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# UNIFORM CONVERGENCE AND MESH INDEPENDENCE OF NEWTON'S METHOD FOR DISCRETIZED VARIATIONAL PROBLEMS*

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**Abstract.** In an abstract framework, we study local convergence properties of Newton's method for a sequence of generalized equations which models a discretized variational inequality. We identify conditions under which the method is locally quadratically convergent, uniformly in the discretization. Moreover, we show that the distance between the Newton sequence for the continuous problem and the Newton sequence for the discretized problem is bounded by the norm of a residual. As an application, we present mesh-independence results for an optimal control problem with control constraints.

**Key words.** Newton's method, variational inequality, optimal control, sequential quadratic programming, discrete approximation, mesh independence

**AMS subject classifications.** 49M25, 65J15, 65K10

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## 1. Introduction.

In this paper we study local convergence properties of Newton-type methods applied to discretized variational problems. Our target problem is the variational inequality representing the first-order optimality conditions in constrained optimal control. In an abstract framework, the optimality conditions are modeled by a "generalized equation," a term coined by S. Robinson [12], where the normal cone mapping is replaced by an arbitrary map with closed graph. In this setting, Newton's method solves at each step a linearized generalized equation. When the generalized equation describes first-order optimality conditions, Newton's method becomes the well-known sequential quadratic programming (SQP) method.

We identify conditions under which Newton's method is not only locally quadratically convergent, but the convergence is uniform with respect to the discretization. Moreover, we derive an estimate for the number of steps required to achieve a given accuracy. Under some additional assumptions, which are natural in the context of the target problem, we prove that the distance between the Newton sequence for the continuous problem and the Newton sequence for the discretized problem, measured in the discrete metric, can be estimated by the norm of a residual. Normally, the residual tends to zero when the approximation becomes finer, and the two Newton sequences approach each other. In the context of the target optimal control problem, the residual is proportional to the mesh spacing $h$, uniformly along the Newton sequence. In particular, this implies that the least number of steps needed to reach a point at distance $\varepsilon$ from the solution of the discrete problem does not depend on the mesh spacing; that is, the method is mesh independent.

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The convergence of the SQP method applied to nonlinear optimal control problems has been studied in several papers recently. In [5, 6] we proved local convergence of the method for a class of constrained optimal control problems. In parallel, Alt and Malanowski obtained related results for state constrained problems [3]. Along the same lines, Tröltzsch [13] studied the SQP method for a problem involving a parabolic partial differential equation.

Kelley and Sachs [10] were the first to obtain a mesh-independence result in constrained optimal control; they studied the gradient projection method. More recently, uniform convergence and mesh-independence results for an augmented Lagrangian version of the SQP method, applied to a discretization of an abstract optimization problem with equality constraints, were presented by Kunisch and Volkwein [11]. Alt [2] studied the mesh independence of Newton's method for generalized equations in the framework of the analysis of operator equations in Allgower et al. [1]. An abstract theory of mesh independence for infinite-dimensional optimization problems with equality constraints, together with applications to optimal control of partial differential equations and an extended survey of the field, can be found in the thesis of Volkwein [14].

The local convergence analysis of numerical procedures is closely tied to the problem's stability. The analysis is complicated for optimization problems with inequality constraints or for related variational inequalities. In this context, the problem solution typically depends on perturbation parameters in a nonsmooth way. In section 2 we present an implicit function theorem which provides a basis for our further analysis. In section 3 we obtain a result on uniform convergence of Newton's method applied to a sequence of generalized equations, while section 4 presents our mesh-independence results. Although in part parallel, our approach is different from the one used by Alt in [2], who adopted the framework of [1]. First, we study the uniform local convergence of Newton's method, which is not considered in [2]. In the mesh-independence analysis, we avoid consistency conditions for the solutions of the continuous and the discretized problems; instead, we consider the residual obtained when the Newton sequence of the continuous problem is substituted into the discrete necessary conditions. This allows us to obtain mesh independence under conditions weaker than those in [2] and, at the same time, to significantly simplify the analysis.

In section 5 we apply the abstract results to the constrained optimal control problem studied in our previous paper [5]. We show that under the smoothness and coercivity conditions given in [5] and assuming that the optimal control of the continuous problem is a Lipschitz continuous function of time, the SQP method applied to the discretized problem is $Q$-quadratically convergent, and the region of attraction and the constant of the convergence are independent of discretization, for a sufficiently small mesh size. Moreover, the $l_\infty$ distance between the Newton sequence for the continuous problem at the mesh points and the Newton sequence for the discretized problem is of order $O(h)$. In particular, this estimate implies the mesh-independence result in Alt [2].

## 2. Lipschitzian localization.

Let $X$ and $Y$ be metric spaces. We denote both metrics by $\rho(\cdot,\cdot)$; it will be clear from the context which metric we are using. $B_r(x)$ denotes the closed ball with center $x$ and radius $r$. In writing "$f$ maps $X$ into $Y$" we adopt the convention that the domain of $f$ is a (possibly proper) subset of $X$. Accordingly, a set-valued map $F$ from $X$ to the subsets of $Y$ may have empty values.

DEFINITION 2.1. *Let $\Gamma$ map $Y$ to the subsets of $X$ and let $x^* \in \Gamma(y^*)$. We say that $\Gamma$ has a Lipschitzian localization with constants $a$, $b$, and $M$ around $(y^*, x^*)$, if* *the map $y \mapsto \Gamma(y) \cap B_a(x^*)$ is single valued (a function) and Lipschitz continuous in $B_b(y^*)$ with a Lipschitz constant $M$.*

THEOREM 2.1. *Let $G$ map $X$ into the subsets of $Y$ and let $y^* \in G(x^*)$. Let $G^{-1}$ have a Lipschitzian localization with constants $a$, $b$, and $M$ around $(y^*, x^*)$. In addition, suppose that the intersection of the graph of $G$ with $B_a(x^*) \times B_b(y^*)$ is closed and $B_a(x^*)$ is complete. Let the real numbers $\lambda$, $\bar{M}$, $\bar{a}$, $m$, and $\delta$ satisfy the relations*

$$\lambda M < 1, \quad \bar{M} = \frac{M}{1 - \lambda M}, \quad m + \delta < b, \quad \text{and} \quad \bar{a} + \bar{M}\delta < a. \tag{1}$$

*Suppose that the function $g : B_a(x^*) \mapsto Y$ is Lipschitz continuous with a constant $\lambda$ in the ball $B_a(x^*)$, that*

$$\sup_{x \in B_a(x^*)} \rho(g(x), y^*) \le m, \tag{2}$$

*and that the set*

$$\Delta := \{x \in B_{\bar{a}}(x^*) : \operatorname{dist}(g(x), G(x)) \le \delta\} \tag{3}$$

*is nonempty.*

*Then the set $\{x \in B_a(x^*) \mid g(x) \in G(x)\}$ consists of exactly one point, $\hat{x}$, and for each $x' \in \Delta$ we have*

$$\rho(x', \hat{x}) \le \bar{M} \operatorname{dist}(g(x'), G(x')). \tag{4}$$

*Proof.* Let us choose positive $\lambda, \bar{M}, m, \bar{a}$, and $\delta$ such that the relations in (1) hold. We first show that the set $T := \{x \in B_a(x^*) \mid g(x) \in G(x)\}$ is nonempty. Let $x' \in \Delta$ and put $x_0 = x'$. Take an arbitrary $\varepsilon > 0$ such that

$$m + \delta + \varepsilon \le b \quad \text{and} \quad \bar{a} + \bar{M}(\delta + \varepsilon) \le a.$$

Choose an $y' \in G(x')$ such that $\rho(y', g(x')) \le \operatorname{dist}(g(x'), G(x')) + \varepsilon$. Since

$$\rho(y', y^*) \le \rho(y^*, g(x')) + \operatorname{dist}(g(x'), G(x')) + \varepsilon \le m + \delta + \varepsilon \le b$$

and

$$\rho(g(x_0), y^*) \le m \le b,$$

from the Lipschitzian localization property, there exists $x_1$ such that

$$g(x_0) \in G(x_1), \quad \rho(x_1, x_0) \le M\rho(y', g(x_0)) \le M(\operatorname{dist}(g(x'), G(x')) + \varepsilon). \tag{5}$$

We define inductively a sequence $x_k$ in the following way. Let $x_0, \dots, x_k$ be already defined for some $k \ge 1$ in such a way that

$$\rho(x_i, x_{i-1}) \le (\lambda M)^{i-1} \rho(x_1, x_0), \quad i = 1, \dots, k, \tag{6}$$

and

$$g(x_{k-1}) \in G(x_k). \tag{7}$$

Clearly, $x_0$ and $x_1$ satisfy these relations. Using the second inequality in (5), we estimate

$$\begin{aligned} \rho(x_i, x^*) &\le \rho(x_0, x^*) + \sum_{j=1}^{i} \rho(x_j, x_{j-1}) \le \rho(x', x^*) + \sum_{j=0}^{\infty} (\lambda M)^j \rho(x_1, x_0) \\ &\le \bar{a} + \frac{1}{1 - \lambda M} M(\operatorname{dist}(g(x'), G(x')) + \varepsilon) \le \bar{a} + \bar{M}(\delta + \varepsilon) \le a. \end{aligned}$$

Thus both $x_{k-1}$ and $x_k$ are in $B_a(x^*)$ from which we obtain by (2)

$$\rho(g(x_i), y^*) \leq m \leq b$$

for $i = k-1$ and for $i = k$. Due to the assumed Lipschitzian localization property of $G$, there exists $x_{k+1}$ such that (7), with $k$ replaced by $k+1$, is satisfied and

$$\rho(x_{k+1}, x_k) \leq M\rho(g(x_k), g(x_{k-1})).$$

By (6) we obtain

$$\rho(x_{k+1}, x_k) \leq M\lambda\rho(x_k, x_{k-1}) \leq (\lambda M)^k \rho(x_1, x_0),$$

and hence (6) with $k$ replaced by $k+1$, is satisfied. The definition of the sequence $x_k$ is complete.

