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On Quantitative Stability in Optimization and Optimal Control*

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Abstract. We study two continuity concepts for set-valued maps that play central roles in quantitative stability analysis of optimization problems: Aubin continuity and Lipschitzian localization. We show that various inverse function theorems involving these concepts can be deduced from a single general result on existence of solutions to an inclusion in metric spaces. As applications, we analyze the stability with respect to canonical perturbations of a mathematical program in a Hilbert space and an optimal control problem with inequality control constraints. For stationary points of these problems, Aubin continuity and Lipschitzian localization coincide; moreover, both properties are equivalent to surjectivity of the map of the gradients of the active constraints combined with a strong second-order sufficient optimality condition.

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1. Introduction

In this paper we consider two continuity concepts for set-valued maps which play leading roles in stability analysis of optimization problems. The first is Aubin continuity; in optimization this concept goes back to the Lyusternik theorem and is usually identified with a regularity property for the constraints guaranteeing the existence of (normal) Lagrange multipliers. The second is Lipschitzian localization, a concept which appears in implicit function type theorems. We prove here that these two concepts are the same for optimality maps of an infinite-dimensional

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program and an optimal control problem, both subject to canonical perturbations. This extends a result in [9] where this equivalence was shown for a variational inequality over a polyhedral convex set in $\mathbf{R}^n$. It is an open question exactly when these two concept are equivalent.

The concept of Aubin continuity can be extracted from the proof of the classical Lyusternik theorem and is even more explicit in the proof of the Graves theorem, see [3] for a discussion. In mathematical programming it has appeared as 'metric regularity'; in a topological framework, it was called 'openness with linear rate'. J.-P. Aubin was the first to define this concept as a continuity property of set-valued maps, calling it 'pseudo-Lipschitz' continuity ([1, 2]). Following [9] we use the name 'Aubin continuity'. There are many studies of this concept; for a discussion see e.g. the bibliographical comments in [13].

The Lipschitzian localization property, a name we use after [13], simply says that when restricted to a neighborhood of a point in its graph, a set-valued map becomes a Lipschitz continuous (single-valued) function. Both Aubin continuity and Lipschitzian localization have the remarkable property of invariance under (non)linearization, which makes them very instrumental in stability analysis. This invariance property was cast in [5] in the following general form. Let F be a set-valued map with closed graph acting from a complete metric space X into the subsets of the linear normed space Y, and let $f\colon X \to Y$ be a function which is 'strictly stationary', e.g. its strict derivative is equal to zero. Then $(f + F)^{-1}$ is Aubin continuous [resp., has a Lipschitzian localization] if and only if F^{-1} is Aubin continuous [resp., has a Lipschitzian localization]. In this paper we show that inverse mapping theorems of this type follow from a single general result (Theorem 2.1) on existence of solutions to an inclusion in metric spaces. Not surprisingly, the proof of this result uses an abstract version of Newton's method in the spirit of the original proofs of Lyusternik and Graves.

We give this result in Section 2 followed by several corollaries. Section 3 contains corresponding results for the Lipschitzian localization property.

The rest of the paper is dedicated to specific optimization problems. In Section 4 we consider a mathematical program in a Hilbert space in the presence of canonical perturbations. We show that, for the map of Karush–Kuhn–Tucker points with the primal variables being optimal solutions, Aubin continuity and Lipschitzian localization coincide; moreover, both properties are equivalent to surjectivity of the gradients of the active constraints combined with the strong second-order sufficient condition. This generalizes [9], Theorem 6. In Section 5, we establish a similar result for an optimal control problem with inequality control constraints, extending the work [8].

2. Aubin Continuous Maps

Let X and Y be metric spaces with both metrics denoted $\rho(\cdot, \cdot)$ and let $B_r(x)$ be the closed ball with center x and radius r. In writing 'f maps X into Y' we adopt the convention that the domain of f is a (possibly proper) subset of X. Accordingly, a set-valued map F from X to the subsets of Y may have empty values for some points of X. Given a map F from X to the subsets of Y, we define graph $F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ and $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$. We denote by $\operatorname{dist}(x, A)$ the distance from a point x to a set A.

DEFINITION 2.1. Let Γ map Y to the subsets of X and let $(y^*, x^*) \in \operatorname{graph}\Gamma$. We say that Γ is Aubin continuous at (y^*, x^*) with constants a, b and M if for every $y', y'' \in B_b(y^*)$ and every $x' \in \Gamma(y') \cap B_a(x^*)$, there exists $x'' \in \Gamma(y'')$ such that $\rho(x', x'') \leqslant M\rho(y', y'')$.

