10,000 YEN PRIZE PROBLEM *

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Let p be a polynomial of degree at most n and let $-1 < \tau_1 < \tau_2 < \ldots < \tau_n < 1$ be either of the following two choices:

1. The Gauss quadrature points or equivalently the zeros of the Legendre polynomial of degree n or equivalently the zeros of the Jacobi polynomial $P_n^{(0,0)}$.
2. The Radau quadrature points or equivalently the zeros of the Jacobi polynomial $P_{n-1}^{(1,0)}$ augmented by $\tau_n = 1$

Suppose that $p(-1) = 0$ and $|p'(\tau_i)| \le 1$ for all $1 \le i \le n$. We conjecture that $|p(\tau_i)| \le 2$ for all $1 \le i \le n$. This is an open problem that arises in [1] and [2]. For the Gauss quadrature points, a more precise upper bound is $|p(\tau_i)| \le 1 + \tau_n$ for all $1 \le i \le n$. For either the Gauss or the Radau quadrature points, the polynomial $p(\tau) = 1+\tau$ reaches the upper bound. Walter Gautschi (see doi:10.4231/R74Q7RX0) points out this result is a sort of reverse Markov inequality. Markov's inequality bounds the sup-norm of the derivative of a polynomial on $[-1, +1]$ in terms of the sup norm of the polynomial, while here we bound the sup norm on the point set τ_i, $1 \le i \le n$, in terms of the derivatives on the point set. In the references below, we check this result numerically for n up to 300. **Note:** Prize only awarded for first correct solution.

REFERENCES

[1] W. W. Hager, H. Hou, and A. V. Rao, *Convergence rate for a Radau collocation method applied to unconstrained optimal control*, (2015, arXiv.org/abs/1508.03783).

[2] ———, *Convergence rate for a Gauss collocation method applied to unconstrained optimal control*, J. Optim. Theory Appl., 169 (2016), pp. 801–824.

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