From (6) and the condition $\lambda M < 1$, $\{x_k\}$ is a Cauchy sequence. Since all $x_k \in B_a(x^*)$, the sequence $\{x_k\}$ has a limit $x'' \in B_a(x^*)$. Passing to the limit in (7), we obtain $g(x'') \in G(x'')$. Hence $x'' \in T$ and the set $T$ is nonempty. Note that $x''$ may depend on the choice of $\varepsilon$. If we prove that the set $T$ is a singleton, say $\hat{x}$, the point $x'' = \hat{x}$ would not depend on $\varepsilon$.

Suppose that there exist $x'' \in T$ and $\bar{x}'' \in T$ with $\rho(x'', \bar{x}'') > 0$. It follows that $\rho(g(x), y^*) \leq m \leq b$ for $x = x''$ and $x = \bar{x}''$. From the definition of the Lipschitzian localization, we obtain

$$\rho(x'', \bar{x}'') \leq M\rho(g(x''), g(\bar{x}'')) \leq M\lambda\rho(x'', \bar{x}'') < \rho(x'', \bar{x}''),$$

which is a contradiction. Thus $T$ consists of exactly one point, $\hat{x}$, which does not depend on $\varepsilon$. To prove (4) observe that for any choice of $k > 1$,

$$\begin{aligned}\rho(x', x'') &\leq \rho(x_0, x_k) + \rho(x_k, x'') \leq \sum_{i=0}^{k-1} \rho(x_{i+1}, x_i) + \rho(x_k, x'') \\ &\leq \sum_{i=0}^{k-1} (\lambda M)^i \rho(x_1, x_0) + \rho(x_k, x'').\end{aligned}$$

Passing to the limit in the latter inequality and using (5), we obtain

$$\rho(x', x'') \leq \bar{M}(\operatorname{dist}(g(x'), G(x')) + \varepsilon). \tag{8}$$

Since $x'' = \hat{x}$ does not depend on the choice of $\varepsilon$, one can take $\varepsilon = 0$ in (8) and the proof is complete. □

**3. Newton's method.** Theorem 2.1 provides a basis for the analysis of the error of approximation and the convergence of numerical procedures for solving variational problems. In this and the following section we consider a sequence of so-called generalized equations. Specifically, for each $N = 1, 2, \ldots$, let $X^N$ be a closed and convex subset of a Banach space, let $Y^N$ be a linear normed space, let $f_N : X^N \mapsto Y^N$ be a function, and let $F_N : X^N \mapsto 2^{Y^N}$ be a set-valued map with closed graph. We denote by $\|\cdot\|_N$ the norms of both $X^N$ and $Y^N$. We study the following sequence of problems:

$$\text{Find } x \in X^N \text{ such that } 0 \in f_N(x) + F_N(x). \tag{9}$$

We assume that there exist constants $\alpha$, $\beta$, $\gamma$, and $L$, as well as points $x_N^* \in X^N$ and $z_N^* \in Y^N$, that satisfy the following conditions for each $N$:

(A1) $z_N^* \in f_N(x_N^*) + F_N(x_N^*)$.
(A2) The function $f_N$ is Frechét differentiable in $B_\alpha(x_N^*)$ and the derivative $\nabla f_N$ is Lipschitz continuous in $B_\alpha(x_N^*)$ with a Lipschitz constant $L$.
(A3) The map

$$y \mapsto \left( f_N(x_N^*) + \nabla f_N(x_N^*)(\cdot - x_N^*) + F_N(\cdot) \right)^{-1}(y)$$

has a Lipschitzian localization with constants $\alpha$, $\beta$, and $\gamma$ around the point $(z_N^*, x_N^*)$.

We study the Newton method for solving (9) for a fixed $N$ which has the following form: If $x^k$ is the current iterate, the next iterate $x^{k+1}$ satisfies

$$0 \in f_N(x^k) + \nabla f_N(x^k)(x^{k+1} - x^k) + F_N(x^{k+1}), \quad k = 0, 1, \ldots, \tag{10}$$

where $x^0$ is a given starting point. If the range of the map $F$ is just the origin, then (9) is an equation and (10) becomes the standard Newton method. If $F$ is the normal cone mapping in a variational inequality describing first-order optimality conditions, then (10) represents the first-order optimality condition for the auxiliary quadratic program associated with the SQP method.

In the following theorem, by applying Theorem 2.1, we obtain the existence of a locally unique solution of the problem (9) which is at a distance from the reference point proportional to the norm of the residual $z_N^*$. We also show that the method (10) converges $Q$-quadratically and this convergence is uniform in $N$ and in the choice of the initial point from a ball around the reference point $x_N^*$ with radius independent of $N$. Note that for obtaining this result we do not pass to a limit and consequently we do not need to consider sequences of generalized equations.

THEOREM 3.1. *For every $\gamma' > \gamma$ there exist positive constants $\kappa$ and $\sigma$ such that if $\|z_N^*\| \le \sigma$, then the generalized equation* (9) *has a unique solution $x_N$ in $B_\kappa(x_N^*)$; moreover, $x_N$ satisfies*

$$\|x_N - x_N^*\|_N \le \gamma' \|z_N^*\|_N. \tag{11}$$

*Furthermore, for every initial point $x^0 \in B_\kappa(x_N^*)$ there is a unique Newton sequence $\{x^k\}$, with $x^k \in B_\kappa(x_N^*)$, $k = 1, 2, \ldots,$ and this Newton sequence is $Q$-quadratically convergent to $x_N$, that is,*

$$\|x^{k+1} - x_N\|_N \le \Theta \|x^k - x_N\|_N^2, \quad k = 0, 1, \ldots, \tag{12}$$

*where $\Theta$ is independent of $k, N$ and $x^0 \in B_\kappa(x_N^*)$.*

*Proof.* Define

$$\kappa = \min\left\{\alpha, \gamma\beta, \frac{\gamma' - \gamma}{L\gamma\gamma'}, \frac{1}{5L\gamma'}\right\}, \quad \sigma = \frac{1}{\gamma'} \min\left\{\frac{\kappa}{4}, \sqrt{\frac{\kappa}{3L\gamma'}}, \frac{1}{6L\gamma'}\right\}, \quad \Theta = \frac{\gamma' L}{2}.$$

We will prove the existence and uniqueness of $x_N$ by using Theorem 2.1 with

$$a = \kappa,\ b = \kappa/\gamma,\ M = \gamma,\ \lambda = \kappa L,\ \bar{M} = \gamma',\ \bar{a} = 0,\ m = \kappa^2 L/2 + \sigma,\ \delta = \sigma$$

and

$$g(x) = -f_N(x) + f_N(x_N^*) + \nabla f_N(x_N^*)(x - x_N^*),$$
$$G(x) = f_N(x_N^*) + \nabla f_N(x_N^*)(x - x_N^*) + F_N(x).$$

Observe that $a \le \alpha, b \le \beta$ and $\gamma b \le a$. By (A3) the map $G$ has a Lipschitzian localization around $(x_N^*, z_N^*)$ with constants $a$, $b$, and $\gamma$. One can check that the relations (1) are satisfied. Further, for $x, x'$, and $x'' \in B_\kappa(x_N^*)$, we have

$$\|g(x) - z_N^*\|_N \le \|z_N^*\|_N + L\|x - x_N^*\|_N^2/2 \le \sigma + L\kappa^2/2 = m,$$

$$\begin{aligned}\|g(x') - g(x'')\|_N &\le \| - f_N(x') + f_N(x'') + \nabla f(x_N^*)(x' - x'')\|_N \\ &\le L\kappa\|x' - x''\|_N = \lambda\|x' - x''\|_N,\end{aligned}$$

$$\mathrm{dist}(g(x_N^*), G(x_N^*)) = \mathrm{dist}(0, f_N(x_N^*) + F(x_N^*)) \le \|z_N^*\|_N \le \sigma = \delta.$$

Obviously, $x_N^* \in B_0(x_N^*)$ and $x_N^* \in \Delta$, with $\Delta$ defined in (3). The assumptions of Theorem 2.1 are satisfied; hence there exists a unique $x_N$ in $B_\kappa(x_N^*)$ for which $g(x_N) \in G(x_N)$. Hence $x_N$ is a unique solution of (9) in $B_\kappa(x_N^*)$ and (11) holds. This completes the first part of the proof.

