

OPTIMIZATION OF GENERALIZED MEAN-SQUARE ERROR IN NOISY LINEAR ESTIMATION*

WILLIAM W. HAGER[†] AND JIANGTAO LUO[†]

Abstract. A class of least squares problems that arises in linear Bayesian estimation is analyzed. The data vector \mathbf{y} is given by the model $\mathbf{y} = \mathbf{P}(\mathbf{H}\boldsymbol{\theta} + \boldsymbol{\eta}) + \mathbf{w}$, where \mathbf{H} is a known matrix, while $\boldsymbol{\theta}$, $\boldsymbol{\eta}$, and \mathbf{w} are uncorrelated random vectors. The goal is to obtain the best estimate for $\boldsymbol{\theta}$ from the measured data. Applications of this estimation problem arise in multisensor data fusion problems and in wireless communication. The unknown matrix \mathbf{P} is chosen to minimize the expected mean-squared error $\mathbf{E}(\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2)$ subject to a power constraint “trace $(\mathbf{P}\mathbf{P}^*) \leq P$,” where $\hat{\boldsymbol{\theta}}$ is the best affine estimate of $\boldsymbol{\theta}$. Earlier work characterized an optimal \mathbf{P} in the case where the noise term $\boldsymbol{\eta}$ vanished, while this paper analyzes the effect of $\boldsymbol{\eta}$, assuming its covariance is a multiple of \mathbf{I} . The singular value decomposition of an optimal \mathbf{P} is expressed in the form $\mathbf{V}\boldsymbol{\Sigma}\mathbf{\Pi}\mathbf{U}^*$ where \mathbf{V} and \mathbf{U} are unitary matrices related to the covariance of either $\boldsymbol{\theta}$ or \mathbf{w} , and singular vectors of \mathbf{H} , $\boldsymbol{\Sigma}$ is diagonal, and $\mathbf{\Pi}$ is a permutation matrix. The analysis is carried out in two special cases: (i) $\mathbf{H} = \mathbf{I}$ and (ii) covariance of $\boldsymbol{\theta}$ is \mathbf{I} . In case (i), $\mathbf{\Pi}$ does not depend on the power P . In case (ii), $\mathbf{\Pi}$ generally depends on P . The optimal $\mathbf{\Pi}$ is determined in the limit as the power tends to zero or infinity; a good approximation to an optimal $\mathbf{\Pi}$ is found for general P .

Key words. linear Bayesian estimation, mean-square error, MSE, CDMA systems, wireless communication

AMS subject classifications. 60G35, 93E10, 94A15

DOI. 10.1137/060676830

1. Introduction. Suppose that $\mathbf{y} \in \mathbb{C}^m$ is a random vector that obeys the model

$$(1.1) \quad \mathbf{y} = \mathbf{P}(\mathbf{H}\boldsymbol{\theta} + \boldsymbol{\eta}) + \mathbf{w},$$

where $\mathbf{H} \in \mathbb{C}^{n \times l}$ is a known matrix, while $\boldsymbol{\theta} \in \mathbb{C}^l$, $\boldsymbol{\eta} \in \mathbb{C}^n$, and $\mathbf{w} \in \mathbb{C}^m$ are uncorrelated random vectors with the property that $\boldsymbol{\eta}$ and \mathbf{w} have zero mean. The matrix $\mathbf{P} \in \mathbb{C}^{m \times n}$ is a “filter” which is applied to the noisy measurement $\mathbf{H}\boldsymbol{\theta} + \boldsymbol{\eta}$, and which is chosen to achieve an optimal estimate of the signal $\boldsymbol{\theta}$. We consider affine estimators of the form

$$\hat{\boldsymbol{\theta}} = \mathbf{A}\mathbf{y} + \mathbf{a},$$

where $\mathbf{A} \in \mathbb{C}^{l \times m}$ is a constant matrix and $\mathbf{a} \in \mathbb{C}^l$ is a constant vector. For any random vector \mathbf{v} , let \mathbf{C}_v denote the covariance defined by

$$\mathbf{C}_v = \mathbf{E}((\mathbf{v} - \mathbf{E}(\mathbf{v}))(\mathbf{v} - \mathbf{E}(\mathbf{v}))^*),$$

where \mathbf{E} is expectation and $*$ is conjugate transpose. According to [4, Thm. 12.1], the affine estimator that minimizes the expected mean-square error $\mathbf{E}(\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2)$ is given by

$$\hat{\boldsymbol{\theta}} = \mathbf{E}(\boldsymbol{\theta}) + (\mathbf{C}_\theta^{-1} + \mathbf{H}^*\mathbf{P}^*\mathbf{C}_{\eta\mathbf{w}}^{-1}\mathbf{P}\mathbf{H})^{-1}(\mathbf{P}\mathbf{H})^*\mathbf{C}_{\eta\mathbf{w}}^{-1}(\mathbf{y} - \mathbf{P}\mathbf{H}\mathbf{E}(\boldsymbol{\theta})),$$

*Received by the editors December 6, 2006; accepted for publication (in revised form) by A. S. Lewis April 28, 2008; published electronically September 4, 2008. This material is based upon work supported by the National Science Foundation under grants 0203270, 0619080, 0620286, and 0724750.
<http://www.siam.org/journals/simax/30-2/67683.html>

[†]Department of Mathematics, University of Florida, Gainesville, FL 32611-8105 (hager@math.ufl.edu, <http://www.math.ufl.edu/~hager>, luo@math.ufl.edu, <http://www.math.ufl.edu/~luo3>).

where $\mathbf{C}_{\eta w} = \mathbf{P}\mathbf{C}_\eta\mathbf{P}^* + \mathbf{C}_w$. Moreover, the error $\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$ has zero mean and covariance

$$(1.2) \quad \mathbf{C} = \mathbf{E}((\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^*) = ((\mathbf{P}\mathbf{H})^*(\mathbf{C}_w + \mathbf{P}\mathbf{C}_\eta\mathbf{P}^*)^{-1}\mathbf{P}\mathbf{H} + \mathbf{C}_\theta^{-1})^{-1}.$$

Since $\hat{\boldsymbol{\theta}}$ depends on \mathbf{P} , the estimation error $\mathbf{E}(\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2)$ depends on \mathbf{P} . The filter \mathbf{P} is chosen to minimize the estimation error, subject to the constraint $\text{tr}(\mathbf{P}\mathbf{P}^*) \leq P$, where P is a positive scalar and “tr” denotes the trace of a matrix. The constraint $\text{tr}(\mathbf{P}\mathbf{P}^*) \leq P$ represents a bound on the power associated with \mathbf{P} .

Multisensor data fusion problems (see [6, 10, 15] and the references therein) fit within the framework of the model (1.1). In these applications, $\boldsymbol{\theta}$ is a random parameter vector that is being measured by a collection of sensors. The sensor measurements correspond to the observation matrix \mathbf{H} . The term $\boldsymbol{\eta}$ in the model (1.1) could represent sensor noise. The output of the observer is sent to the “fusion center” which leads to the final output \mathbf{y} in (1.1). If the dimension of the column space of \mathbf{P} is less than the dimension of the row space, then there is a reduction in dimensionality of the data. The term \mathbf{w} might represent either noise or quantization error in the transmission to the fusion center. The constraint $\text{tr}(\mathbf{P}\mathbf{P}^*) \leq P$ might also be viewed as a constraint on the amplifier gain to prevent the amplified observations from exceeding the dynamic range of the quantizer.

Another application which fits the model (1.1) concerns spreading sequence optimization for code division multiple access (CDMA) communication systems [12, 13]. In CDMA systems, many users simultaneously share a communication channel. In modeling the uplink (communication from the mobile units to the base station), \mathbf{y} is the signal received at the base station, the j th column of \mathbf{P} is the “spreading sequence” assigned to the j th user, and θ_j is the symbol transmitted from the j th user. \mathbf{H} is a diagonal matrix corresponding to channel gains.

The problem of estimating the channel matrix for a multiple input, single output (MISO) system can be expressed in the form (1.1) as observed in [2]. In this context, there are multiple transmit antennas and a single receiver. The j th column of \mathbf{P} is the training signal to transmit from the j th antenna to obtain the best estimate for the communication channel gains; the matrix \mathbf{H} is the square root of the correlation between the transmit antennas. The noise in the channel gains and in the transmitted signal associated with atmospheric conditions is modeled by $\boldsymbol{\eta}$ and \mathbf{w} .

The model (1.1) is related to the channel estimation problem for multiple input, multiple output (MIMO) systems [5, 14]. That is, in [5] it is shown that when \mathbf{H} and \mathbf{C}_w have a special Kronecker product form and when $\mathbf{C}_\eta = \mathbf{0}$, then the covariance of the best channel estimate is a multiple of \mathbf{C} in (1.2). The model (1.1) is loosely connected with joint linear transmitter-receiver design in MIMO communication [8, 9]. In MIMO communication, the precoder \mathbf{P} precedes the channel matrix \mathbf{H} . Hence, in the special case $\mathbf{H} = \mathbf{I}$, the model (1.1) corresponds to a MIMO communication channel with two noise terms.

Since $\mathbf{E}(\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2)$ is the trace of \mathbf{C} , minimizing the trace of the covariance \mathbf{C} is equivalent to minimizing $\mathbf{E}(\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2)$. Hence, the \mathbf{P} that minimizes the expected mean-square error $\mathbf{E}(\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2)$ is a solution of the problem

$$(1.3) \quad \begin{aligned} \min_{\mathbf{P}} \quad & \text{tr}((\mathbf{P}\mathbf{H})^*(\mathbf{W} + \mathbf{P}\mathbf{N}\mathbf{P}^*)^{-1}\mathbf{P}\mathbf{H} + \mathbf{T})^{-1} \\ \text{subject to} \quad & \text{tr}(\mathbf{P}\mathbf{P}^*) \leq P, \quad \mathbf{P} \in \mathbb{C}^{m \times n}, \end{aligned}$$

where $\mathbf{W} = \mathbf{C}_w$, $\mathbf{N} = \mathbf{C}_\eta$, and $\mathbf{T} = \mathbf{C}_\theta^{-1}$. Both \mathbf{C}_w and \mathbf{C}_θ are assumed positive definite. This holds, for example, if the probability density function associated with

$\boldsymbol{\theta}$ and \mathbf{w} is continuous. The wireless communication applications in [2] correspond to $\mathbf{N} = \mathbf{0}$ and $\mathbf{T} = \mathbf{I}$. The application in [5] corresponds to $\mathbf{N} = \mathbf{0}$ and $\mathbf{H} = \mathbf{I}$. In this paper, we again consider the cases (i) $\mathbf{H} = \mathbf{I}$ or (ii) $\mathbf{T} = \mathbf{I}$; however, the noise covariance \mathbf{N} is no longer zero, but a multiple of \mathbf{I} . By a rescaling of the variables \mathbf{P} , \mathbf{H} , and the power P , there is no loss of generality in assuming that $\mathbf{N} = \mathbf{I}$. In the multisensor data fusion problem studied in [6], it is pointed out that when the sensor noise is uncorrelated with the signal $\boldsymbol{\theta}$ and when the sensor noise is spatially uncorrelated with zero mean, then by appropriate pre- and post-whitening if necessary, there is no loss of generality in assuming that $\mathbf{N} = \mathbf{I}$ and $\mathbf{T} = \mathbf{I}$. Hence, we focus on the problem

$$(1.4) \quad \min_{\mathbf{P}} \operatorname{tr} \left((\mathbf{P}\mathbf{H})^* (\mathbf{W} + \mathbf{P}\mathbf{P}^*)^{-1} \mathbf{P}\mathbf{H} + \mathbf{T} \right)^{-1}$$

subject to $\operatorname{tr} (\mathbf{P}\mathbf{P}^*) \leq P, \quad \mathbf{P} \in \mathbb{C}^{m \times n}.$

A unified analysis is developed for (1.4) which handles the special cases (i) or (ii) and which exposes the similarities and differences in these two problems. In both cases, the singular value decomposition of an optimal solution is expressed in the form $\mathbf{V}\boldsymbol{\Sigma}\boldsymbol{\Pi}\mathbf{U}^*$ where \mathbf{V} and \mathbf{U} are unitary matrices related to eigenvectors of \mathbf{W} or \mathbf{T} or singular vectors of \mathbf{H} , with a specific ordering for the columns described below. The matrix $\boldsymbol{\Sigma}$ is diagonal, and $\boldsymbol{\Pi}$ is a permutation matrix. A fundamental difference in these problems is that in case (i) ($\mathbf{H} = \mathbf{I}$), the permutation is independent of P , which is the same result obtained in [5] for $\boldsymbol{\eta} = \mathbf{0}$. For case (ii) ($\mathbf{T} = \mathbf{I}$), $\boldsymbol{\Pi}$ depends on the choice of P and the singular values of \mathbf{H} . When the noise term $\boldsymbol{\eta}$ vanishes and P is large, the permutation arranges the singular values of \mathbf{H} in increasing order, as obtained in [2]; when noise $\boldsymbol{\eta}$ with covariance \mathbf{I} is included in the model, the singular values of \mathbf{H} smaller than one are arranged in decreasing order, while the singular values greater than one are arranged in increasing order (see Theorem 5.1). As a result, depending on the size of the power P and the distribution of the singular values of \mathbf{H} , the noise term $\boldsymbol{\eta}$ can have a significant effect on the structure of the optimal solution.

