-1}$ and $t_i \in [\tau_q + \eta, \tau_{q+1}]$, then (68) holds by (124).

Observe that if

$$(K_i x_i + \sigma_i)_{J_q} = 0 \quad \text{for} \quad i = p, \tag{126}$$

where p is the smallest integer i such that $t_i \in [\tau_q, \tau_{q+1}]$, and if

$$(K_i x_i + \sigma_i)'_{J_q} = 0 \text{ for all } \; t_i \in [\tau_q, \tau_{q+1}], \tag{127}$$

then $(K_i x_i + \sigma_i)_{J_q} = 0$ for all $t_i \in [\tau_q, \tau_{q+1}]$. Carrying out the differencing in (127) and substituting for x_{i+1} using the state equation (56), we obtain a linear equation for u_i. By Lemma 9.1, this equation has a solution, and the minimum norm solution can be written:

$$u_i(x_i) = V_i[-\sigma_i' + K_{i+1} a_i - K_i' x_i - K_{i+1} A_i x_i]_{J_q}, \tag{128}$$

where

$$V_i = (K_{i+1}B_i)_{J_q}^{\mathsf{T}}[(K_{i+1}B_i)_{J_q}(K_{i+1}B_i)_{J_q}^{\mathsf{T}}]^{-1}.$$

(Recall that the J_q subscript attached to a matrix denotes the submatrix consisting of those rows associated with indices in J_q.) In the special case where J_q is empty, we simply set $u_i(x) = 0$.

Using these observations, we now explain how to construct x and u in order to satisfy (124) and (125). On the initial interval, $u_i = 0$ for each $t_i \in [0, \tau_2]$, and x_i is obtained from the state equation (56). Assuming the components of x and u have been determined on the interval $[0, \tau_q]$, their values on $[\tau_q, \tau_{q+1}]$ are obtained in the following way: The control is given in feedback form by (128), where for $j \in J_i \cap J_{i-1}$,

$$(\sigma_i)_j = (b_i)_j \quad \text{for} \quad t_i \in [\tau_q, \tau_{q+1}]. \tag{129}$$

For $j \in J_q \setminus J_{q-1}$, $(\sigma_i)_j = (b_i)_j$ for $t_i \in [\tau_q + \eta, \tau_{q+1}]$, while σ_j is linear in i on $[\tau_q, \tau_q + \eta]$ with

$$(\sigma_i)_j = -(K_i x_i)_j \quad \text{for} \quad i = p \quad \text{and} \quad (\sigma_i)_j = (b_i)_j \quad \text{for} \quad i = \bar{p}, \tag{130}$$

where $\bar{p}$ is the largest integer i such that $t_i \in [\tau_q, \tau_q + \eta]$. With this choice for σ, (126) is satisfied by (130) for $j \in J_q \setminus J_{q-1}$ and by induction for $j \in J_q \cap J_{q-1}$. With x and u given by (56) and (128) respectively, we have $(K_i x_i + \sigma_i)_{J_q} = 0$ for all $t_i \in [\tau_q, \tau_{q+1}]$ since (127) is satisfied. Also, by the choice of σ in (129),

$$(K_i x_i + \sigma_i)_j = (K_i x_i + b_i)_j = 0$$

for each $j \in J_q \cap J_{q-1}$ and $t_i \in [\tau_q, \tau_{q+1}]$, and for each $j \in J_q \setminus J_{q-1}$ and $t_i \in [\tau_q + \eta, \tau_{q+1}]$. Hence, (124) and (125) hold, which yields (68).

By the equations (56) for the state, (128) for the control, and (129)–(130) for σ, (x, u) is an affine function of (a, b). Moreover, the change $(\delta x, \delta u)$ in the state and control associated with the change $(\delta a, \delta b)$ in the parameters satisfies:

$$\|\delta x\|_{H^1([0,\tau_q])} + \|\delta u\|_{L^2([0,\tau_q])} \le c(\|\delta a\|_{L^2([0,\tau_q])} + \|\delta \sigma'\|_{L^2([0,\tau_q])}) \tag{131}$$

for each q, where σ is specified in (129)–(130). To complete the proof, we need to relate the σ term of (131) to the b term of (69). Note that $(\delta\sigma_i)_j = (\delta b_i)_j$ if $j \in J_q$ and $t_i \in [\tau_q + \eta, \tau_{q+1}]$ or if $j \in J_q \cap J_{q-1}$ and $t_i \in [\tau_q, \tau_q + \eta]$. Hence, for this j, we have

$$|(\delta\sigma_i')_j| = |(\delta b_i')_j|. \tag{132}$$

Assuming h is small enough that $2h < \eta/2$, we have for $j \in J_q \setminus J_{q-1}$ and $t_i \in [\tau_q, \tau_q + \eta]$,

$$\begin{aligned} |(\delta\sigma_i')_j| &\le (|(\delta b_{\bar{p}})_j| + |(K_p)_j \delta x_p|)/(\eta - 2h) \\ &\le (2/\eta)(|(\delta b_{\bar{p}})_j| + |(K_p)_j \delta x_p|) \\ &\le c(\|\delta b\|_{L^\infty} + |\delta x_p|) \le c(\|\delta b\|_{H^1} + |\delta x_p|). \end{aligned}$$

Combining this with (132), we have, for each $t_i \in [\tau_q, \tau_{q+1}]$,

$$|\delta\sigma_i'| \le c(\|\delta b\|_{H^1} + |\delta x_p| + |\delta b_i'|). \tag{133}$$

We obtain (69) by an inductive argument. Initially, on the interval $[\tau_1, \tau_2] = [0, \tau_2]$, $p = 0$ and $\delta x_0 = 0$ due to the initial condition. Hence, (131) and (133) give us

$$\|\delta x\|_{H^1([0,\tau_2])} + \|\delta u\|_{L^2([0,\tau_2])} \le c(\|\delta a\|_{L^2([0,\tau_2])} + \|\delta b\|_{H^1([0,\tau_2])}).$$

Proceeding by induction, suppose that

$$\|\delta x\|_{H^1([0,t_q])} + \|\delta u\|_{L^2([0,t_q])} \le c(\|\delta a\|_{L^2([0,t_q])} + \|\delta b\|_{H^1([0,t_q])}),$$

and let p be the smallest integer i such that $t_i \in [\tau_q, \tau_{q+1}]$. Since $|\delta x_p| \le \|\delta x\|_{H^1([0,t_q])}$, it follows that

$$|\delta x_p| \le c(\|\delta a\|_{L^2([0,t_q])} + \|\delta b\|_{H^1([0,t_q])}).$$

Combining this with (133) and utilizing (131) with q replaced by $q+1$, the induction is complete. □