Given $x^k \in B_\kappa(x_N^*)$, a point $x$ is a Newton step from $x^k$ if and only if $x$ satisfies the inclusion

$$g(x) \in G(x), \tag{13}$$

where $G$ is the same as above, but now

$$g(x) = -f_N(x^k) - \nabla f_N(x^k)(x - x^k) + f_N(x_N^*) + \nabla f_N(x_N^*)(x - x_N^*).$$

The proof will be completed if we show that (13) has a unique solution $x^{k+1}$ in $B_\kappa(x_N^*)$ and this solution satisfies (12). To this end we apply again Theorem 2.1 with $a, b, M, \bar{M}$, and $\lambda$ the same as in the first part of the proof and with

$$\bar{a} = \sigma\gamma', \quad m = \sigma + \frac{5L\kappa^2}{2}, \quad \delta = \frac{L}{2}(\gamma'\sigma + \kappa)^2.$$

With these identifications, it can be checked that the assumptions (1) and (2) hold, and that $g$ is Lipschitz continuous in $B_\kappa(x_N^*)$ with a Lipschitz constant $\lambda$. Further, by using the solution $x_N$ obtained in the first part of the proof, we have

$$\begin{aligned}\mathrm{dist}(g(x_N), G(x_N)) &= \mathrm{dist}(0, f_N(x^k) + \nabla f_N(x^k)(x_N - x^k) + F_N(x_N)) \\ &\le \frac{L}{2}\|x_N - x^k\|_N^2 + \mathrm{dist}(0, f(x_N) + F_N(x_N)) = \frac{L}{2}\|x_N - x^k\|_N^2. \qquad (14)\end{aligned}$$

The last expression has the estimate

$$\frac{L}{2}\|x_N - x^k\|_N^2 \le \frac{L}{2}\bigg(\|x_N - x_N^*\|_N + \|x_N^* - x^k\|_N\bigg)^2 \le \frac{L}{2}(\gamma'\sigma + \kappa)^2 = \delta.$$

Thus $x_N \in \Delta \ne \emptyset$ and the assumptions of Theorem 2.1 are satisfied. Hence, there exists a unique Newton step $x^{k+1}$ in $B_\kappa(x_N^*)$ and by Theorem 2.1 and (14) it satisfies

$$\|x^{k+1} - x_N\|_N \le \frac{\gamma' L}{2}\|x^k - x_N\|_N^2 = \Theta\|x^k - x_N\|_N^2. \qquad \square$$

**4. Mesh independence.** Consider the generalized equation (9) under the assumptions (A1)–(A3). We present first a lemma in which, for simplicity, we suppress the dependence of $N$.

LEMMA 4.1. *For every $\gamma' > \gamma$, every $\mu > 0$, and every sufficiently small $\xi > 0$, there exists a positive $\eta$ such that the map*

$$(y, w) \mapsto P(y, w) := (f(w) + \nabla f(w)(\cdot - w) + F(\cdot))^{-1}(y) \cap B_\alpha(x^*) \tag{15}$$

*is a Lipschitz continuous function from $B_\eta(z^*) \times B_\xi(x^*)$ into $B_\xi(x^*)$ with Lipschitz constants $\gamma'$ for $y$ and $\mu$ for $w$.*

*Proof.* Let $\gamma' > \gamma$ and $\mu > 0$. We choose the positive constants $\xi$ and $\eta$ as a solution of the following system of inequalities:

$$\gamma L\xi < 1, \quad \xi \leq \frac{\gamma - \gamma'}{\gamma\gamma' L}, \quad 3\eta + \frac{15}{2}L\xi^2 + L\xi\alpha \leq \beta,$$

$$\xi + \gamma'(2\eta + 6L\xi^2) \leq \alpha, \quad 3L\xi\gamma' \leq \mu, \quad \gamma'(\eta + 3L\xi^2) \leq \xi.$$

This system of inequalities is satisfied by first taking $\xi$ sufficiently small and then taking $\eta$ sufficiently small. In particular, we have $\xi \leq \alpha$ and $\eta \leq \beta$.

Take $(y'', w'') \in B_\eta(z^*) \times B_\xi(x^*)$. We apply Theorem 2.1 with $a = \alpha$, $b = \beta$, $M = \gamma$, $\bar{a} = \xi$, $\bar{b} = \eta$, $\bar{M} = \gamma'$, $\lambda = L\xi$, $m = \eta + \frac{3}{2}L\xi^2 + L\xi\alpha$, $\delta = 2\eta + 6L\xi^2$,

$$g(x) = y'' + f(x^*) + \nabla f(x^*)(x - x^*) - f(w'') - \nabla f(w'')(x - w''),$$

and

$$G(x) = f(x^*) + \nabla f(x^*)(x - x^*) + F(x).$$

We have

$$\begin{aligned} \|g(x_1) - g(x_2)\| &= \|(\nabla f(x^*) - \nabla f(w''))(x_1 - x_2)\| \\ &\leq L\|w'' - x^*\|\|x_1 - x_2\| \leq L\xi\|x_1 - x_2\| \end{aligned}$$

for all $x_1, x_2 \in B_\alpha(x^*)$. Hence the function $g$ is Lipschitz continuous with a Lipschitz constant $\lambda$. For $x \in B_\alpha(x^*)$ we have

$$\begin{aligned} \|g(x) - z^*\| &\leq \|y'' - z^*\| + \|f(w'') - f(x^*) - \nabla f(x^*)(w'' - x^*)\| \\ &\quad + \|(\nabla f(x^*) - \nabla f(w''))(x - w'')\| \\ &\leq \eta + \frac{L}{2}\|w'' - x^*\|^2 + L\|w'' - x^*\|\|x - w''\| \\ &\leq \eta + \frac{1}{2}L\xi^2 + L\xi(\xi + \alpha) = m. \end{aligned}$$

Note that a point $x$ is in the set $P(y'', w'')$ if and only if $g(x) \in G(x)$. Since

$$\begin{aligned} \operatorname{dist}(g(x^*), G(x^*)) &\leq \|y'' - z^*\| + \operatorname{dist}(z^* - f(w'') - \nabla f(w'')(x^* - w''), F(x^*)) \\ &\leq \eta + \operatorname{dist}(z^*, f(x^*) + F(x^*)) + \frac{L}{2}\|x^* - w''\|^2 \leq \eta + \frac{L}{2}\xi^2 < \delta, \end{aligned}$$

the set $\Delta$ defined in (3) is not empty. Hence, from Theorem 2.1 the set $P(y'', w'') \cap B_\alpha(x^*)$ consists of exactly one point. Let us call it $x''$. Applying the same argument to an arbitrary point $(y', w') \in B_\eta(z^*) \times B_\xi(x^*)$, we obtain that there is exactly one point $x' \in P(y', w') \cap B_\alpha(x^*)$. Furthermore,

$$\begin{aligned}
\operatorname{dist}(g(x'), G(x')) &\le \|y' - y''\| + \|f(w'') - \nabla f(w'')(x' - w'') - f(w') - \nabla f(w')(x' - w')\| \\
&\le \|y' - y''\| + \|f(w'') - f(w') - \nabla f(w')(w'' - w')\| \\
&\qquad\qquad + \|\nabla f(w'') - \nabla f(w')\|\|x' - w''\| \\
&\le \|y' - y''\| + \frac{L}{2}\|w' - w''\|^2 + 2L\xi\|w' - w''\| \\
&\le \|y' - y''\| + 3L\xi\|w' - w''\|.
\end{aligned}$$

Hence $x' \in \Delta$ and we obtain

$$\rho(x', x'') \le \gamma'(\|y' - y''\| + 3L\xi\|w' - w''\|) \le \gamma'\|y' - y''\| + \mu\|w' - w''\|.$$

It remains to prove that $P$ maps $B_\eta(z^*) \times B_\xi(x^*)$ into $B_\xi(x^*)$. From the last inequality with $x' = x^*$ and $w' = x^*$, we have

$$\rho(x'', x^*) \le \gamma'(\|y'' - z^*\| + 3L\xi\|w'' - x^*\|) \le \gamma'(\eta + 3L\xi^2) \le \xi.$$

Thus $x'' \in B_\xi(x^*)$. ◻

In the remaining part of this section, we fix $\gamma' > \gamma$ and $0 < \mu < 1$, and we choose the constants $\kappa$ and $\sigma$ according to Theorem 3.1. For a positive $\xi$ with $\xi \le \kappa$, let $\eta$ be the constant whose existence is claimed in Lemma 4.1. Note that $\eta$ can be chosen arbitrarily small; we take $0 < \eta \le \sigma$. Also, we assume that $\|z_N^*\| \le \eta$ and consider Newton sequences with initial points $x^0 \in B_\xi(x_N^*)$. In such a way, the assumptions of Theorem 3.1 hold and we have a unique Newton sequence which is convergent quadratically to a solution.