The paper is organized as follows: In section 2 we derive the singular value decomposition $\mathbf{V}\boldsymbol{\Sigma}\boldsymbol{\Pi}\mathbf{U}^*$ of a solution to (1.4). Section 3 gives the optimal solution, assuming the permutation $\boldsymbol{\Pi}$ is known. In section 4, we evaluate $\boldsymbol{\Pi}$ in the special case $\mathbf{H} = \mathbf{I}$. When $\mathbf{H} \neq \mathbf{I}$, $\boldsymbol{\Pi}$ depends on P . Section 5 evaluates the permutation in the limit as P tends to infinity, while section 6 analyzes the limit as P tends to zero. Finally, section 7 explores the dependence of the permutation on P using randomly generated test problems. For general P , we present a family of permutations which often contains the optimal $\boldsymbol{\Pi}$.

Notation. Throughout the paper, we use the following notation: $\mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^*$ is the singular value decomposition of \mathbf{H} (see Figure 1.1) and $\boldsymbol{\lambda}$ is the diagonal of $\boldsymbol{\Lambda}$. The diagonal of a rectangular matrix $\boldsymbol{\Lambda}$ are the entries $\Lambda_{ii}, i = 1, 2, \dots, \min\{m, n\}$. $\mathbf{V}_w\boldsymbol{\Omega}\mathbf{V}_w^*$ and $\mathbf{V}_t\boldsymbol{\Theta}\mathbf{V}_t^*$ are diagonalizations of the Hermitian matrices \mathbf{W} and \mathbf{T} , respectively, while $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ are the diagonals of $\boldsymbol{\Omega}$ and $\boldsymbol{\Theta}$. The diagonal elements are ordered as follows:

$$(1.5) \quad \lambda_i \geq \lambda_{i+1}, \quad \theta_i \leq \theta_{i+1}, \quad \text{and} \quad \omega_i \leq \omega_{i+1}.$$

M denotes the minimum of m and the rank of \mathbf{H} . The trace of a matrix is denoted “tr,” “*” denotes conjugate transpose, \mathcal{S}^c denotes complement of the set \mathcal{S} , and $|\mathcal{S}|$ is the number of elements in \mathcal{S} . A diagonal matrix \mathbf{D} is said to be nondegenerate if

Decomposition	Dimension	Description
$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^*$	$n \times l$	Singular value decomposition of observer
$\mathbf{W} = \mathbf{V}_w\mathbf{\Omega}\mathbf{V}_w^*$	$m \times m$	Diagonalization of covariance of \mathbf{w}
$\mathbf{T} = \mathbf{V}_t\mathbf{\Theta}\mathbf{V}_t^*$	$l \times l$	Diagonalization of inverse of covariance of $\boldsymbol{\theta}$
$\mathbf{P} = \mathbf{V}_w\mathbf{S}\mathbf{V}_t^*$	$m \times n$	Change of variables when $\mathbf{H} = \mathbf{I}$
$\mathbf{P} = \mathbf{V}_w\mathbf{S}\mathbf{U}^*$	$m \times n$	Change of variables when $\mathbf{\Theta} = \mathbf{I}$

FIG. 1.1. Summary of decompositions, $l = n$ without loss of generality.

the following condition is satisfied:

$$(1.6) \quad d_{ii} \neq d_{jj} > 0 \text{ for all } i \neq j.$$

For any matrix \mathbf{A} , $\text{Col}_k(\mathbf{A})$ denotes the submatrix formed by the first k columns, while $\text{Prin}_k(\mathbf{A})$ denotes the k by k leading principal submatrix. \mathcal{P}_m is the set of bijections of $\{1, 2, \dots, m\}$ onto itself (the set of all permutations of the integers between 1 and m).

2. Solution structure. We begin by analyzing the structure of an optimal solution to (1.4). Let us make the following change of variables:

$$(2.1) \quad \mathbf{P} = \mathbf{V}_w\mathbf{S}\mathbf{U}^* \text{ (if } \mathbf{\Theta} = \mathbf{I}) \text{ or } \mathbf{P} = \mathbf{V}_w\mathbf{S}\mathbf{V}_t^* \text{ (if } \mathbf{H} = \mathbf{I}).$$

With these substitutions, (1.4) reduces to the following problem in the cases $\mathbf{H} = \mathbf{I}$ or $\mathbf{T} = \mathbf{I}$:

$$(2.2) \quad \min_{\mathbf{S}} \text{tr} ((\mathbf{S}\mathbf{\Lambda})^*(\mathbf{\Omega} + \mathbf{S}\mathbf{S}^*)^{-1}\mathbf{S}\mathbf{\Lambda} + \mathbf{\Theta})^{-1}$$

subject to $\text{tr} (\mathbf{S}\mathbf{S}^*) \leq P, \quad \mathbf{S} \in \mathbb{C}^{m \times n}.$

If $\mathbf{H} = \mathbf{I}$, then $l = n$. We now show that in general (2.2) can always be transformed to an equivalent problem with $l = n$. Note though that the transformed problem may have zero singular values in \mathbf{H} even when the singular values of the original \mathbf{H} are strictly positive. If $l > n$, then define $\bar{\mathbf{\Lambda}} = \text{Col}_n(\mathbf{\Lambda})$, the submatrix formed by the first n columns of $\mathbf{\Lambda}$, and define

$$\bar{\mathbf{C}} = ((\mathbf{S}\bar{\mathbf{\Lambda}})^*(\mathbf{\Omega} + \mathbf{S}\mathbf{S}^*)^{-1}\mathbf{S}\bar{\mathbf{\Lambda}} + \mathbf{\Theta}_1)^{-1},$$

where $\mathbf{\Theta}_1 = \text{Prin}_n(\mathbf{\Theta})$, the leading n by n principal submatrix of $\mathbf{\Theta}$. Since the last $l - n$ columns of $\mathbf{\Lambda}$ are zero, the covariance matrix

$$\mathbf{C} = ((\mathbf{S}\mathbf{\Lambda})^*(\mathbf{\Omega} + \mathbf{S}\mathbf{S}^*)^{-1}\mathbf{S}\mathbf{\Lambda} + \mathbf{\Theta})^{-1}$$

has the structure

$$\mathbf{C} = \begin{bmatrix} \bar{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta}_2 \end{bmatrix},$$

where $\mathbf{\Theta}_2$ is the trailing $l - n$ by $l - n$ submatrix of $\mathbf{\Theta}$. Hence,

$$\text{tr} (\mathbf{C}) = \text{tr} (\bar{\mathbf{C}}) + \text{tr} (\mathbf{\Theta}_2^{-1}),$$

and minimizing the trace of \mathbf{C} is equivalent to minimizing the trace of $\bar{\mathbf{C}}$ (since $\mathbf{\Theta}_2$ does not depend on \mathbf{S}).

On the other hand, suppose that $l < n$. Let $\mathbf{\Lambda}_0$ be the matrix obtained by appending $n - l$ columns of zeros to the right side of $\mathbf{\Lambda}$, let $\overline{\mathbf{\Theta}}$ be the matrix obtained by appending $n - l$ trailing ones on the diagonal of $\mathbf{\Theta}$, and define

$$\mathbf{C}_0 = ((\mathbf{S}\mathbf{\Lambda}_0)^*(\mathbf{\Omega} + \mathbf{S}\mathbf{S}^*)^{-1}\mathbf{S}\mathbf{\Lambda}_0 + \overline{\mathbf{\Theta}})^{-1}.$$

The matrix \mathbf{C}_0 has the following structure:

$$\mathbf{C}_0 = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Hence, $\text{tr}(\mathbf{C}_0) = \text{tr}(\mathbf{C}) + n - l$, and minimizing the trace of \mathbf{C} is equivalent to minimizing the trace of \mathbf{C}_0 . In either case $l > n$ or $l < n$, we are able to formulate a problem with the associated $\mathbf{\Lambda}$ square and with the same solution as the original problem. Consequently, it is assumed henceforth that $l = n$. We begin by formulating the first-order optimality conditions for (2.2).

LEMMA 2.1. *If \mathbf{S} is a solution of (2.2) and $\boldsymbol{\lambda} > \mathbf{0}$, then there exists $\mu > 0$ such that*

$$(2.3) \quad (\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\boldsymbol{\Lambda}\mathbf{M}^{-2}\boldsymbol{\Lambda}\mathbf{S}^*\mathbf{L}^{-1} = \mu\mathbf{S}^*,$$

where

$$(2.4) \quad \mathbf{L} = \mathbf{\Omega} + \mathbf{S}\mathbf{S}^* \quad \text{and} \quad \mathbf{M} = \boldsymbol{\Lambda}\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}\boldsymbol{\Lambda} + \mathbf{\Theta}.$$

Moreover, the matrices $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ and $\boldsymbol{\Lambda}\mathbf{M}^{-2}\boldsymbol{\Lambda}$ commute.

Proof. If $\mathbf{S} = \mathbf{0}$, then the results hold trivially. Suppose that $\mathbf{S} \neq \mathbf{0}$. The first-order necessary optimality conditions are satisfied at any nonzero solution of (2.2) since the gradient of the constraint does not vanish. Hence, if \mathbf{S} is a solution of (2.2), then there exists a $\mu \geq 0$ such that the Fréchet derivative of the Lagrangian vanishes at \mathbf{S} . The Lagrangian associated with the optimization problem (2.2) is

$$(2.5) \quad \text{tr}((\boldsymbol{\Lambda}\mathbf{S}^*(\mathbf{\Omega} + \mathbf{S}\mathbf{S}^*)^{-1}\mathbf{S}\boldsymbol{\Lambda} + \mathbf{\Theta})^{-1} + \mu\mathbf{S}\mathbf{S}^*),$$

where the multiplier $\mu \geq 0$ is a real scalar. As shown in the Appendix, when we equate to zero the derivative of the Lagrangian, we obtain (2.3).

We now show that the multiplier μ is strictly positive. Suppose $\mu = 0$. Since $\mathbf{\Omega}$ and $\mathbf{\Theta}$ are positive definite (see section 1), the factors \mathbf{L} and \mathbf{M} in (2.3) are positive definite. Since $\boldsymbol{\lambda} > \mathbf{0}$, $\boldsymbol{\Lambda}$ is positive definite. If we can show that $(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})$ is invertible, then $\mu = 0$ implies that $\mathbf{S} = \mathbf{0}$, which is a contradiction. Thus $\mu > 0$.