Suppose that Newton's method (10) is supplied with the following stopping test: Given $\varepsilon > 0$, at the $k$th step the point $x^k$ is accepted as an approximate solution if

$$\operatorname{dist}(0, f_N(x^k) + F_N(x^k)) < \varepsilon. \tag{16}$$

Denote by $k_\varepsilon$ the first step at which the stopping test (16) is satisfied.

THEOREM 4.1. *For any positive $\varepsilon < \eta$, if $x^{k_\varepsilon}$ is the approximate solution obtained using the stopping test* (16) *at the step $k = k_\varepsilon$, then*

$$\|x^{k_\varepsilon} - x_N\|_N \le \frac{\gamma'\varepsilon}{1 - \mu} \tag{17}$$

*and*

$$k_\varepsilon \le 2 + \frac{1}{2}\log_\mu\left(\frac{\varepsilon}{2L\xi^2}\right). \tag{18}$$

*Proof.* Choose an $\varepsilon$ such that $0 < \varepsilon < \eta$. If the stopping test (16) is satisfied at $x^{k_\varepsilon}$, then there exists $v_\varepsilon^k$ with $\| v_\varepsilon^k \|_N \le \varepsilon$ such that

$$v_\varepsilon^k \in f_N(x^{k_\varepsilon}) + F_N(x^{k_\varepsilon}).$$

Let $P^N$ be defined as in (15) on the basis of $f_N$ and $F_N$. Since

$$x^{k_\varepsilon} = P^N(v_\varepsilon^k, x^{k_\varepsilon}) \quad \text{and} \quad x_N \in P^N(0, x_N),$$

Lemma 4.1 implies that

$$\|x^{k_\varepsilon} - x_N\|_N \leq \gamma' \parallel v_\varepsilon^k \parallel_N + \mu \|x^{k_\varepsilon} - x_N\|_N.$$

The latter inequality yields (17). For all $k < k_\varepsilon$, we obtain

$$\varepsilon \leq \mathrm{dist}(0, f_N(x^k) + F_N(x^k)).$$

Since $x^k$ is a Newton iterate, we have

$$f_N(x^k) - f_N(x^{k-1}) - \nabla f_N(x^{k-1})(x^k - x^{k-1}) \in f_N(x^k) + F_N(x^k).$$

Hence

$$\begin{aligned} \mathrm{dist}(0, f_N(x^k) + F_N(x^k)) &\leq \parallel f_N(x^k) - f_N(x^{k-1}) - \nabla f_N(x^{k-1})(x^k - x^{k-1}) \parallel_N \\ &\leq L\|x^k - x^{k-1}\|_N^2/2. \end{aligned} \tag{19}$$

By the definition of the map $P^N$, the Newton step $x^1$ from $x^0$ satisfies

$$x^1 = P^N(0, x^0),$$

while the Newton step $x^2$ from $x^1$ is

$$x^2 = P^N(0, x^1).$$

Since $P^N$ is Lipschitz continuous with a constant $\mu$, we have

$$\|x^2 - x^1\|_N \leq \mu \|x^1 - x^0\|_N.$$

By induction, the $(k+1)$st Newton step $x^{k+1}$ satisfies

$$\|x^{k+1} - x^k\|_N \leq \mu^k \|x^1 - x^0\|_N. \tag{20}$$

Combining (19) and (20) and we obtain the estimate

$$\varepsilon \leq 2L\xi^2 \mu^{2(k-1)},$$

which yields (18). ∎

Our next result provides a basis for establishing the mesh independence of Newton's method (10). Namely, we compare the Newton sequence $x_N^k$ for the "discrete" problem (9) and the Newton sequence for a "continuous" problem which is again described by (9) but with index $N = 0$. Let us assume that the conditions (A1)–(A3) hold for the generalized equation (9) with $N = 0$. According to Theorem 3.1, for each starting point $x_0^0$ close to a solution $x_0$, there is a unique Newton sequence $x_0^k$ which converges to $x_0$ $Q$-quadratically. To relate the continuous problem to the discrete one, we introduce a mapping $\pi_N$ from $X^0$ to $X^N$. Having in mind the application to optimal control presented in the following section, $X^0$ can be thought as a space of continuous functions $x(\cdot)$ in $[0, 1]$ and, for a given natural number $N$, $t_0 = 0$ and $t_i = i/N$, $X^N$ will be the space of sequences $\{x_i, i = 0, 1, \ldots, N\}$. In this case the operator $\pi_N$ is the interpolation map $\pi_N(x(\cdot)) = (x(t_0), \ldots, x(t_N))$.

THEOREM 4.2. *Suppose that for every $k$ and $N$ there exists $r_N^k \in Y^N$ such that*

$$r_N^k \in f_N(\pi_N(x_0^k)) + \nabla f_N(\pi_N(x_0^k))(\pi_N(x_0^{k+1}) - \pi_N(x_0^k)) + F_N(\pi_N(x_0^{k+1}))$$

*and*

$$\omega_N := \sup_k \| r_N^k \|_N < \eta. \tag{21}$$

*In addition, let*

$$\|\pi_N(x_0^k) - x_N^*\|_N \leq \xi$$

*for all $k$ and $N$. Then for all $k = 1, 2, \ldots$ and $N$*

$$\|x_N^{k+1} - \pi_N(x_0^{k+1})\|_N \leq \frac{\gamma'}{1-\mu}\omega_N + \mu^{k+1}\|x_N^0 - \pi_N(x_0^0)\|_N. \tag{22}$$

*Proof.* By definition, we have

$$\pi_N(x_0^{k+1}) = P^N(r_N^k, \pi_N(x_0^k)) \quad \text{and} \quad x_N^{k+1} = P^N(0, x_N^k).$$

Using Lemma 4.1 we have

$$\|x_N^{k+1} - \pi_N(x_0^{k+1})\|_N \leq \gamma' \| r_N^k \|_N + \mu\|x_N^k - \pi_N(x_0^k)\|_N \leq \gamma'\omega_N + \mu\|x_N^k - \pi_N(x_0^k)\|_N.$$

By induction we obtain (22). ∎

The above result means that, under our assumptions, the distance between the Newton sequence for the continuous problem and the Newton sequence for the discretized problem, measured in the discrete metric, can be estimated by the sup-norm $\omega_N$ of the residual obtained when the Newton sequence for the continuous problem is inserted into the discretized generalized equations. If the sup-norm of the residual tends to zero when the approximation becomes finer, that is, when $N \to \infty$, then the two Newton sequences approach each other. In the next section, we will present an application of the abstract analysis to an optimal control problem for which the residual is proportional to the mesh spacing $h$, uniformly along the Newton sequence. For this particular problem Theorem 4.2 implies that the distance between the Newton sequences for the continuous problem and the Newton sequence for the discretized problem is $O(h)$.

For simplicity, let us assume that if the continuous Newton process starts from the point $x_N^0$, then the discrete Newton process starts from $\pi_N(x_0^0)$. Also, suppose that for any fixed $w, v \in X^0$,

$$\|\pi_N(w) - \pi_N(v)\|_N \to \|w - v\|_0 \quad \text{as} \quad N \to \infty. \tag{23}$$

In addition, let

$$\omega_N \to 0 \quad \text{as} \quad N \to \infty, \tag{24}$$

where $\omega_N$ is defined in (21). Letting $k$ tend to infinity and assuming that $\pi_N$ is a continuous mapping for each $N$, Theorem 4.2 gives us the following estimate for the distance between the solution $x_N$ of the discrete problem and the discrete representation $\pi_N(x_0)$ of the solution $x_0$ of the continuous problem:

$$\|x_N - \pi_N(x_0)\|_N \leq \frac{\gamma'}{1-\mu}\omega_N. \tag{25}$$

Choose a real number $\varepsilon$ satisfying

$$0 < \varepsilon < 1/(5\Theta), \tag{26}$$

where $\Theta$ is as in Theorem 3.1. Theorem 4.2 yields the following result.

THEOREM 4.3. *Let* (23) *and* (24) *hold and let* $\varepsilon$ *satisfy* (26). *Then for all* $N$ *sufficiently large,*

$$|\min\left\{k \in \mathbf{N} : \|x_0^k - x_0\|_0 < \varepsilon\right\} - \min\left\{k \in \mathbf{N} : \|x_N^k - x_N\|_N < \varepsilon\right\}| \le 1. \tag{27}$$

*Proof.* Let $m$ be such that

$$\|x_0^{m+1} - x_0\|_0 < \varepsilon \le \|x_0^m - x_0\|_0. \tag{28}$$

Choose $N$ so large that

$$\frac{\gamma'}{1-\mu}\omega_N < \varepsilon/2$$

and

$$\|\pi_N(x_0^{m+1}) - \pi_N(x_0)\|_N \le \varepsilon.$$

Using Theorem 3.1, Theorem 4.2, (25), and (29), we obtain

$$\begin{aligned}\|x_N^{m+2} - x_N\|_N &\le \Theta\|x_N^{m+1} - x_N\|_N^2 \\ &\le \Theta\left(\|x_N^{m+1} - \pi_N(x_0^{m+1})\|_N + \|\pi_N(x_0^{m+1}) - \pi_N(x_0)\|_N + \|\pi_N(x_0) - x_N\|_N\right)^2 \\ &\le \Theta(\varepsilon/2 + \varepsilon + \varepsilon/2)^2 = 4\Theta\varepsilon^2 < \varepsilon.\end{aligned}$$

This means that if the continuous Newton sequence achieves accuracy $\varepsilon$ (measured by the distance to the exact solution) at the $m$th step, then the discrete Newton sequences should achieve the same accuracy $\varepsilon$ at the $(m+1)$st step or earlier. Now we show that the latter cannot happen earlier than at the $(m-1)$st step. Choose $N$ so large that

$$\|x_0^{m-1} - x_0\|_0^2 \le \|\pi_N(x_0^{m-1}) - \pi_N(x_0)\|_N^2 + \varepsilon^2 \tag{29}$$

and suppose that

$$\|x_N^{m-1} - x_N\|_N < \varepsilon.$$

From Theorem 3.1, (22), (25), (28), and (29), we get

$$\begin{aligned}\varepsilon \le \|x_0^m - x_0\|_0 &\le \Theta\|x_0^{m-1} - x_0\|_0^2 \le \Theta\|\pi_N(x_0^{m-1}) - \pi_N(x_0)\|_N^2 + \varepsilon^2 \\ &\le \Theta\left(\|\pi_N(x_0^{m-1}) - x_N^{m-1}\|_N + \|x_N^{m-1} - x_N\|_N + \|x_N - \pi_N(x_0)\|_N\right)^2 + \varepsilon^2 \\ &\le \Theta(\varepsilon/2 + \varepsilon + \varepsilon/2)^2 + \varepsilon^2 = 5\Theta\varepsilon^2,\end{aligned}$$

which contradicts the choice of $\varepsilon$ in (26). □

## 5. Application to optimal control.

We consider the following optimal control problem:

$$\text{minimize } G(y(1)) + \int_0^1 \varphi(y(t), u(t))\, dt \tag{30}$$

$$\begin{aligned}&\text{subject to} \quad \dot{y}(t) = g(y(t), u(t)) \text{ and } u(t) \in U \text{ for almost every (a.e.) } t \in [0,1], \\ &\qquad y(0) = y_0,\ y \in W^{1,\infty}(\mathbb{R}^n), \text{ and } u \in L^\infty(\mathbb{R}^m),\end{aligned}$$

where $\varphi : \mathbb{R}^{n+m} \to \mathbb{R}$, $g : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}$, $U$ is a nonempty, closed and convex set in $\mathbb{R}^m$, and $y_0$ is a fixed vector in $\mathbb{R}^n$. $L^\infty(\mathbb{R}^m)$ denotes the space of essentially bounded and measurable functions with values in $\mathbb{R}^m$ and $W^{1,\infty}(\mathbb{R}^n)$ is the space of Lipschitz continuous functions with values in $\mathbb{R}^n$.

We are concerned with local analysis of the problem (30) around a fixed local minimizer $(y^*, u^*)$ for which we assume certain regularity. Our first standing assumption is the following:

SMOOTHNESS. *The optimal control $u^*$ is Lipschitz continuous in $[0,1]$. There exists a positive number $\delta$ such that the first three derivatives of $\varphi$ and $g$ exist and are continuous in the set $\{(y,u) \in \mathbb{R}^{n+m} : |y - y^*(t)| + |u - u^*(t)| \le \delta$ for all $t \in [0,1]\}$.*

Defining the Hamiltonian $H$ by

$$H(y,u,\psi) = \varphi(y,u) + \psi^{\mathsf{T}} g(y,u),$$

it is well known that the first-order necessary optimality conditions at the solution $(y^*, u^*)$ can be expressed in the following way: There exists $\psi^* \in W^{1,\infty}(\mathbb{R}^n)$ such that $(y^*, u^*, \psi^*)$ is a solution of the variational inequality

$$\dot{y}(t) = g(y(t), u(t)), \quad y(0) = y_0, \tag{31}$$

$$\dot{\psi}(t) = -\nabla_y H(y(t), u(t), \psi(t)), \quad \psi(1) = \nabla G(y(1)), \tag{32}$$

$$0 \in \nabla_u H(y(t), u(t), \psi(t)) + N_U(u(t)) \quad \text{for a.e. } t \in [0,1], \tag{33}$$

where $N_U(u)$ is the normal cone to the set $U$ at the point $u$; that is, $N_U(u)$ is empty if $u \notin U$, while for $u \in U$,

$$N_U(u) = \{p \in \mathbb{R}^m : p^{\mathsf{T}}(q - u) \le 0 \text{ for all } q \in U\}.$$

Although the problem (30) is posed in $L^\infty$ and the optimality system (31)–(33) is satisfied a.e. in $[0,1]$, the regularity we assume for the particular optimal solution implies that at $(y^*, u^*, \psi^*)$ the relations (31)–(33) hold everywhere in $[0,1]$.

Defining the matrices

$$A(t) = \nabla_y g(z^*(t)), \quad B(t) = \nabla_u g(z^*(t)), \quad V = \nabla^2 G(y^*(1)),$$
$$Q(t) = \nabla_{yy}^2 H(x^*(t)), \quad R(t) = \nabla_{uu}^2 H(x^*(t)), \quad S(t) = \nabla_{yu}^2 H(x^*(t)),$$

where $z^* = (y^*, u^*)$ and $x^* = (y^*, u^*, \psi^*)$, we employ the following coercivity condition.

COERCIVITY. *There exists $\alpha > 0$ such that*

$$y(1)^{\mathsf{T}} V y(1) + \int_0^1 [y(t)^{\mathsf{T}} Q(t) y(t) + u(t)^{\mathsf{T}} R(t) u(t) + 2y(t)^{\mathsf{T}} S(t) u(t)]\, dt \ge \alpha \int_0^1 |u(t)|^2\, dt \tag{34}$$

*whenever $y \in W^{1,2}(\mathbb{R}^n)$, $y(0) = 0, u \in L^2(\mathbb{R}^n), \dot{y} = Ay + Bu, u(t) \in U - U$ for a.e. $t \in [0,1]$.*

Let $N$ be a natural number, let $h = 1/N$ be the mesh spacing, let $t_i = ih$, and let $y_i'$ denote the forward difference operator defined by

$$y_i' = \frac{y_{i+1} - y_i}{h}.$$

We consider the following Euler discretization of the optimality system (31)–(33):

$$y_i' = \nabla_\psi H(y_i, u_i, \psi_i), \tag{35}$$

$$\psi_{i-1}' = -\nabla_y H(y_i, u_i, \psi_i), \quad \psi_{N-1} = \nabla G(y_N), \tag{36}$$

$$0 \in \nabla_u H(y_i, u_i, \psi_i) + N_U(u_i), \quad i = 0, 1, \ldots, N-1. \tag{37}$$

The system (35)–(37) is a discrete-time variational inequality depending on the step size $h$. It represents the first-order necessary optimality condition for the following discretization of the original problem (30):

$$\begin{aligned} &\text{minimize} \quad G(y_N) + \sum_{i=0}^{N-1} h\varphi(y_i, u_i) \\ &\text{subject to} \;\; y_i' = g(y_i, u_i),\; u_i \in U,\; i = 0, 1, \ldots, N-1. \end{aligned} \tag{38}$$

In this section we examine the following version of the Newton method for solving the variational system (35)–(37), which correspond to the SQP method for solving the optimization problem (38). Let $x^k = (y^k, u^k, \psi^k)$ denote the $k$th iterate. Let the superscript $k$ and the subscript $i$ attached to the derivatives of $H$ and $G$ denote their values at $x_i^k$. Then the new iterate $x^{k+1} = (y^{k+1}, u^{k+1}, \psi^{k+1})$ is a solution of the following linear variational inequality for the variable $x = (y, u, \psi)$:

$$y_i' = \nabla_\psi H_i^k + \nabla_{\psi x}^2 H_i^k (x_i - x_i^k), \tag{39}$$

$$\psi_{i-1}' = -\nabla_y H_i^k - \nabla_{yx}^2 H_i^k (x_i - x_i^k), \quad \psi_{N-1} = \nabla G^k + \nabla^2 G^k (y_N - y_N^k), \tag{40}$$

$$0 \in \nabla_u H_i^k + \nabla_{ux}^2 H_i^k (x_i - x_i^k) + N_U(u_i), \quad i = 0, 1, \ldots, N-1. \tag{41}$$