To show that $(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})$ is invertible, we apply the matrix modification formula [3]

$$(2.6) \quad (\mathbf{I} + \mathbf{Z}\mathbf{Z}^*)^{-1} = \mathbf{I} - \mathbf{Z}(\mathbf{I} + \mathbf{Z}^*\mathbf{Z})^{-1}\mathbf{Z}^*$$

with $\mathbf{Z}^* = \mathbf{\Omega}^{-1/2}\mathbf{S}$ to obtain

$$\begin{aligned} \mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S} &= \mathbf{I} - \mathbf{S}^*(\mathbf{\Omega} + \mathbf{S}\mathbf{S}^*)^{-1}\mathbf{S} \\ &= \mathbf{I} - \mathbf{S}^*\mathbf{\Omega}^{-1/2}(\mathbf{I} + \mathbf{\Omega}^{-1/2}\mathbf{S}\mathbf{S}^*\mathbf{\Omega}^{-1/2})^{-1}\mathbf{\Omega}^{-1/2}\mathbf{S} \\ &= \mathbf{I} - \mathbf{Z}(\mathbf{I} + \mathbf{Z}^*\mathbf{Z})^{-1}\mathbf{Z}^* = (\mathbf{I} + \mathbf{Z}\mathbf{Z}^*)^{-1} \\ &= (\mathbf{I} + \mathbf{S}^*\mathbf{\Omega}^{-1}\mathbf{S})^{-1}. \end{aligned}$$

Hence, the matrix $\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ is positive definite and invertible. This completes the proof that $\mu > 0$.

We multiply (2.3) by \mathbf{S} to obtain

$$(2.7) \quad \mu\mathbf{S}^*\mathbf{S} = (\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}.$$

Forming the conjugate transpose of (2.7) gives

$$(2.8) \quad \mu\mathbf{S}^*\mathbf{S} = \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}).$$

Equating the right sides of (2.7) and (2.8) yields

$$(2.9) \quad (\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})(\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}) = (\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda})(\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}).$$

Hence, the matrices $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ and $\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}$ commute. \square

We now present cases where (2.2) has a solution with at most one nonzero in each row and column.

LEMMA 2.2. *Suppose \mathbf{S} is a solution of (2.2) with the property that $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ is block diagonal and $\text{Prin}_k(\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})$ is diagonal for some $k > 0$. If $\mathbf{\Omega}$ is nondegenerate and $\mathbf{\Lambda}$ is positive definite, then $\text{Col}_k(\mathbf{S}) = \mathbf{\Pi}_1\mathbf{\Sigma}\mathbf{\Pi}_2$ where $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ are permutation matrices and $\mathbf{\Sigma}$ is diagonal.*

Proof. Since \mathbf{S} is a solution of (2.2), (2.3) holds. We multiply (2.3) by \mathbf{L} to obtain

$$(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}\mathbf{S}^* = \mu\mathbf{S}^*\mathbf{L} = \mu\mathbf{S}^*(\mathbf{\Omega} + \mathbf{S}\mathbf{S}^*).$$

Rearranging this, we have

$$(2.10) \quad \mathbf{S}^*\mathbf{\Omega} = \mathbf{E}\mathbf{S}^*,$$

where

$$\mathbf{E} = \frac{1}{\mu}(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})(\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}) - \mathbf{S}^*\mathbf{S}.$$

Since $\mathbf{\Lambda}$, $\mathbf{\Omega}$, and $\mathbf{\Theta}$ are diagonal and $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ is block diagonal, it follows that $\mathbf{M} = \mathbf{\Lambda}\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}\mathbf{\Lambda} + \mathbf{\Theta}$ is block diagonal and $\text{Prin}_k(\mathbf{M})$ is diagonal. By (2.7), $\mathbf{S}^*\mathbf{S}$ is block diagonal and $\text{Prin}_k(\mathbf{S}^*\mathbf{S})$ is diagonal. Hence, \mathbf{E} is block diagonal and $\text{Prin}_k(\mathbf{E})$ is diagonal. Let e_i denote the i th diagonal element of \mathbf{E} . For $1 \leq i \leq k$ and $1 \leq j \leq m$, we equate the (i, j) elements in (2.10) to obtain

$$(\mathbf{S}^*)_{ij}\omega_j = e_i(\mathbf{S}^*)_{ij} \quad \text{or} \quad (\mathbf{S}^*)_{ij}(\omega_j - e_i) = 0.$$

If $(\mathbf{S}^*)_{ij} \neq 0$, then $\omega_j = e_i$. By the nondegeneracy assumption, the ω_j , $1 \leq j \leq m$, are all distinct. Consequently, there is at most one j for which $(\mathbf{S}^*)_{ij} \neq 0$. In other words, each of the first k columns of \mathbf{S} has at most one nonzero. Since $\text{Prin}_k(\mathbf{S}^*\mathbf{S})$ is diagonal, no two of the leading k columns of \mathbf{S} can have their single nonzero in the same row. A suitable permutation of the rows and the first k columns of \mathbf{S} yields a diagonal matrix $\mathbf{\Sigma}$. \square

We now apply Lemma 2.2 to the case $\mathbf{\Lambda} = \mathbf{I}$:

LEMMA 2.3. *If $\mathbf{\Lambda} = \mathbf{I}$, then there exists a solution of (2.2) of the form $\mathbf{S} = \mathbf{\Pi}_1\mathbf{\Sigma}\mathbf{\Pi}_2$ where $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ are permutation matrices and $\mathbf{\Sigma}$ is diagonal.*

Proof. Since any $\mathbf{\Omega}$ and $\mathbf{\Theta}$ can be approximated arbitrarily closely by nondegenerate matrices, there is no loss of generality in assuming that $\mathbf{\Omega}$ and $\mathbf{\Theta}$ are nondegenerate

(see [2]). There exists an optimal solution of (2.2) since the feasible set is compact and the cost function is a continuous function of \mathbf{S} .

By Lemma 2.1, the matrices $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ and $\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}$ commute. Since $\mathbf{\Lambda} = \mathbf{I}$, it follows that $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ and \mathbf{M}^{-2} commute. Since commuting matrices share a common set of eigenvectors [11, p. 249], and since the eigenvectors of \mathbf{M}^{-2} and \mathbf{M} are the same, it follows that $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ and \mathbf{M} commute:

$$(\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\mathbf{M} = \mathbf{M}(\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}).$$

This implies that

$$(\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})(\mathbf{\Theta} + \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}) = (\mathbf{\Theta} + \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})(\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}),$$

which reduces to

$$(\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\mathbf{\Theta} = \mathbf{\Theta}(\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}).$$

Since $\mathbf{\Theta}$ satisfies the nondegeneracy condition, we conclude that $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ is diagonal. Taking $k = n$ (the number of columns in \mathbf{S}) in Lemma 2.2, $\mathbf{S} = \text{Prin}_k(\mathbf{S}) = \mathbf{\Pi}_1\mathbf{\Sigma}\mathbf{\Pi}_2$. \square

The case $\mathbf{\Theta} = \mathbf{I}$ and $\mathbf{\Lambda} \neq \mathbf{I}$ is tougher to analyze. In an effort to simplify the structure of a solution to (2.2), we will apply a permutation to our problem. Let $\mathbf{\Pi}$ be a permutation matrix which we will apply to the columns of \mathbf{S} . The permuted matrix is $\mathbf{S}_p = \mathbf{S}\mathbf{\Pi}$. Let $\mathbf{\Lambda}_p = \mathbf{\Pi}^*\mathbf{\Lambda}\mathbf{\Pi}$ be the symmetric permutation of the rows and columns of $\mathbf{\Lambda}$. In essence, $\mathbf{\Lambda}_p$ is obtained from $\mathbf{\Lambda}$ by interchanging diagonal elements. Similarly, $\mathbf{\Theta}_p = \mathbf{\Pi}^*\mathbf{\Theta}\mathbf{\Pi}$ denotes the symmetric permutation of $\mathbf{\Theta}$. We replace \mathbf{S} , $\mathbf{\Lambda}$, and $\mathbf{\Theta}$ by their representation in terms of the permuted quantities to obtain the following equivalent form of (2.2) (after taking into account the fact that the trace is invariant under a similarity transformation):

$$(2.11) \quad \min_{\mathbf{S}_p} \text{tr} ((\mathbf{S}_p\mathbf{\Lambda}_p)^*(\mathbf{\Omega} + \mathbf{S}_p\mathbf{S}_p^*)^{-1}\mathbf{S}_p\mathbf{\Lambda}_p + \mathbf{\Theta}_p)^{-1}$$

subject to $\text{tr} (\mathbf{S}_p\mathbf{S}_p^*) \leq P, \quad \mathbf{S}_p \in \mathbb{C}^{m \times n}.$

We begin with the following result:

LEMMA 2.4. *Let \mathbf{D} be defined by*

$$(2.12) \quad \mathbf{D} = \mathbf{S}^*\mathbf{L}^{-1/2}(\mathbf{I} + \mathbf{W}^*\mathbf{W})^{-2}\mathbf{L}^{-1/2}\mathbf{S},$$

where $\mathbf{W} = \mathbf{\Lambda}\mathbf{S}^*\mathbf{L}^{-1/2}$ and $\mathbf{L} = \mathbf{\Omega} + \mathbf{S}\mathbf{S}^*$. If $d_{ii} = 0$, then column i of \mathbf{S} vanishes. If $\mathbf{\Theta} = \mathbf{I}$, \mathbf{S} is a solution of (2.2), and $\mathbf{\Lambda}$ is nondegenerate, then \mathbf{D} is diagonal. Let $\mathbf{D}_p = \mathbf{\Pi}^*\mathbf{D}\mathbf{\Pi}$ be the symmetrically permuted \mathbf{D} where $\mathbf{\Pi}$ is chosen so that the diagonal elements of $\mathbf{\Lambda}_p^2\mathbf{D}_p$ are in decreasing order. Then the matrix $\mathbf{S}_p^*\mathbf{L}^{-1}\mathbf{S}_p$ is block diagonal and the size of the diagonal blocks is equal to the number of times the associated diagonal elements of $\mathbf{\Lambda}_p^2\mathbf{D}_p$ repeat.

Proof. By the definition of \mathbf{D} , we have

$$d_{ii} = \|(\mathbf{I} + \mathbf{W}^*\mathbf{W})^{-1}\mathbf{L}^{-1/2}\mathbf{s}_i\|^2,$$

where \mathbf{s}_i is the i th column of \mathbf{S} . If $d_{ii} = 0$, then $\mathbf{s}_i = \mathbf{0}$.