In [5, Appendix 2], it was proved that the coercivity condition (34) is stable under the Euler discretization, then the variational system (39)–(41) is equivalent, for $x^k$ near $x^* = (y^*, u^*, \psi^*)$, to the following linear-quadratic discrete-time optimal control problem which is expressed in terms of the variables $y$, $u$, and $z = (y, u)$:

$$\begin{aligned} \text{minimize} \quad & \left(\nabla G^k + \frac{1}{2}\nabla^2 G^k (y_N - y_N^k)\right)^{\mathsf{T}} (y_N - y_N^k) \\ & + \; h\sum_{i=0}^{N-1} \left(\nabla_z \varphi_i^k + \frac{1}{2}\nabla_{zz}^2 H_i^k (z_i - z_i^k)\right)^{\mathsf{T}} (z_i - z_i^k) \\ \text{subject to} \quad & y_i' = g_i^k + \nabla_z g_i^k (z_i - z_i^k), \quad u_i \in U, \quad i = 0, 1, \ldots, N-1. \end{aligned}$$

A natural stopping criterion for the problem at hand is the following: Given $\varepsilon > 0$, a control $\tilde{u}^k$ obtained at the $k$th iteration is considered an $\varepsilon$-optimal solution if

$$\max_{0 \le i \le N-1} \operatorname{dist}(\nabla_u H(\tilde{y}_i^k, \tilde{u}_i^k, \tilde{\psi}_i^k), N_U(\tilde{u}_i^k)) \le \varepsilon, \tag{42}$$

where $\tilde{y}_i^k$ and $\tilde{\psi}_i^k$ are the solutions of the state and the adjoint equations (35) and (36) correspond to $u = \tilde{u}^k$.

We now apply the general approach developed in the previous section to the discrete-time variational inequality (35)–(36). The discrete $L_N^\infty$ norm is defined by

$$\|v\|_N^\infty = \max_{0 \le i \le N-1} |v_i|.$$

The variable $x$ is the triple $(y, u, \psi)$ while $X^N$ is the space of all finite sequences $x_0, x_1, \ldots, x_{N-1}$, with $y_0$ given, equipped with the $L_N^\infty$ norm. The space $Y^N$ is the Cartesian product $L_N^\infty \times L_N^\infty \times \mathbb{R}^n \times L_N^\infty$ corresponding to the four components of the function $f_N$ defined by

$$f_N(x)_i = \begin{pmatrix} y_i' - g(y_i, u_i) \\ -\psi_{i-1}' + \nabla_y H(y_i, u_i, \psi_i) \\ \psi_{N-1} - \nabla G(y_N) \\ -\nabla_u H(y_i, u_i, \psi_i) \end{pmatrix} \quad \text{and} \quad F_N(x)_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ N_U(u_i) \end{pmatrix}.$$

With the choice $(x_N^*)_i = (y^*(t_i), u^*(t_i), \psi^*(t_i))$, the general condition (A1) is satisfied by taking

$$(z_N^*)_i = \begin{pmatrix} (y^*(t_{i+1}) - y^*(t_i))/h - g(y^*(t_i), u^*(t_i)) \\ (\psi^*(t_{i-1}) - \psi^*(t_i))/h - \nabla_x H(y^*(t_i), u^*(t_i), \psi^*(t_i)) \\ 0 \\ 0 \end{pmatrix}.$$

The first component of $z_N^*$ is estimated in the following way:

$$\sup_i \left| \frac{y^*(t_{i+1}) - y^*(t_i)}{h} - g(y^*(t_i), u^*(t_i)) \right| \\ \leq \sup_i \frac{1}{h} \int_{t_i}^{t_{i+1}} |g(y^*(t_i), u^*(t_i)) - g(y^*(t), u^*(t))| dt.$$

Since $g$ is smooth and both $y^*$ and $u^*$ are Lipschitz continuous, the above expression is bounded by $O(h)$. The same bound applies for the second component of $z_N^*$, while the third and fourth components are zero. Thus the norm of $z_N^*$ can be made arbitrarily small for all sufficiently large $N$. Condition (A2) follows from the smoothness assumption. A proof of condition (A3) is contained in the proof of Theorem 6 in [5]. Applying Theorems 3.1 and 4.1 and using the result from [5, Appendix 2], that the discretized coercivity condition is a second-order sufficient condition for the discrete problem, we obtain the following theorem.

THEOREM 5.1. *If smoothness and coercivity hold, then there exist positive constants $K$, $c$, $\sigma$, $\bar{\varepsilon}$, and $\bar{N}$ with the property that for every $N > \bar{N}$ there is a unique solution $(y_h, u_h, \psi_h)$ of the variational system* (35)–(37) *and $(y_h, u_h)$ is a local minimizer for the discrete problem* (38). *For every starting point $(y^0, u^0, \psi^0)$ with*

$$\max_{0 \leq i \leq N} \left( |(y^0)_i - y^*(t_i)| + |(u^0)_i - u^*(t_i)| + |(\psi^0)_i - \psi^*(t_i)| \right) \leq \sigma,$$

*there is a unique SQP sequence $(y^k, u^k, \psi^k)$ and it is Q-quadratically convergent, with a constant $K$, to the solution $(y_h, u_h, \psi_h)$. In particular, for the sequence of controls we have*

$$\max_{0 \leq i \leq N-1} |(u^{k+1})_i - (u_h)_i| \leq K \left( \max_{0 \leq i \leq N-1} |(u^k)_i - (u_h)_i| \right)^2.$$

*Moreover, if the stopping test* (42) *is applied with an $\varepsilon \in [0, \bar{\varepsilon}]$, then the resulting $\varepsilon$-optimal control $u^{k_\varepsilon}$ satisfies*

$$\max_{0 \leq i \leq N-1} |u_i^{k_\varepsilon} - u^*(t_i)| \leq c(\varepsilon + h).$$

Note that the termination step $k_\varepsilon$ corresponding to the assumed accuracy of the stopping test can be estimated by Theorem 4.1. Combining the error in the discrete control with the discrete state equation (35) and the discrete adjoint equation (36), yield corresponding estimates for discrete state and adjoint variables.

*Remark.* Following the approach developed in [5] one can obtain an analogue of Theorem 5.1 by assuming that the optimal control $u^*$ is merely bounded and Riemann integrable in $[0, 1]$ and employing the so-called averaged modulus of smoothness to obtain error estimates.. The stronger Lipschitz continuity condition for the optimal control is, however, needed in our analysis of the mesh independence.

The SQP method applied to the continuous-time optimal control problem (30) has the following form: If $x^0 = (y^0, u^0, \psi^0)$ is a starting point, the iterate $x^{k+1} = (y^{k+1}, u^{k+1}, \psi^{k+1})$ satisfies

$$\dot{y}(t) = \nabla_\psi H^k(t) + \nabla^2_{\psi x} H^k(t)(x(t) - x^k(t)),\ y(0) = y_0, \tag{43}$$

$$\dot{\psi}(t) = -\nabla_y H^k(t) - \nabla^2_{yx} H^k(t)(x(t) - x^k(t)), \tag{44}$$

$$\psi(1) = \nabla G^k(1) + \nabla^2 G^k(y(1) - y^k(1)), \tag{45}$$

$$0 \in \nabla_u H^k(t) + \nabla^2_{ux} H^k(t)(x(t) - x^k(t)) + N_U(u(t)) \tag{46}$$

for a.e. $t \in [0,1]$, where the superscript $k$ attached to the derivatives of $H$ and $G$ denotes their values at $x^k$. In particular, (43)–(46) is a variational inequality to which we can apply the general theory from the previous sections. We attach the index $N = 0$ to the continuous problem and for $x = (y, u, \psi)$ we choose $X^0 = C^1_0(\mathbb{R}^n) \times C(\mathbb{R}^m) \times C^1(\mathbb{R}^n)$, where $C^1_0 = \{y \in C^1 \mid y(0) = y_0\}$, and $Y^0 = C(\mathbb{R}^n) \times C(\mathbb{R}^n) \times \mathbb{R}^n \times C(\mathbb{R}^m)$. Condition (A1) is clearly satisfied with $x^*_0 = x^* := (y^*, u^*, \psi^*)$ and $z^*_0 = 0$. Condition (A2) follows from the smoothness assumption. Condition (A3) follows from the coercivity assumption as proved in [9, Lemma 3] (see also [4, section 2.3.4], for an earlier version of this result in the convex case). Hence, we can apply Theorem 3.1 obtaining that for any sufficiently small ball $\mathcal{B}$ around $x^*$ (in the norm of $X^0$), if the starting point $x^0$ is chosen from $\mathcal{B}$, then the SQP method produces a unique sequence $x^k \in \mathcal{B}$ which is $Q$-quadratically convergent to $x^*$ (in the norm of $X^0$). Moreover, from Theorem 4.1 we obtain an estimate for the number of steps needed to achieve a given accuracy.

In order to derive a mesh-independence result from the general theory, we first study the regularity of the SQP sequence for the continuous problem.

LEMMA 5.1. *There exist positive constants $p$ and $q$ such that for every $x^0 \in B_p(x^*)$ with $u^0(\cdot)$ Lipschitz continuous in $[0,1]$, for every $k = 1, 2, \ldots,$ and for every $t_1, t_2 \in [0,1]$,*

$$|u^k(t_1) - u^k(t_2)| \le q|t_1 - t_2|.$$

*Proof.* In [5, section 6], extending a previous result in [7], see also [6], Lemma 2, we showed that the coercivity condition implies pointwise coercivity almost everywhere. In the present circumstances, the latter condition is satisfied everywhere in $[0,1]$; that is, there exists a constant $\alpha > 0$ such that for every $v \in U - U$ and for all $t \in [0,1]$,

$$v^{\mathsf{T}} R(t) v \ge \alpha v^{\mathsf{T}} v. \tag{47}$$

For a positive parameter $p$, consider the SQP sequence $x^k$ starting from $x^0 \in B_p(x^*)$ such that the initial control $u^0$ is a Lipschitz continuous function in $[0,1]$. Throughout the proof we will choose $p$ sufficiently small and check the dependence of the constants of $p$. By (46) the iterate $x^k$ satisfies

$$\begin{aligned}(\nabla_u H^k(t) &+ \nabla^2_{uu} H(x^k(t))(u^{k+1}(t) - u^k(t)) + \nabla^2_{uy} H(x^k(t))(y^{k+1}(t) - y^k(t)) \\ &+ \nabla^2_{u\psi} H(x^k(t))(\psi^{k+1}(t) - \psi^k(t)))^{\mathsf{T}}(u - u^{k+1}(t)) \ge 0\end{aligned} \tag{48}$$

for every $t \in [0,1]$ and for every $u \in U$. Let $t_1, t_2 \in [0,1]$. Note that $x^k$ are contained in $B_p(x^*)$ for all $k$ and therefore both $y'^k$ and $\psi'^k$ are bounded by a constant independent of $k$; hence, $y^k$ and $\psi^k$ are Lipschitz continuous functions in time with Lipschitz constants independent of $k$. We have from (48)

$$(\nabla_u H^k(t_1) + \nabla^2_{uu} H(x^k(t_1))(u^{k+1}(t_1) - u^k(t_1)) + \nabla^2_{uy} H(x^k(t_1))(y^{k+1}(t_1) - y^k(t_1)) + \nabla^2_{u\psi} H(x^k(t_1))(\psi^{k+1}(t_1) - \psi^k(t_1)))^{\mathsf{T}}(u^{k+1}(t_2) - u^{k+1}(t_1)) \geq 0$$

and the analogous inequality with $t_1$ and $t_2$ interchanged. Adding these two inequalities and adding and subtracting the expressions $\nabla^2_{uu} H(x^k(t_1))u^{k+1}(t_2)$ and $\nabla^2_{uu} H(x^k(t_1))u^*(t_1) - \nabla^2_{uu} H(x^k(t_2))u^*(t_2)$ we obtain

$$\begin{aligned}
&(\theta^k(t_1) - \theta^k(t_2) - \nabla^2_{uu} H(x^k(t_1))u^*(t_1) + \nabla^2_{uu} H(x^k(t_2))u^*(t_2) \\
&\quad + (\nabla^2_{uu} H(x^k(t_1)) - \nabla^2_{uu} H(x^k(t_2)))u^{k+1}(t_2) \\
&\quad + \nabla^2_{uy} H(x^k(t_1))(y^{k+1}(t_1) - y^k(t_1)) \\
&\quad + \nabla^2_{u\psi} H(x^k(t_1))(\psi^{k+1}(t_1) - \psi^k(t_1)))^{\mathsf{T}}(u^{k+1}(t_2) - u^{k+1}(t_1)) \\
&\quad \geq (\nabla^2_{uu} H(x^k(t_1))(u^{k+1}(t_1) - u^{k+1}(t_2)))^{\mathsf{T}}(u^{k+1}(t_1) - u^{k+1}(t_2))
\end{aligned} \tag{49}$$

where the function $\theta^k$ is defined as

$$\theta^k(t) = \nabla_u H^k(t) + \nabla^2_{uu} H(x^k(t))(u^k(t) - u^*(t)).$$

By (47), for a sufficiently small $p$ the right-hand side of the inequality (49) satisfies

$$(\nabla^2_{uu} H(x^k(t_1))(u^{k+1}(t_1) - u^{k+1}(t_2)))^{\mathsf{T}}(u^{k+1}(t_1) - u^{k+1}(t_2)) \geq \frac{\alpha}{2}|u^{k+1}(t_1) - u^{k+1}(t_2)|^2. \tag{50}$$

Combining (49) and (50) we obtain

$$\begin{aligned}
&\frac{\alpha}{2}|u^{k+1}(t_1) - u^{k+1}(t_2)| \leq |\theta^k(t_1) - \theta^k(t_2)| \\
&+ |(\nabla^2_{uu} H(x^k(t_1)) - \nabla^2_{uu} H(x^k(t_2)))(u^{k+1}(t_2) - u^*(t_2))| \\
&+ |\nabla^2_{uu} H(x^k(t_1))(u^*(t_1) - u^*(t_2))| \\
&+ |(\nabla^2_{uy} H(x^k(t_1)) - \nabla^2_{uy} H(x^k(t_2)))(y^{k+1}(t_1) - y^k(t_1))| \\
&+ |\nabla^2_{uy} H(x^k(t_2))((y^{k+1}(t_1) - y^{k+1}(t_2)) - (y^k(t_1) - y^k(t_2)))| \\
&+ |(\nabla^2_{u\psi} H(x^k(t_1)) - \nabla^2_{u\psi} H(x^k(t_2)))(\psi^{k+1}(t_1) - \psi^k(t_1))| \\
&+ |\nabla^2_{u\psi} H(x^k(t_2))((\psi^{k+1}(t_1) - \psi^{k+1}(t_2)) - (\psi^k(t_1) - \psi^k(t_2)))|.
\end{aligned} \tag{51}$$

Let $u_k$ be Lipschitz continuous in time with a constant $L_k$. Then the function $\theta^k$ is almost everywhere differentiable and its derivative is given by

$$\begin{aligned}
\dot{\theta}(t) &= \nabla^2_{uy} H^k(t)\dot{y}_k(t) + \nabla^2_{u\psi} H^k(t)\dot{\psi}_k(t) - \nabla^3_{uuu} H^k(t)\dot{u}^k(t)(u^k(t) - u^*(t)) \\
&\quad - \nabla^3_{uuy} H^k(t)\dot{y}^k(t)(u^k(t) - u^*(t)) - \nabla^3_{uu\psi} H^k(t)\dot{\psi}^k(t)(u^k(t) - u^*(t)) - \nabla^2_{uu} H^k(t)\dot{u}^*(t).
\end{aligned}$$

From this expression we obtain that there exists a constant $c_1$, independent of $k$ and $t$ and bounded from above when $p \to 0$, such that

$$\|\dot{\theta}\|_{L^\infty} \leq cp\|\dot{u^k}\|_{L^\infty} + c_1 \leq c_1(pL_k + 1).$$

Estimating the expressions in the right-hand side of (51) we obtain that there exists a constant $c_2$, independent of $k$ and $t$ and bounded from above when $p \to 0$, such that

$$|u^{k+1}(t_1) - u^{k+1}(t_2)| \le c_2(pL_k + 1)|t_1 - t_2|.$$

Hence, $u^{k+1}$ is Lipschitz continuous and, for some constants $c$ of the same kind as $c_1, c_2$, its Lipschitz constant $L_{k+1}$ satisfies

$$L_{k+1} \le c(pL_k + 1).$$

Since $p$ can be chosen arbitrarily small, the sequence $L_i, i = 1, 2, \dots,$ is bounded, i.e., by a constant $q$. The proof is complete. $\square$

To apply the general mesh-independence result presented in Theorem 4.2 we need to estimate the residual $r_N^k$ obtained when the SQP sequence of the continuous problem is substituted into the relations determining the SQP sequence of the discretized problem. Specifically, the residual is the remainder term associated with the Euler scheme applied to (43)–(46); that is,

$$r_N^k = \begin{pmatrix} \frac{1}{h}\int_{t_i}^{t_{i+1}} (\nabla_\psi H^k(t) + \nabla^2_{\psi x} H^k(t)(x^{k+1}(t) - x^k(t)) \\ \qquad - (\nabla_\psi H_i^k + \nabla^2_{\psi x} H_i^k (x_i^{k+1} - x_i^k)))dt \\ \frac{1}{h}\int_{t_i}^{t_{i+1}} (-\nabla_x H^k(t) - \nabla^2_{xx} H^k(t)(x^{k+1}(t) - x^k(t)) \\ \qquad - (-\nabla_x H_i^k - \nabla^2_{xx} H_i^k (x_i^{k+1} - x_i^k)))dt \\ \psi^{k+1}(1-h) - \psi^{k+1}(1) \\ 0 \end{pmatrix},$$

where the subscript $i$ denotes the value at $t_i$. From the regularity of the Newton sequence established in Lemma 5.1, the uniform norm of the residual is bounded by $ch$, where $c$ is independent of $k$. Note that the map $\pi_N(x)$ defined in section 4, acting on a function $x \in X^0$, gives the sequence $x(t_i), i = 0, 1, \dots N$. Condition (23) is satisfied because the space $X^0$ is a subset of the space of continuous functions. Summarizing, we obtain the following result.