If $\mathbf{\Theta} = \mathbf{I}$, then \mathbf{M} in (2.4) has the form $\mathbf{I} + \mathbf{W}\mathbf{W}^*$ with $\mathbf{W} = \mathbf{\Lambda}\mathbf{S}^*\mathbf{L}^{-1/2}$. It follows from the matrix modification formula (2.6) that

$$\begin{aligned} \mathbf{M}^{-1}\mathbf{W} &= (\mathbf{I} - \mathbf{W}(\mathbf{I} + \mathbf{W}^*\mathbf{W})^{-1}\mathbf{W}^*)\mathbf{W} \\ &= \mathbf{W} - \mathbf{W}(\mathbf{I} + \mathbf{W}^*\mathbf{W})^{-1}\mathbf{W}^*\mathbf{W} \\ &= \mathbf{W}(\mathbf{I} + \mathbf{W}^*\mathbf{W})^{-1}. \end{aligned}$$

Hence, we have

$$\mathbf{M}^{-2}\mathbf{W} = \mathbf{W}(\mathbf{I} + \mathbf{W}^*\mathbf{W})^{-2},$$

which implies that

$$\begin{aligned} \mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S} &= \mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{W}\mathbf{L}^{-1/2}\mathbf{S} \\ &= \mathbf{\Lambda}\mathbf{W}(\mathbf{I} + \mathbf{W}^*\mathbf{W})^{-2}\mathbf{L}^{-1/2}\mathbf{S} \\ (2.13) \quad &= \mathbf{\Lambda}^2\mathbf{S}^*\mathbf{L}^{-1/2}(\mathbf{I} + \mathbf{W}^*\mathbf{W})^{-2}\mathbf{L}^{-1/2}\mathbf{S} = \mathbf{\Lambda}^2\mathbf{D}, \end{aligned}$$

where \mathbf{D} is defined in (2.12). By (2.13), $\mathbf{\Lambda}^2\mathbf{D}$ is the product of Hermitian matrices $\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}$ and $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$. The matrices $\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}$ and $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ commute by Lemma 2.1. Consequently, $\mathbf{\Lambda}^2\mathbf{D}$ is Hermitian. Since \mathbf{D} and $\mathbf{\Lambda}$ are also Hermitian, we have

$$(\mathbf{\Lambda}^2\mathbf{D}) = (\mathbf{\Lambda}^2\mathbf{D})^* = \mathbf{D}\mathbf{\Lambda}^2.$$

Since the diagonal elements of $\mathbf{\Lambda}$ are distinct, \mathbf{D} is diagonal.

By Lemma 2.1, we can commute the factors $\mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}$ and $\mathbf{\Lambda}\mathbf{M}^{-2}\mathbf{\Lambda}$ in (2.8). Utilizing (2.13) in (2.8) gives

$$(2.14) \quad \mu\mathbf{S}^*\mathbf{S} = \mathbf{\Lambda}^2\mathbf{D}(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}).$$

Inserting (2.13) in (2.7) gives

$$(2.15) \quad \mu\mathbf{S}^*\mathbf{S} = (\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\mathbf{\Lambda}^2\mathbf{D}.$$

We equate the right sides of (2.14) and (2.15) to deduce that

$$(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\mathbf{\Lambda}^2\mathbf{D} = \mathbf{\Lambda}^2\mathbf{D}(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S}).$$

Hence, the matrix $(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})$ and the diagonal matrix $\mathbf{\Lambda}^2\mathbf{D}$ commute, and they share a common set of eigenvectors.

Suppose that $\mathbf{\Pi}$ is chosen so that the diagonal elements of $\mathbf{\Lambda}_p^2\mathbf{D}_p$ are in decreasing order. Hence, zero diagonal elements in \mathbf{D}_p trail at the end of the diagonal, and the corresponding (trailing) columns of \mathbf{S}_p vanish, as shown at the start of the proof. Suppose that $\lambda_i^2 d_{ii} = \lambda_j^2 d_{jj}$ for i and $j \in [p, q]$. The eigenvectors of $\mathbf{\Lambda}_p^2\mathbf{D}_p$ correspond to columns p through q of the identity matrix. Since $\mathbf{\Lambda}_p^2\mathbf{D}_p$ and $(\mathbf{I} - \mathbf{S}_p^*\mathbf{L}^{-1}\mathbf{S}_p)$ share a common set of eigenvectors, the corresponding eigenvectors of $(\mathbf{I} - \mathbf{S}_p^*\mathbf{L}^{-1}\mathbf{S}_p)$ are linear combinations of columns p through q of the identity matrix. Hence, $\mathbf{S}_p^*\mathbf{L}^{-1}\mathbf{S}_p$ is block diagonal and the size of the blocks is equal to the number of times a positive diagonal element of $\mathbf{\Lambda}_p^2\mathbf{D}_p$ repeats. \square

Let $\mathbf{\Lambda}_k$ and $\mathbf{\Omega}_k$, $k = 1, 2, \dots$, be nondegenerate matrices which approach limits $\mathbf{\Lambda}$ and $\mathbf{\Omega}$, respectively. Let \mathbf{S}_k be a solution to (2.2) corresponding to $(\mathbf{\Lambda}_k, \mathbf{\Omega}_k)$. Such a solution exists for each k since the objective function in (2.2) is continuous when $\mathbf{\Omega}$ and $\mathbf{\Theta}$ are positive definite and the feasible set is a compact set. By suitable pruning of the sequence $(\mathbf{\Lambda}_k, \mathbf{\Omega}_k)$ if necessary, there is no loss in generality in assuming that the sequence \mathbf{S}_k , $k = 1, 2, \dots$, converges to a limit \mathbf{S} , which is a solution of (2.2) (by the continuity of the objective function). We now show, under suitable hypothesis, that the limit \mathbf{S} is a permutation of a diagonal matrix.

LEMMA 2.5. *Suppose $\mathbf{\Theta} = \mathbf{I}$ and let \mathbf{S} be a solution of (2.2), which is a limit of a sequence of solutions \mathbf{S}_k , $k = 1, 2, \dots$, associated with nondegenerate matrices $\mathbf{\Lambda}_k$ and*

Ω_k . If the positive diagonal elements of $\Lambda^2 \mathbf{D}$ are distinct, then \mathbf{S} can be expressed $\mathbf{\Pi}_1 \mathbf{\Sigma} \mathbf{\Pi}_2$ where $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ are permutation matrices and $\mathbf{\Sigma}$ is diagonal.

In randomly generated test problems, the “distinct diagonal” property of Lemma 2.5 was always satisfied.

Proof. Since the matrices Λ_k and Ω_k are nondegenerate, the associated matrices \mathbf{D}_k (see (2.12)) are diagonal. Since the \mathbf{D}_k converge to \mathbf{D} , the limit \mathbf{D} is diagonal. Since the positive diagonal elements of $\Lambda^2 \mathbf{D}$ are distinct, the associated diagonal elements of $\Lambda_k^2 \mathbf{D}_k$ are distinct for k sufficiently large. Assume that the columns of \mathbf{S} and the rows of Λ are permuted so that the diagonal elements of the limit $\Lambda^2 \mathbf{D}$ are in decreasing order. Let p be the number of positive diagonal elements of \mathbf{D} . By Lemma 2.4, $\text{Prin}_p(\mathbf{S}_k^* \mathbf{L}_k^{-1} \mathbf{S}_k)$ is diagonal. By Lemma 2.2, $\text{Col}_p(\mathbf{S}_k) = \mathbf{\Pi}_{1k} \mathbf{\Sigma}_k \mathbf{\Pi}_{2k}$. Also, by Lemma 2.4, columns $l + 1$ through n of \mathbf{S} vanish. Hence, the limit \mathbf{S} can be expressed as a product $\mathbf{\Pi}_1 \mathbf{\Sigma} \mathbf{\Pi}_2$. \square

Due to the ordering (1.5), one of the permutations in Lemma 2.3 or Lemma 2.5 can be eliminated.

THEOREM 2.6. *If either $\Theta = \mathbf{I}$ or $\Lambda = \mathbf{I}$ and (2.2) has a solution of the form $\mathbf{S} = \mathbf{\Pi}_1 \mathbf{\Sigma} \mathbf{\Pi}_2$, where $\mathbf{\Sigma}$ is diagonal and the $\mathbf{\Pi}_i$ are permutation matrices, then (2.2) has a solution of the form $\mathbf{S} = \mathbf{\Sigma} \mathbf{\Pi}$ where $\mathbf{\Sigma}$ is diagonal and $\mathbf{\Pi}$ is a permutation matrix. Moreover, if the diagonal σ of an optimal $\mathbf{\Sigma}$ has p positive components, then p is less than or equal to the rank of Λ and $\mathbf{\Pi}$ permutes only the first p column of $\mathbf{\Sigma}$. There also exists a solution of (2.2) of the form $\mathbf{S} = \mathbf{\Pi} \mathbf{\Sigma}$.*

Proof. The substitution $\mathbf{S} = \mathbf{\Pi}_1 \mathbf{\Sigma} \mathbf{\Pi}_2$ in (2.2) yields the following equivalent problem (assuming $l = n$):

$$(2.16) \quad \min_{\sigma, \mathbf{\Pi}_1, \mathbf{\Pi}_2} \text{tr} \left((\mathbf{\Pi}_2 \mathbf{\Lambda} \mathbf{\Pi}_2^*) \mathbf{\Sigma}^* (\mathbf{\Pi}_1^* \mathbf{\Omega} \mathbf{\Pi}_1 + \mathbf{\Sigma} \mathbf{\Sigma}^*)^{-1} \mathbf{\Sigma} (\mathbf{\Pi}_2 \mathbf{\Lambda} \mathbf{\Pi}_2^*) + \mathbf{\Pi}_2 \mathbf{\Theta} \mathbf{\Pi}_2^* \right)^{-1}$$

subject to $\sum_{i=1}^N \sigma_i^2 \leq P,$

where N is the minimum of m and n . Here the minimization is over diagonal matrices $\mathbf{\Sigma}$ with σ on the diagonal, and permutation matrices $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$.

A symmetric permutation such as $\mathbf{\Pi}_2 \mathbf{\Lambda} \mathbf{\Pi}_2^*$ interchanges diagonal elements. Hence, (2.16) is equivalent to

$$(2.17) \quad \min_{\sigma, \pi_1, \pi_2} \sum_{i=1}^N \frac{\omega_{\pi_1(i)} + \sigma_i^2}{\theta_{\pi_2(i)} \omega_{\pi_1(i)} + (\theta_{\pi_2(i)} + \lambda_{\pi_2(i)}^2) \sigma_i^2}$$

subject to $\sum_{i=1}^N \sigma_i^2 \leq P, \quad \pi_1 \in \mathcal{P}_m, \quad \pi_2 \in \mathcal{P}_n,$

where \mathcal{P}_m is the set of bijections of $\{1, 2, \dots, m\}$ onto itself.

Let σ denote an optimal solution of (2.17). If $\lambda_{\pi_2(i)} = 0$, then the associated term in the objective function of (2.17) reduces to $1/\theta_{\pi_2(i)}$, independent of σ_i . In this case $\sigma_i = 0$ is optimal (see Theorem 3.1 and (3.2) in the next section). Hence, the number of positive components of σ is less than or equal to the rank of Λ . Define the set

$$\mathcal{S} = \{i : \sigma_i > 0\},$$

and let $p = |\mathcal{S}|$. The function

$$\frac{\omega + x}{\omega \theta + (\theta + \lambda^2)x}, \quad x > 0$$

is monotone increasing in $\omega \geq 0$ and monotone decreasing in $\lambda \geq 0$. Since the objective function is being minimized in (2.17), it follows that $\omega_{\pi_1(i)}$ is one of the p smallest elements of ω . In the same fashion, if $\Theta = \mathbf{I}$ and $i \in \mathcal{S}$, then $\lambda_{\pi_2(i)}$ is one of the p largest elements of λ .

Finally, let us consider the case $\mathbf{A} = \mathbf{I}$. The cost function in (2.17) is the sum of two expressions:

$$(2.18) \quad \sum_{i \in \mathcal{S}} \frac{\omega_{\pi_1(i)} + \sigma_i^2}{\theta_{\pi_2(i)}\omega_{\pi_1(i)} + (\theta_{\pi_2(i)} + \lambda_{\pi_2(i)}^2)\sigma_i^2} + \sum_{j \in \mathcal{S}^c} \frac{1}{\theta_{\pi_2(j)}}.$$

We now show that if $i \in \mathcal{S}$, but $\theta_{\pi_2(i)}$ is not one of the p smallest elements of θ , then the cost function is decreased by exchanging $\pi_2(i)$ with $\pi_2(j)$ where $j \in \mathcal{S}^c$ and $\theta_{\pi_2(j)} < \theta_{\pi_2(i)}$. Let $\beta_1 = \theta_{\pi_2(i)}$ and $\beta_2 = \theta_{\pi_2(j)}$, and define

$$V_1 = \frac{\omega + \sigma^2}{\omega\beta_1 + \sigma^2(\beta_1 + 1)} + \frac{1}{\beta_2} \quad \text{and} \quad V_2 = \frac{\omega + \sigma^2}{\omega\beta_2 + \sigma^2(\beta_2 + 1)} + \frac{1}{\beta_1}.$$

Here V_1 represents the i and j terms in (2.18) after substituting $\lambda_{\pi_2(i)} = 1$, while V_2 reflects the corresponding terms after the exchange of $\pi_2(i)$ with $\pi_2(j)$. Since $\beta_1 > \beta_2$, it can be shown that $V_1 - V_2 \geq 0$ (cross multiply and cancel terms). Hence, by exchanging $\pi_2(j)$ with $\pi_2(i)$, the cost function is decreased. In summary, if either $\Theta = \mathbf{I}$ or $\mathbf{A} = \mathbf{I}$, then for $i \in \mathcal{S}$, $\lambda_{\pi_2(i)}$ is one of the p largest elements in λ while $\theta_{\pi_2(i)}$ and $\omega_{\pi_1(i)}$ are among the p smallest elements in θ and ω , respectively. Due to the ordering (1.5),

$$\{\pi_1(i) : i \in \mathcal{S}\} \subset \{1, 2, \dots, p\} \quad \text{and} \quad \{\pi_2(i) : i \in \mathcal{S}\} \subset \{1, 2, \dots, p\}.$$

Let $\pi_3 \in \mathcal{P}_N$ be chosen so that

$$\sigma_{\pi_3(1)} \geq \sigma_{\pi_3(2)} \geq \dots \geq \sigma_{\pi_3(N)}.$$

Since \mathcal{S} is the set of indices of positive components of σ , we have

$$\mathcal{S} = \{\pi_3(i) : i = 1, 2, \dots, p\}.$$

Define $\hat{\pi}_1 = \pi_1(\pi_3)$, $\hat{\pi}_2 = \pi_2(\pi_3)$, and $\hat{\sigma}_i = \sigma_{\pi_3(i)}$. The optimal cost (2.18) can be written

$$\sum_{i=1}^p \frac{\omega_{\hat{\pi}_1(i)} + \hat{\sigma}_i^2}{\theta_{\hat{\pi}_2(i)}\omega_{\hat{\pi}_1(i)} + (\theta_{\hat{\pi}_2(i)} + \lambda_{\hat{\pi}_2(i)}^2)\hat{\sigma}_i^2} + \sum_{i>p} \frac{1}{\theta_{\hat{\pi}_2(i)}}.$$

Hence, (2.2) has a solution of the form $\hat{\mathbf{S}} = \hat{\mathbf{\Pi}}_1 \hat{\mathbf{\Sigma}} \hat{\mathbf{\Pi}}_2$ where $\hat{\mathbf{\Pi}}_1$ permutes only the first p rows and $\hat{\mathbf{\Pi}}_2$ permutes only the first p columns of $\hat{\mathbf{\Sigma}}$. Let $\bar{\mathbf{\Pi}}_1$ be a permutation matrix which is the same as $\hat{\mathbf{\Pi}}_1$ except that it has been expanded (by an identity matrix) or chopped ($\bar{\mathbf{\Pi}}_1 = \text{Prin}_n(\hat{\mathbf{\Pi}}_1)$) to match the number of columns of \mathbf{S} . Define $\mathbf{\Sigma}' = \bar{\mathbf{\Pi}}_1 \hat{\mathbf{\Sigma}} \bar{\mathbf{\Pi}}_1^*$. $\mathbf{\Sigma}'$ is diagonal since it is a symmetric permutation of a diagonal matrix. Consequently, we have

$$\hat{\mathbf{S}} = \hat{\mathbf{\Pi}}_1 \hat{\mathbf{\Sigma}} \hat{\mathbf{\Pi}}_2 = \hat{\mathbf{\Pi}}_1 \hat{\mathbf{\Sigma}} \bar{\mathbf{\Pi}}_1^* \bar{\mathbf{\Pi}}_1 \hat{\mathbf{\Pi}}_2 = \mathbf{\Sigma}' \mathbf{\Pi},$$

where $\mathbf{\Pi} = \bar{\mathbf{\Pi}}_1 \hat{\mathbf{\Pi}}_2$. In a similar manner, we obtain $\mathbf{S} = \mathbf{\Pi} \mathbf{\Sigma}$ for a different choice of $\mathbf{\Pi}$ and $\mathbf{\Sigma}$. \square

Based on Theorem 2.6, one of the permutation in (2.17) can be deleted when $\mathbf{\Lambda} = \mathbf{I}$ or $\mathbf{\Theta} = \mathbf{I}$. We delete the permutation π_2 to obtain the following problem:

$$(2.19) \quad \min_{\mathbf{s}, \pi} \sum_{i=1}^M \frac{\omega_{\pi(i)} + s_i}{\theta_i \omega_{\pi(i)} + (\theta_i + \lambda_i) s_i}$$

$$\text{subject to } \sum_{i=1}^M s_i \leq P, \quad \mathbf{s} \geq \mathbf{0}, \quad \pi \in \mathcal{P}_M,$$

where $s_i = \sigma_i^2$ and M is the minimum of m and the rank of $\mathbf{\Lambda}$. If \mathbf{s} and π are solutions of (2.19), then $\mathbf{S} = \mathbf{\Pi}\mathbf{\Sigma}$ where $\sigma_i^2 = s_i$ is a solution of (2.2). We now combine Theorem 2.6 with the change of variables (2.1).

COROLLARY 2.7. *If $\mathbf{H} = \mathbf{I}$, then (1.4) has a solution of the form $\mathbf{P} = \mathbf{V}_w \mathbf{\Sigma} \mathbf{\Pi} \mathbf{V}_t^*$, where $\mathbf{\Pi}$ is a permutation matrix and $\mathbf{\Sigma}$ is diagonal. If $\mathbf{\Theta} = \mathbf{I}$ and (2.2) has a nondegenerate solution as described in Lemma 2.5, then (1.4) has a solution of the form $\mathbf{P} = \mathbf{V}_w \mathbf{\Sigma} \mathbf{\Pi} \mathbf{U}^*$.*

Remark. As in Theorem 2.6, the factor $\mathbf{\Sigma}\mathbf{\Pi}$ in Corollary 2.7 can be replaced by $\mathbf{\Pi}\mathbf{\Sigma}$.

3. The optimal $\mathbf{\Sigma}$. Assuming the permutation π in (2.19) is given, let us now consider the problem of optimizing over σ . To simplify the indexing, the permutation is suppressed and we consider the problem:

$$(3.1) \quad \min_{\sigma} \sum_{i=1}^M \frac{\omega_i + s_i}{\theta_i \omega_i + (\theta_i + \lambda_i^2) s_i} \quad \text{subject to } \sum_{i=1}^M s_i \leq P, \quad \mathbf{s} \geq \mathbf{0}.$$

The solution of (3.1) can be expressed in terms of a Lagrange multiplier for the constraint (this solution technique is often called “water filling” [1] in the communication literature).

THEOREM 3.1. *The optimal solution of (3.1) is given by*

$$(3.2) \quad s_i = \frac{1}{\theta_i + \lambda_i^2} \max \left\{ \sqrt{\frac{\omega_i \lambda_i^2}{\mu}} - \theta_i \omega_i, 0 \right\},$$

where the parameter μ is chosen so that

$$(3.3) \quad \sum_{i=1}^M s_i = P.$$

Proof. Since the minimization in (3.1) takes place over a closed, bounded set, there exists a solution. Since the function $(\omega_i + x)/(\theta_i \omega_i + (\theta_i + \lambda_i^2)x)$ is a decreasing function of $x \geq 0$, the objective function decreases when s_i increases. Hence, there exists a solution of (3.1) with the inequality constraint active. Due to the strict convexity of the cost function and the convexity of the constraints, (3.1) has a unique solution.

The first-order optimality conditions (KKT conditions) for an optimal solution of (3.1) are the following: There exists a scalar $\mu \geq 0$ and a vector $\nu \in \mathbb{R}^M$ such that

$$(3.4) \quad \mu - \nu_i - \frac{\omega_i \lambda_i^2}{(\theta_i \omega_i + (\theta_i + \lambda_i^2) s_i)^2} = 0, \quad \nu_i \geq 0, \quad s_i \geq 0, \quad \text{and } \nu_i s_i = 0,$$

$1 \leq i \leq M$. Any solution of (3.4) is the unique optimal solution of (3.1).

A solution to (3.4) is obtained as follows: Define the function

$$(3.5) \quad s_i(\mu) = \frac{1}{\theta_i + \lambda_i^2} \left(\sqrt{\frac{\omega_i \lambda_i^2}{\mu}} - \theta_i \omega_i \right)^+.$$

Here $x^+ = \max\{x, 0\}$. This particular value for s_i is obtained by setting $\nu_i = 0$ in (3.4), solving for s_i , and replacing the solution by 0 when it is negative. Observe that $s_i(\mu)$ is a decreasing function of μ that approaches $+\infty$ as μ approaches 0 and that approaches 0 as μ tends to $+\infty$. Hence, the equation

$$(3.6) \quad \sum_{i=1}^M s_i(\mu) = P$$

has a unique positive solution. Observe that $s_i(\mu) = 0$ if and only if $\mu \geq \lambda_i^2 / (\theta_i^2 \omega_i)$. Moreover, if $\mu \geq \lambda_i^2 / (\omega_i \theta_i^2)$, then

$$\mu - \frac{\omega_i \lambda_i^2}{(\theta_i \omega_i + (\theta_i + \lambda_i^2) s_i(\mu))^2} = \mu - \frac{\lambda_i^2}{\omega_i \theta_i^2} \geq 0.$$

It follows that the KKT conditions are satisfied by the positive solution of (3.6). \square

4. Optimal permutation for $\Lambda = \mathbf{I}$. Starting with this section, we will determine optimal permutations π in (2.19). When $\Lambda = \mathbf{I}$, the optimal permutation is the identity (due to the ordering (1.5)):

THEOREM 4.1. *If $\Lambda = \mathbf{I}$, then $\pi(i) = i$, for all i is optimal in (2.19).*

Proof. Recall that the components of ω and θ are in increasing order. Let p be the number of positive components of an optimal \mathbf{s} in (2.19). By Theorem 2.6, an optimal permutation π permutes only the first p components of ω ; moreover, $s_i > 0$ for $i \leq p$ and $s_i = 0$ for $i > p$.

Suppose that there exists a permutation π which is optimal in (2.19) and with the property that $\omega_{\pi(i)} > \omega_{\pi(j)}$ for some $i < j \leq p$. Since the components of θ are in increasing order, $\theta_i \leq \theta_j$. We will show that by interchanging components i and j of π , the objective function value does not increase. Consequently, after a finite number of pairwise exchanges, and without increasing the cost, it can be arranged so that $\omega_{\pi(i)}$ is an increasing function of i . Since $1 \leq \pi(i) \leq p$ for $i \leq p$ and since the components of ω are in increasing order, we conclude that $\pi(i) = i$ for all i is optimal in (2.19).