THEOREM 5.2. *Suppose that smoothness and coercivity conditions hold. Then there exists a neighborhood $\mathcal{W}$, in the norm of $X^0$, of the solution $x^* = (y^*, u^*, \psi^*)$ such that for all sufficiently small step-sizes $h$, the following mesh-independence property holds:*

$$\sup_k \max_{0 \le i \le N-1} |u^k(t_i) - (u_h^k)_i| = O(h), \tag{52}$$

*where $u^k(\cdot)$ is the control in the SQP sequence $(y^k(\cdot), u^k(\cdot), \psi^k(\cdot))$ for the continuous problem starting from a point $x^0 = (y^0, u^0, \psi^0) \in \mathcal{W}$ with $u^0(\cdot)$ Lipschitz continuous in $[0,1]$, and $u_h^k$ is the control in the SQP sequence $(y_h^k, u_h^k, \psi_h^k)$ for the discretized problem starting from the point $\pi_N(x^0)$.*

Applying Theorem 4.3 to the optimal control problem considered we obtain the mesh-independence property (27) which relates the number of steps for the continuous and the discretized problem needed to achieve certain accuracy. The latter property can be also easily deduced from the estimate (52) in Theorem 5.2, in a way analogous to the proof of Theorem 4.3. Therefore the estimate (52) is a stronger mesh-independence property than (27).

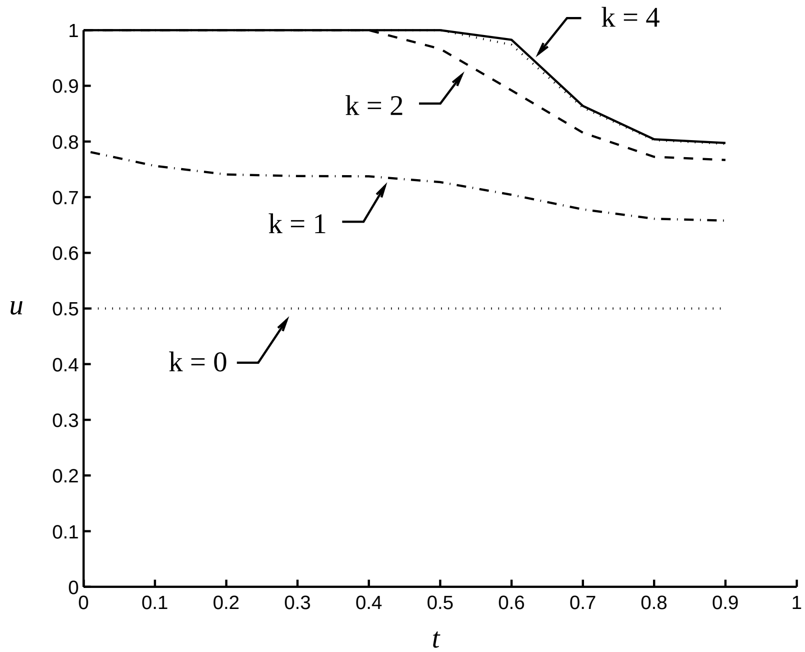


FIG. 1. *SQP iterates for the control with* $N = 10$.

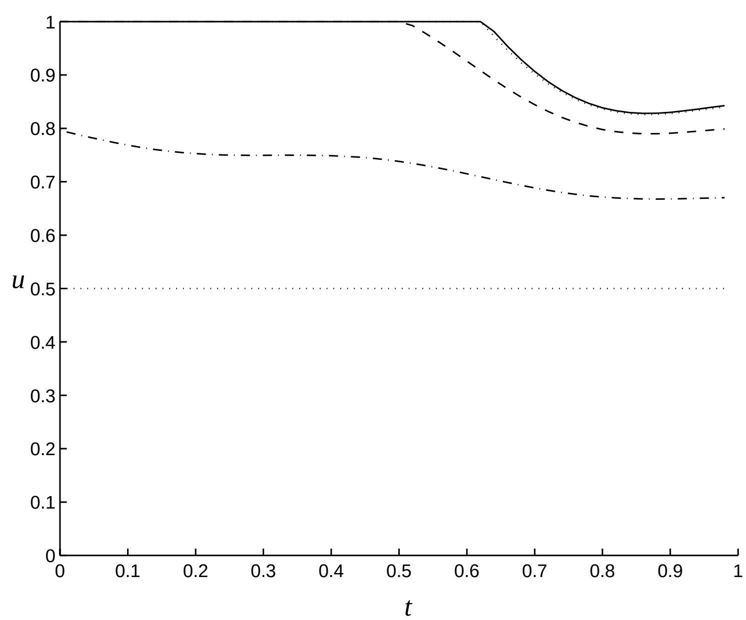


FIG. 2. *SQP iterates for the control with* $N = 50$.

**6. Numerical examples.** The convergence estimate of Theorem 5.2 is illustrated using the following example:

$$\text{minimize} \int_0^1 \left(\tfrac{1}{2}(y(t)^4 + u(t)^2 + u(t)y(t)) + \tfrac{1}{4}\sin(10t)u(t) + u(t)^{-1}\right) dt$$

$$\text{subject to } \dot{y}(t) = -u(t)/(2y(t)),\ y(0) = \sqrt{\tfrac{1+3e}{2(e-1)}},\ u(t) \le 1.$$

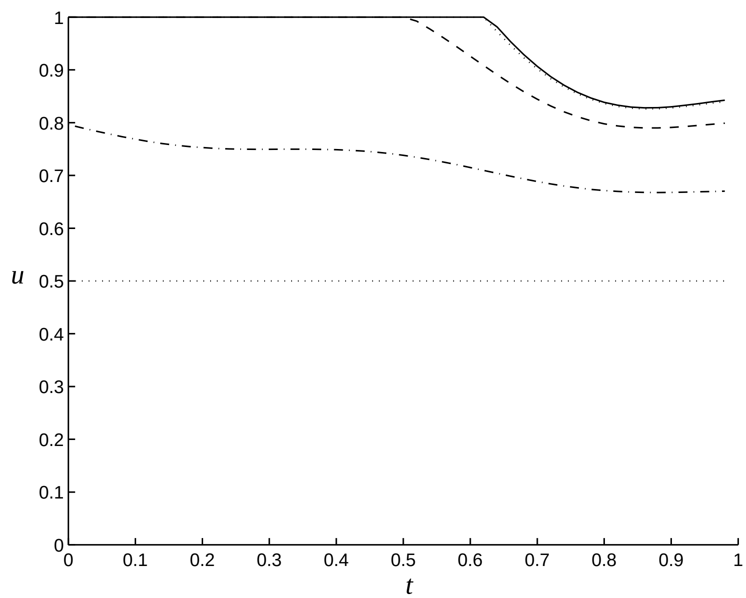


FIG. 3. *SQP iterates for the control with N=250.*

TABLE 1
*$L^\infty$ error in the control for various choices of the mesh.*

| Iteration | $N = 10$ | $N = 50$ | $N = 250$ |
|---|---|---|---|
| 0 | .500000 | .500000 | .500000 |
| 1 | .278473 | .290428 | .291671 |
| 2 | .090857 | .091727 | .097923 |
| 3 | .008928 | .008971 | .010185 |
| 4 | .000082 | .000084 | .000105 |

TABLE 2
*Error in current iterate divided by error in prior iterate squared.*

| Iteration | $N = 10$ | $N = 50$ | $N = 250$ |
|---|---|---|---|
| 1 | 1.113 | 1.161 | 1.166 |
| 2 | 1.171 | 1.087 | 1.151 |
| 3 | 1.081 | 1.066 | 1.062 |
| 4 | 1.027 | 1.039 | 1.013 |

This problem is a variation of Problem I in [8] that has been converted from a linear-quadratic problem to a fully nonlinear problem by making the substitution $x = -y^2$ and by adding additional terms to the cost function that degrade the speed of the SQP iteration so that the convergence is readily visible (without these additional terms, the SQP iteration converges to computing precision within 2 iterations). Figures 1–3 show the control iterates for successively finer meshes. The control corresponding to $k = 3$ is barely visible beneath the $k = 4$ iterate. Observe that the SQP iterations are relatively insensitive to the choice of the mesh. Specifically, $N = 10$ is already sufficiently large to obtain mesh independence. In Table 1 we give the $L^\infty$ error in the successive iterates. In Table 2 we observe that the ratio of the error in the current iterate to the error in the prior iterate squared is slightly larger than 1.

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