Let \mathbf{s} denote a solution of (2.19) associated with the permutation π and suppose that $\omega_{\pi(i)} > \omega_{\pi(j)}$ for some $i < j \leq p$. For notational convenience, let us take $i = 1$, $j = 2$, $\pi(1) = 2$, and $\pi(2) = 1$. Define $\omega'_1 = \omega_{\pi(1)} = \omega_2$ and $\omega'_2 = \omega_{\pi(2)} = \omega_1$. Due to the optimality of \mathbf{s} and π , $t_1 = s_1 > 0$ and $t_2 = s_2 > 0$ is an optimal solution of the following 2-variable problem:

$$(4.1) \quad \min_{\mathbf{t}} \sum_{i=1}^2 \frac{\omega'_i + t_i}{\theta_i \omega'_i + (\theta_i + 1)t_i}$$

$$\text{subject to } \sum_{i=1}^2 t_i \leq \bar{P} := s_1 + s_2, \quad \mathbf{t} \geq \mathbf{0}.$$

We will show that the optimal objective function value for the following unpermuted problem is less than or equal to the objective function value for (4.1):

$$(4.2) \quad \min_{\mathbf{t}} \sum_{i=1}^2 \frac{\omega_i + t_i}{\theta_i \omega_i + (\theta_i + 1)t_i}$$

subject to $\sum_{i=1}^2 t_i \leq \bar{P}, \quad \mathbf{t} \geq \mathbf{0}.$

By assumption, the solution of (4.1) is strictly positive. We now show that this implies the solution of (4.2) is strictly positive. By Theorem 3.1, the condition $\mathbf{s} > \mathbf{0}$ is equivalent to

$$(4.3) \quad 1/\sqrt{\mu} > \theta_i \sqrt{\omega'_i}.$$

The multiplier μ is given by

$$(4.4) \quad \frac{1}{\sqrt{\mu}} = \frac{\bar{P} + \sum_{j=1}^2 \frac{\theta_j \omega'_j}{(\theta_j + 1)}}{\sum_{j=1}^2 \frac{\omega_j'^{1/2}}{(\theta_j + 1)}}.$$

Combining (4.3) and (4.4) gives

$$(4.5) \quad \bar{P} > \frac{\theta_1 \sqrt{\omega'_1 \omega'_2}}{(1 + \theta_2)} - \frac{\theta_2 \omega'_1}{(1 + \theta_2)} = \frac{\theta_1 \sqrt{\omega_1 \omega_2}}{(1 + \theta_2)} - \frac{\theta_2 \omega_2}{(1 + \theta_2)} \text{ and}$$

$$(4.6) \quad \bar{P} > \frac{\theta_2 \sqrt{\omega'_1 \omega'_2}}{(1 + \theta_1)} - \frac{\theta_1 \omega'_2}{(1 + \theta_1)} = \frac{\theta_2 \sqrt{\omega_1 \omega_2}}{(1 + \theta_1)} - \frac{\theta_1 \omega_1}{(1 + \theta_1)}.$$

Above, the first inequality corresponds to the condition $s_1 > 0$ while the second corresponds to $s_2 > 0$. Similarly, the optimal \mathbf{t} in (4.2) is positive if and only if

$$(4.7) \quad \bar{P} > \frac{\theta_1 \sqrt{\omega_1 \omega_2}}{(1 + \theta_2)} - \frac{\theta_2 \omega_1}{(1 + \theta_2)} \text{ and}$$

$$(4.8) \quad \bar{P} > \frac{\theta_2 \sqrt{\omega_1 \omega_2}}{(1 + \theta_1)} - \frac{\theta_1 \omega_2}{(1 + \theta_1)}.$$

Since $\omega_1 \leq \omega_2$, (4.6) implies that (4.8) holds. Since $\theta_1 \leq \theta_2$, we have

$$(4.9) \quad \frac{1}{1 + \theta_1} \geq \frac{1}{1 + \theta_2} \quad \text{and} \quad \frac{\theta_1}{1 + \theta_1} \leq \frac{\theta_2}{1 + \theta_2}.$$

Combining this with (4.6) gives

$$\begin{aligned} \bar{P} &> \frac{\theta_2 \sqrt{\omega_1 \omega_2}}{(1 + \theta_1)} - \frac{\theta_1 \omega_1}{(1 + \theta_1)} \geq \frac{\theta_2 \sqrt{\omega_1 \omega_2}}{(1 + \theta_2)} - \frac{\theta_2 \omega_1}{(1 + \theta_2)} \\ &\geq \frac{\theta_1 \sqrt{\omega_1 \omega_2}}{(1 + \theta_2)} - \frac{\theta_2 \omega_1}{(1 + \theta_2)}. \end{aligned}$$

Hence, (4.7) is satisfied. Since both (4.7) and (4.8) are satisfied, it follows that the solution to (4.2) is strictly positive.

Using the solution given by Theorem 3.1 and the multiplier (4.4), we obtain the following expression for the optimal objective function value C' for (4.1) (the algebra is omitted):

$$(4.10) \quad \begin{aligned} C' &= \frac{1}{1+\theta_1} + \frac{1}{1+\theta_2} + \frac{\left(\frac{\sqrt{\omega'_1}}{(1+\theta_1)} + \frac{\sqrt{\omega'_2}}{(1+\theta_2)}\right)^2}{\bar{P} + \frac{\theta_1\omega'_1}{(1+\theta_1)} + \frac{\theta_2\omega'_2}{(1+\theta_2)}} \\ &= \frac{1}{1+\theta_1} + \frac{1}{1+\theta_2} + \frac{\left(\frac{\sqrt{\omega_2}}{(1+\theta_1)} + \frac{\sqrt{\omega_1}}{(1+\theta_2)}\right)^2}{\bar{P} + \frac{\theta_1\omega_2}{(1+\theta_1)} + \frac{\theta_2\omega_1}{(1+\theta_2)}}. \end{aligned}$$

Similarly, the optimal objective function value C for (4.2) is obtained by erasing the primes in (4.10):

$$(4.11) \quad C = \frac{1}{1+\theta_1} + \frac{1}{1+\theta_2} + \frac{\left(\frac{\sqrt{\omega_1}}{(1+\theta_1)} + \frac{\sqrt{\omega_2}}{(1+\theta_2)}\right)^2}{\bar{P} + \frac{\theta_1\omega_1}{(1+\theta_1)} + \frac{\theta_2\omega_2}{(1+\theta_2)}}.$$

We will show that $C \leq C'$.

Recall the following majorization property [7, p. 141]: If \mathbf{a} and $\mathbf{b} \in \mathbb{R}^n$, then

$$\sum_{i=1}^n a_{[i]} b_{[n-i+1]} \leq \sum_{i=1}^n a_i b_i \leq \sum_{i=1}^n a_{[i]} b_{[i]},$$

where $a_{[i]}$ denotes the i th largest component of \mathbf{a} . We apply the inequality (4.9) and $\omega_1 \leq \omega_2$ and the majorization property to the numerators in (4.10) and (4.11) to obtain

$$\left(\frac{\sqrt{\omega_1}}{1+\theta_1} + \frac{\sqrt{\omega_2}}{1+\theta_2}\right)^2 \leq \left(\frac{\sqrt{\omega_2}}{1+\theta_1} + \frac{\sqrt{\omega_1}}{1+\theta_2}\right)^2 = \left(\frac{\sqrt{\omega'_1}}{1+\theta_1} + \frac{\sqrt{\omega'_2}}{1+\theta_2}\right)^2.$$

Also, by (4.9) and the majorization property, the denominators in (4.10) and (4.11) satisfy

$$\left(\frac{\theta_1\omega_1}{1+\theta_1} + \frac{\theta_2\omega_2}{1+\theta_2}\right) \geq \left(\frac{\theta_1\omega_2}{1+\theta_1} + \frac{\theta_2\omega_1}{1+\theta_2}\right).$$

Hence, $C \leq C'$. This completes the proof. \square

5. Optimal permutation for $\Theta = \mathbf{I}$ and large power. When $\Theta = \mathbf{I}$, the optimal permutation depends on P . In this section, we determine the optimal permutation when P is large, while the next section analyzes the case of small P . As shown in Theorem 2.6, the solution to (2.2) can be written as either $\mathbf{\Pi\Sigma}$ or $\mathbf{\Sigma\Pi}$. When

$\Theta = \mathbf{I}$, the analysis is simpler when we take $\mathbf{S} = \Sigma\Pi$, in which case (2.2) reduces to (see (2.17))

$$(5.1) \quad \min_{\mathbf{s}, \pi} \sum_{i=1}^M \frac{\omega_i + s_i}{\omega_i + (1 + \lambda_{\pi(i)}^2) s_i}$$

subject to $\sum_{i=1}^M s_i \leq P, \quad \mathbf{s} \geq \mathbf{0}, \quad \pi \in \mathcal{P}_M,$

where M is the minimum of m and the rank of $\mathbf{\Lambda}$.

THEOREM 5.1. *For P sufficiently large, an optimal permutation π in (5.1) is given by*

$$(5.2) \quad \frac{\lambda_{\pi(1)}}{1 + \lambda_{\pi(1)}^2} \geq \frac{\lambda_{\pi(2)}}{1 + \lambda_{\pi(2)}^2} \geq \cdots \geq \frac{\lambda_{\pi(M)}}{1 + \lambda_{\pi(M)}^2}.$$

In the noise term $\boldsymbol{\eta}$ vanishes, the optimal permutation arranged the singular values in increasing order for large P . Since the function $\lambda/(1 + \lambda^2)$ is monotone increasing for $\lambda \in [0, 1]$ and monotone decreasing for $\lambda > 1$, it follows that when $\boldsymbol{\eta}$ is included in the model and when its covariance is \mathbf{I} , the singular values smaller than one are in decreasing order, while the singular values larger than one are in increasing order. Hence, when $\Theta = \mathbf{I}$, the solution of the problem when $\boldsymbol{\eta}$ is included in the model is fundamentally different from the solution when the noise $\boldsymbol{\eta}$ is neglected.

Proof. Referring to Theorem 3.1, as P tends to infinity, the optimal multiplier μ tends to zero; consequently, as P tends to infinity, all the components of the optimal \mathbf{s} tend to infinity. We assume that P is large enough that for any permutation of the components of $\boldsymbol{\lambda}$, the \mathbf{s} that satisfies (3.2) and (3.3) is strictly positive.

So far, we have assumed that the components of $\boldsymbol{\lambda}$ are in decreasing order (1.5). In the proof of this theorem, it is more convenient to assume that the components of $\boldsymbol{\lambda}$ are arranged in the order (5.2). In other words,

$$\frac{\lambda_1}{1 + \lambda_1^2} \geq \frac{\lambda_2}{1 + \lambda_2^2} \geq \cdots \geq \frac{\lambda_M}{1 + \lambda_M^2}.$$

Let π by an optimal permutation in (5.1) and define $\lambda'_i = \lambda_{\pi(i)}$. Suppose for some $i < j$, we have $\lambda'_i/(1 + \lambda'^2_i) < \lambda'_j/(1 + \lambda'^2_j)$. We will show that by interchanging the values of $\pi(i)$ and $\pi(j)$, the objective function cannot increase. Hence, after a finite series of pairwise exchanges, we obtain (5.2) without increasing the objective function.

As in Theorem 4.1, we assume for notational convenience that $i = 1, j = 2, \pi(1) = 2$, and $\pi(2) = 1$. To summarize, we have

$$(5.3) \quad \omega_1 \leq \omega_2, \quad \lambda'_1 = \lambda_2, \quad \lambda'_2 = \lambda_1, \quad \text{and} \quad \frac{\lambda_1}{1 + \lambda_1^2} > \frac{\lambda_2}{1 + \lambda_2^2}.$$

If \mathbf{s} is a solution of (5.1), then $t_1 = s_1$ and $t_2 = s_2$ is a solution to

$$(5.4) \quad \min_{\boldsymbol{\sigma}, \pi} \sum_{i=1}^2 \frac{\omega_i + t_i}{\omega_i + (1 + \lambda'^2_i) t_i}$$

subject to $\sum_{i=1}^2 t_i \leq \bar{P} := s_1 + s_2, \quad \mathbf{t} \geq \mathbf{0}.$

The unpermuted problem is obtained by erasing the prime:

$$(5.5) \quad \min_{\sigma, \pi} \sum_{i=1}^2 \frac{\omega_i + t_i}{\omega_i + (1 + \lambda_i^2)t_i}$$

subject to $\sum_{i=1}^2 t_i \leq \bar{P}, \quad \mathbf{t} \geq \mathbf{0}.$

If $\omega_1 = \omega_2$, then the optimal cost C' for the permuted problem (5.4) equals the optimal cost C for the unpermuted problem since the objective functions are identical. Hence, by interchanging the values of $\pi(1)$ and $\pi(2)$, the objective function value does not change.

Now, let us consider the case where $\omega_1 < \omega_2$. We define

$$N' = \frac{\lambda_1' \sqrt{\omega_1}}{1 + \lambda_1'^2} + \frac{\lambda_2' \sqrt{\omega_2}}{1 + \lambda_2'^2} \quad \text{and} \quad D' = \frac{\omega_1}{1 + \lambda_1'^2} + \frac{\omega_2}{1 + \lambda_2'^2} = \frac{\omega_1}{1 + \lambda_2'^2} + \frac{\omega_2}{1 + \lambda_1'^2}.$$

Parameters N and D are obtained by erasing the primes in N' and D' . With this notation the multiplier μ given by Theorem 3.1 for the problem (5.4) can be expressed

$$(5.6) \quad \frac{1}{\sqrt{\mu}} = \frac{\bar{P} + D'}{N'}.$$

Moreover, the optimal objective function value C' for (5.4) is

$$(5.7) \quad C' = \frac{1}{1 + \lambda_1'^2} + \frac{1}{1 + \lambda_2'^2} + \frac{N'^2}{\bar{P} + D'} = \frac{1}{1 + \lambda_1'^2} + \frac{1}{1 + \lambda_2'^2} + \frac{N'^2}{\bar{P} + D'}.$$

Similarly, the optimal objective function value C for the unpermuted problem is obtained by erasing the primes:

$$C = \frac{1}{1 + \lambda_1^2} + \frac{1}{1 + \lambda_2^2} + \frac{N^2}{\bar{P} + D}.$$

The inequality $C < C'$ is equivalent to

$$N^2(\bar{P} + D') < N'^2(\bar{P} + D).$$

Rearranging this, we have

$$\bar{P}(N + N')(N - N') = \bar{P}(N^2 - N'^2) < N'^2 D - N^2 D'.$$

Since $N + N' > 0$, it follows that

$$(5.8) \quad N - N' \leq \frac{N'^2 D - N^2 D'}{\bar{P}(N + N')}.$$

By the definitions of N and N' , we obtain

$$N - N' = (\sqrt{\omega_1} - \sqrt{\omega_2}) \left(\frac{\lambda_1}{1 + \lambda_1^2} - \frac{\lambda_2}{1 + \lambda_2^2} \right) < 0$$

since $\omega_1 < \omega_2$ and (5.3) holds. Since $N - N' < 0$, it follows that (5.8) holds for P sufficiently large. Equivalently, for P sufficiently large, $C < C'$. This completes the proof. \square

6. Optimal permutation for $\Theta = \mathbf{I}$ and small power. We now evaluate the optimal solution to (5.1) when P is small.

THEOREM 6.1. *Let L be the minimum of the multiplicities of γ_1 and λ_1 and let ϵ be the positive separation parameter defined by*

$$\epsilon = \min \left\{ \left| \frac{\sqrt{\omega_k}(\lambda_i \sqrt{\omega_l} - \lambda_j \sqrt{\omega_k})}{(1 + \lambda_i^2)\lambda_j} \right| : i, j, k, l \in [1, M], \lambda_i \sqrt{\omega_l} \neq \lambda_j \sqrt{\omega_k} \right\}.$$

If $P < \epsilon$, then an optimal solution of (5.1) is

$$(6.1) \quad s_i = P/L, \quad 1 \leq i \leq L, \quad s_i = 0, \quad i > L, \quad \pi(i) = i \text{ for all } i.$$

Proof. Let π and \mathbf{s} be optimal in (5.1) and define $\lambda'_i = \lambda_{\pi(i)}$. We now show that if $s_i > 0$, $s_j > 0$, and $P < \epsilon$, then we have $\lambda'_i \sqrt{\omega_j} = \lambda'_j \sqrt{\omega_i}$. To simplify the notation, we take $i = 1$ and $j = 2$, but in general, i and j are distinct integers between 1 and M . Since \mathbf{s} yields an optimal solution of (5.1), it follows that an optimal solution for the following reduced problem is $t_1 = s_1$ and $t_2 = s_2$:

$$(6.2) \quad \min_{t_1, t_2} \frac{\omega_1 + t_1}{\omega_1 + (1 + \lambda_1'^2)t_1} + \frac{\omega_2 + t_2}{\omega_2 + (1 + \lambda_2'^2)t_2}$$

subject to $t_1 + t_2 = \bar{P} := s_1 + s_2, \quad \mathbf{t} \geq \mathbf{0}$.

By Theorem 3.1, the t_i can be expressed:

$$(6.3) \quad t_i = \frac{1}{1 + \lambda_i'^2} \left(\lambda_i' \sqrt{\frac{\omega_i}{\mu}} - \omega_i \right),$$

where μ is obtained from the condition $t_1 + t_2 = \bar{P}$:

$$\mu = \left(\frac{\sum_{i=1}^2 \frac{\lambda_i' \sqrt{\omega_i}}{1 + \lambda_i'^2}}{\bar{P} + \sum_{i=1}^2 \frac{\omega_i}{1 + \lambda_i'^2}} \right)^2.$$

By (6.3), $t_i > 0$ is equivalent to

$$\lambda_i'^2 > \omega_i \mu = \omega_i \left(\frac{\sum_{i=1}^2 \frac{\lambda_i' \sqrt{\omega_i}}{1 + \lambda_i'^2}}{\bar{P} + \sum_{i=1}^2 \frac{\omega_i}{1 + \lambda_i'^2}} \right)^2.$$

We rearrange this to obtain

$$\lambda_i' \left(\bar{P} + \frac{\omega_1}{1 + \lambda_1'^2} + \frac{\omega_2}{1 + \lambda_2'^2} \right) > \sqrt{\omega_i} \left(\frac{\lambda_1' \sqrt{\omega_1}}{1 + \lambda_1'^2} + \frac{\lambda_2' \sqrt{\omega_2}}{1 + \lambda_2'^2} \right),$$

which reduces to

$$\bar{P} \lambda_i' > \sqrt{\omega_i} \left(\frac{\lambda_1' \sqrt{\omega_1}}{1 + \lambda_1'^2} + \frac{\lambda_2' \sqrt{\omega_2}}{1 + \lambda_2'^2} \right) - \lambda_i' \left(\frac{\omega_1}{1 + \lambda_1'^2} + \frac{\omega_2}{1 + \lambda_2'^2} \right).$$

Setting $i = 1$ and $i = 2$, respectively, we get

$$\bar{P} > \frac{\sqrt{\omega_2}(\lambda'_2\sqrt{\omega_1} - \lambda'_1\sqrt{\omega_2})}{(1 + \lambda'^2_2)\lambda'_1}$$

and

$$\bar{P} > \frac{\sqrt{\omega_1}(\lambda'_1\sqrt{\omega_2} - \lambda'_2\sqrt{\omega_1})}{(1 + \lambda'^2_1)\lambda'_2}.$$

Unless $\lambda'_1\sqrt{\omega_2} = \lambda'_2\sqrt{\omega_1}$, the condition $\epsilon \geq P \geq \bar{P}$ is violated. Hence, $\lambda'_1\sqrt{\omega_2} = \lambda'_2\sqrt{\omega_1}$, and in general, $\lambda'_i\sqrt{\omega_j} = \lambda'_j\sqrt{\omega_i}$ for each i and j with $s_i > 0$ and $s_j > 0$.

By the ordering (1.5), we have $\omega_1 \leq \omega_2$. If $\omega_1 < \omega_2$, then we will show that by exchanging $\pi(1)$ and $\pi(2)$ in (5.1), the value of the objective function is strictly decreased, which violates the optimality of π . In general, whenever $s_i > 0$ and $s_j > 0$, we have $\omega_i = \omega_j$. Since $\lambda'_i\sqrt{\omega_j} = \lambda'_j\sqrt{\omega_i}$ for each i and j for which $s_i > 0$ and $s_j > 0$, it follows that $\lambda'_i = \lambda'_j$. By Theorem 2.6, if \mathbf{s} has p positive components, then π permutes only the p largest components of $\boldsymbol{\lambda}$. Since the components of $\boldsymbol{\lambda}$ and $\boldsymbol{\omega}$ associated with the positive components of \mathbf{s} are all equal, we conclude that the positive components of \mathbf{s} correspond to the minimum of the multiplicities of λ_1 and ω_1 , and $\pi(i) = i$ for all i is optimal. Since the L largest components of $\boldsymbol{\lambda}$ and the L smallest components of $\boldsymbol{\omega}$ are all equal, it follows from Theorem 3.1 that the first L components of \mathbf{s} are all equal. Since the s_i sum to P , $s_i = P/L$ for $1 \leq i \leq L$ and $s_i = 0$ for $i > L$, which completes the proof.

Now, let us prove that when $\omega_1 < \omega_2$, the exchange of $\pi(1)$ and $\pi(2)$ yields a strictly smaller value of the objective function, violating the optimality of π (hence, $\omega_1 = \omega_2$). By (5.7) the optimal objective function value C' for (6.2) is

$$(6.4) \quad C' = \frac{\left(\frac{\lambda'_1\sqrt{\omega_1}}{1 + \lambda'^2_1} + \frac{\lambda'_2\sqrt{\omega_2}}{1 + \lambda'^2_2}\right)^2}{\bar{P} + \frac{\omega_1}{1 + \lambda'^2_1} + \frac{\omega_2}{1 + \lambda'^2_2}} + \frac{1}{1 + \lambda'^2_1} + \frac{1}{1 + \lambda'^2_2}.$$

Since $\lambda'_1\sqrt{\omega_2} = \lambda'_2\sqrt{\omega_1}$, (6.4) can be written

$$\begin{aligned} C' &= \frac{\omega_1\lambda'^2_1 \left(\frac{1}{1 + \lambda'^2_1} + \frac{\lambda'^2_2}{\lambda'^2_1(1 + \lambda'^2_2)}\right)^2}{\bar{P} + \omega_1 \left(\frac{1}{1 + \lambda'^2_1} + \frac{\lambda'^2_2}{\lambda'^2_1(1 + \lambda'^2_2)}\right)} + \frac{1}{1 + \lambda'^2_1} + \frac{1}{1 + \lambda'^2_2} \\ &= \frac{\omega_1\lambda'^2_1 x^2}{\bar{P} + \omega_1 x} + \frac{1}{1 + \lambda'^2_1} + \frac{1}{1 + \lambda'^2_2} \\ &= \lambda'^2_1 x - \frac{\bar{P}\lambda'^2_1 x}{\bar{P} + \omega_1 x} + \frac{1}{1 + \lambda'^2_1} + \frac{1}{1 + \lambda'^2_2} \\ &= \lambda'^2_1 x - \frac{\bar{P}\lambda'^2_1}{\omega_1} + \frac{\bar{P}^2\lambda'^2_1}{\omega_1(\bar{P} + \omega_1 x)} + \frac{1}{1 + \lambda'^2_1} + \frac{1}{1 + \lambda'^2_2}, \end{aligned}$$

where

$$x = \frac{1}{1 + \lambda'^2_1} + \frac{\lambda'^2_2}{\lambda'^2_1(1 + \lambda'^2_2)}.$$

Exploiting the identity

$$\lambda_1'^2 x + \frac{1}{1 + \lambda_1'^2} + \frac{1}{1 + \lambda_2'^2} = 2,$$

it follows that

$$C' = 2 - \frac{\bar{P}\lambda_1'^2}{\omega_1} + \frac{\bar{P}^2\lambda_1'^2}{\omega_1(\bar{P} + \omega_1 x)}.$$

Exchanging the values of $\pi(1)$ and $\pi(2)$ leads to the following permuted version of (6.2):

$$(6.5) \quad \min_{t_1, t_2} \frac{\omega_1 + t_1}{\omega_1 + (1 + \lambda_2'^2)t_1} + \frac{\omega_2 + t_2}{\omega_2 + (1 + \lambda_1'^2)t_2}$$

subject to $t_1 + t_2 = \bar{P}$, $t_1 \geq 0$, $t_2 \geq 0$.

The choice $t_1 = \bar{P}$ and $t_2 = 0$ is feasible in (6.5). Hence, an upper bound C^+ for the optimal objective function value is

$$C^+ = 1 + \frac{\omega_1 + \bar{P}}{(\omega_1 + \bar{P}) + \bar{P}\lambda_2'^2} = 2 - \frac{\bar{P}\lambda_2'^2}{\omega_1 + \bar{P}(1 + \lambda_2'^2)} = 2 - \frac{\bar{P}\lambda_2'^2}{\omega_1} + O(\bar{P}^2).$$

Since $\omega_1 < \omega_2$, it follows from the condition $\omega_1\lambda_2'^2 = \omega_2\lambda_1'^2$ that $\lambda_1'^2 < \lambda_2'^2$. Comparing C' and C^+ , we conclude that for P sufficiently small, $C^+ < C'$, which contradicts the optimality of C' . This completes the proof. \square

7. Numerical experiments. Some small test problems were solved to see how P should be chosen in order to observe Theorems 5.1 and 6.1, and to evaluate a conjecture concerning the structure of the optimal permutation in general. In the first experiment, we randomly generate $\omega_i \in [0, 1]$ and $\lambda_i \in [0, 2]$ in the special case $l = m = n = 10$. The interval $[0, 2]$ for λ was chosen so that λ_i would be generated on each side of the maximum $x = 1$ for the function $x/(1 + x^2)$. These dimensions are small enough that we can enumerate all permutations $\pi \in \mathcal{P}_5$ and select the best. Table 7.1 shows how many times the solution given in Theorems 5.1 or 6.1 is correct for 100 randomly generated problems and for various choices of P .

In another series of experiments, we evaluated the quality of the following M permutations: For each $k = 1, 2, \dots, M$, let π_k be the permutation defined by

$$(7.1) \quad \pi_k(i) = i \text{ for } i > k, \quad \pi_k(i) \in [1, k] \text{ for } i \in [1, k],$$

TABLE 7.1

Number of times the permutation given by Theorem 5.1 or 6.1 was exact out of 100 trials ($\omega_i \in [0, 1]$ and $\lambda_i \in [0, 2]$, $l = m = n = 10$).

P	Thm. 5.1 exact (out of 100)	Thm. 6.1 exact (out of 100)
10^4	100	0
10^3	97	0
10^2	68	0
10^1	15	0
10^0	1	0
10^{-1}	0	0
10^{-2}	0	44
10^{-3}	0	98
10^{-4}	0	100

TABLE 7.2

Number of times that one of the permutations $\pi_1, \pi_2, \dots, \pi_M$ was optimal in (5.1) out of 100 trials ($\omega_i \in [0, 1]$ and $\lambda_i \in [0, 2]$, $l = m = n = 10$).

P	Some π_k exact (out of 100)	Relative error (no π_k exact)
10^4	100	0
10^3	98	7.8e-08
10^2	74	2.6e-06
10^1	46	5.0e-05
10^0	92	3.0e-05
10^{-1}	96	4.9e-05
10^{-2}	100	0
10^{-3}	100	0
10^{-4}	100	0

and

$$(7.2) \quad \frac{\lambda_{\pi_k(1)}}{1 + \lambda_{\pi_k(1)}^2} \geq \frac{\lambda_{\pi_k(2)}}{1 + \lambda_{\pi_k(2)}^2} \geq \dots \geq \frac{\lambda_{\pi_k(k)}}{1 + \lambda_{\pi_k(k)}^2}.$$

We optimized (5.1) with the added constraint that π was one of the M permutation π_k , $k = 1, 2, \dots, M$. In Table 7.2 we consider the same set of test problems used for Table 7.1, and we evaluate the number of times that one of these M permutation yields the exact minimizer. When none of these M permutations yields the exact minimum, we evaluate the relative error in the cost (best approximate cost minus exact cost divided by the exact cost). The average relative error for the best inexact approximation in the set π_k , $1 \leq k \leq M$, is shown in the last column of Table 7.2. Thus, one of the π_k often yields the optimal solution of (5.1). When none of the π_k approximations is optimal, the best approximate cost is nearly optimal.

The motivation for considering the permutations π_k is the following: By Theorem 2.6, there exists an integer $p \geq 1$ (p is the number of positive components of σ in an optimal solution) with the property that $\pi(i) = i$ when π is optimal in (5.1) and $i > p$. Hence, we try M different permutations of the form (7.1). When the power P is sufficiently large, we know that the permutation (5.2) is optimal. Thus, we try the same ordering, but applied to the k largest singular values as in (7.2).

8. Conclusions. We analyze the optimization problem (1.4) which arises in linear Bayesian estimation in the presence of noise, and which is relevant to multisensor data fusion problems and wireless communication. Unlike our earlier work [2, 5], we now take into account the noise term η in the model (1.1).

By letting the covariance of η tends to zero, the results given in the present paper include the results given in [2, 5]. In particular, if we take $\mathbf{N} = \alpha \mathbf{I}$ in (1.3), then a rescaling of \mathbf{P} and \mathbf{H} yields (1.4). In the rescaled problem, the singular values of \mathbf{H} are divided by $\sqrt{\alpha}$. Hence, as α tends to zero, all the singular values in the rescaled problem become larger than 1. Since the function $x/(1+x^2)$ is monotone decreasing for $x > 1$, we deduce that for P sufficiently large, the optimal permutation arranges the singular values in increasing order, the same ordering derived in [2] when the noise η was neglected.

For general P , computing the optimal permutation π in (5.1) may not be easy. Nonetheless, we exhibit in section 7 a set of M permutations $\pi_1, \pi_2, \dots, \pi_M$ which often contains the optimal permutation.

In the case $\mathbf{H} = \mathbf{I}$, the analysis in this paper also yields the results in [5] by taking $\mathbf{N} = \alpha\mathbf{I}$ and letting α tend to zero. It is interesting to note that the analysis in [5] for the case $\boldsymbol{\eta} = \mathbf{0}$ was much more difficult than the analysis of the case $\boldsymbol{\eta} \neq \mathbf{0}$ considered in this paper. Hence, by including the noise term $\boldsymbol{\eta}$ in the model and by letting $\boldsymbol{\eta}$ tend to $\mathbf{0}$, we could recover with less effort the solution given in [5].

9. Appendix. First-order optimality condition. We evaluate the derivative of the Lagrangian (2.5) and set it to zero. Since $\text{tr}(\mathbf{A} + \mathbf{A}^*) = 2(\text{Real}[\text{tr}(\mathbf{A})])$ and $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$, it follows that the derivative of \mathbf{SS}^* in the direction $\boldsymbol{\delta}\mathbf{S}$ is

$$(9.1) \quad \text{tr}(\mathbf{S}\boldsymbol{\delta}\mathbf{S}^* + \boldsymbol{\delta}\mathbf{S}\mathbf{S}^*) = 2(\text{Real}[\text{tr}(\boldsymbol{\delta}\mathbf{S}\mathbf{S}^*)]) = 2(\text{Real}[\text{tr}(\mathbf{S}^*\boldsymbol{\delta}\mathbf{S})]).$$

For any invertible matrix \mathbf{M} , we have

$$(9.2) \quad \frac{d\mathbf{M}^{-1}}{d\mathbf{T}} = -\mathbf{M}^{-1} \left(\frac{d\mathbf{M}}{d\mathbf{T}} \right) \mathbf{M}^{-1}.$$

We equate to zero the derivative of the Lagrangian in the direction $\boldsymbol{\delta}\mathbf{S}$ and utilize (9.1) and (9.2) to obtain

$$\text{Real}[\text{tr}((\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\boldsymbol{\Lambda}\mathbf{M}^{-2}\boldsymbol{\Lambda}\mathbf{S}^*\mathbf{L}^{-1} - \mu\mathbf{S}^*)\boldsymbol{\delta}\mathbf{S}] = \mathbf{0},$$

where \mathbf{L} and \mathbf{M} are defined in (2.4). Inserting

$$\boldsymbol{\delta}\mathbf{S} = ((\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\boldsymbol{\Lambda}\mathbf{M}^{-2}\boldsymbol{\Lambda}\mathbf{S}^*\mathbf{L}^{-1} - \mu\mathbf{S}^*)^*$$

gives

$$(\mathbf{I} - \mathbf{S}^*\mathbf{L}^{-1}\mathbf{S})\boldsymbol{\Lambda}\mathbf{M}^{-2}\boldsymbol{\Lambda}\mathbf{S}^*\mathbf{L}^{-1} = \mu\mathbf{S}^*.$$

Acknowledgments. We thank the reviewers for their comments and suggestions which led to a more precise and complete paper.

REFERENCES

- [1] T. M. COVER AND J. A. THOMAS, *Elements of Information Theory*, John Wiley, New York, 1991.
- [2] W. W. HAGER, Y. LIU, AND T. F. WONG, *Optimization of generalized mean square error in signal processing and communication*, *Linear Algebra Appl.*, 416 (2006), pp. 815–834.
- [3] W. W. HAGER, *Updating the inverse of a matrix*, *SIAM Rev.*, 31 (1989), pp. 221–239.
- [4] S. M. KAY, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, Philadelphia, 1993.
- [5] Y. LIU, T. F. WONG, AND W. W. HAGER, *Training signal design for estimation of correlated MIMO channels with colored interference*, *IEEE Trans. Signal Process.*, 55 (2007), pp. 1486–1497.
- [6] Z.-Q. LUO, G. B. GIANNAKIS, AND S. ZHANG, *Optimal linear decentralized estimation in a bandwidth constrained sensor network*, in *Proceedings of the International Conference on Information Theory*, Adelaide, Australia, IEEE, 2005.
- [7] A. W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.
- [8] D. PALOMAR, J. CIOFFI, AND M. LAGUNAS, *Joint Tx-Rx beamforming design for multicarrier MIMO channels: A unified framework for convex optimization*, *IEEE Trans. Signal Process.*, 51 (2003), pp. 2381–2401.
- [9] A. SCAGLIONE, P. STOICA, S. BARBAROSSA, G. B. GIANNAKIS, AND H. SAMPATH, *Optimal designs for space-time linear precoders and decoders*, *IEEE Trans. Signal Process.*, 50 (2002), pp. 1051–1064.

- [10] E. SONG, Y. ZHU, AND J. ZHOU, *Sensors' optimal dimensionality compression matrix in estimation fusion*, *Automatica*, 41 (2005), pp. 2131–2139.
- [11] G. STRANG, *Linear Algebra and Its Applications*, 4th ed., Thomson, Belmont, CA, 2006.
- [12] P. VISWANATH AND V. ANANTHARAM, *Optimal sequences for CDMA under colored noise: A Schur-saddle function property*, *IEEE Trans. Inform. Theory*, 48 (2002), pp. 1295–1318.
- [13] A. J. VITERBI, *CDMA-Principles of Spread Spectrum Communication*, Addison-Wesley, Reading, MA, 1995.
- [14] T. F. WONG AND B. PARK, *Training sequence optimization in MIMO systems with colored interference*, *IEEE Trans. Commun.*, 52 (2004), pp. 1939–1947.
- [15] Y. ZHU, E. SONG, J. ZHOU, AND Z. YOU, *Optimal dimensionality reduction of sensor data in multisensor estimation fusion*, *IEEE Trans. Signal Process.*, 53 (2005), pp. 1631–1639.