The following properties of Aubin continuous maps follows directly from the definition:

PROPERTY 2.1. If Γ is Aubin continuous at (y^*, x^*) with constants a, b and M, then for every $0 < a' \leqslant a$ and $0 < b' \leqslant b$ the map Γ is Aubin continuous at (y^*, x^*) with constants a', b' and M. If, in addition, $b' \leqslant a'/M$, then $\Gamma(y) \cap B_{a'}(x^*) \neq \emptyset$ for all $y \in B_{b'}(y^*)$.

PROPERTY 2.2. If Γ is Aubin continuous at (y^*, x^*) with constants a, b and M, then there exist constants a', b' and c' such that for every $(y, x) \in \operatorname{graph}\Gamma \cap B_{c'}((y^*, x^*))$, the map Γ is Aubin continuous at (y, x) with constants a', b' and M.

PROPERTY 2.3. Let X be a complete metric space and Y be a linear normed space. Let G be a set-valued map from X to Y, and let G^{-1} be Aubin continuous at (y^*, x^*) with constants a, b, and M. Then for every $\varepsilon < b$ the map

$$y \mapsto \{x \in X \mid y \in G(x) + B_\varepsilon(0)\}$$

is Aubin continuous at (y^*, x^*) with constants a, $b - \varepsilon$, and M.

THEOREM 2.1. *Let G map X to the subsets of Y and let $(x^*, y^*) \in \operatorname{graph} G$. Let G^{-1} be Aubin continuous at (y^*, x^*) with constants a, b, and M. Suppose that the set* (graph G) $\cap (B_a(x^*) \times B_b(y^*))$ *is closed and $B_a(x^*)$ is complete. Let the real numbers λ, $\bar{M}$, $\bar{a}$, m and δ satisfy the relations*

$$\lambda M < 1, \quad \bar{M} \geqslant \frac{M}{1 - \lambda M}, \quad m + \delta \leqslant b \quad \text{and} \quad \bar{a} + \bar{M}\delta \leqslant a. \tag{1}$$

Let $g\colon X \to Y$ be a Lipschitz continuous function in the ball $B_a(x^*)$ with Lipschitz constant λ and such that

$$\sup_{x \in B_a(x^*)} \rho(g(x), y^*) \leqslant m. \tag{2}$$

Then for every $(x', y') \in \operatorname{graph} G$ with $x' \in B_{\bar{a}}(x^*)$ and $\rho(y', g(x')) \leqslant \delta$, there exists $x'' \in B_a(x^*)$ such that

$$g(x'') \in G(x'') \quad \text{and} \quad \rho(x', x'') \leqslant \bar{M}\rho(y', g(x')). \tag{3}$$

Proof. Let us choose positive λ, $\bar{M}$, m, $\bar{a}$ and δ such that the inequalities in (1) hold and let $(x', y') \in \operatorname{graph} G$ be such that $x' \in B_{\bar{a}}(x^*)$ and $\rho(y', g(x')) \leqslant \delta$. Then

$$\rho(g(x'), y^*) \leqslant m \leqslant b$$

and

$$\rho(y', y^*) \leqslant \rho(y', g(x')) + \rho(g(x'), y^*) \leqslant \delta + m \leqslant b.$$

From the Aubin continuity of G^{-1}, there exists x_1 such that

$$g(x') \in G(x_1) \quad \text{and} \quad \rho(x_1, x') \leqslant M\rho(y', g(x')). \tag{4}$$

We define inductively a sequence $\{x_k\}$ in the following way. Let $x_0 = x'$ and let for some $k \geqslant 1$, $x_1, \ldots, x_k$ satisfy:

$$\rho(x_i, x_{i-1}) \leqslant (\lambda M)^{i-1}\rho(x_1, x_0), \tag{5}$$

and

$$g(x_{i-1}) \in G(x_i) \quad \text{for } i = 1, \ldots, k. \tag{6}$$

From (4), $x_0 = x'$ and x_1 satisfy these relations. Using the inequality in (4), we estimate

$$\begin{aligned} \rho(x_i, x^*) &\leqslant \rho(x_0, x^*) + \sum_{j=1}^{i} \rho(x_j, x_{j-1}) \leqslant \rho(x', x^*) + \sum_{j=0}^{\infty} (\lambda M)^j \rho(x_1, x_0) \\ &\leqslant \bar{a} + \frac{M}{1 - \lambda M}\rho(y', g(x')) \leqslant \bar{a} + \bar{M}\delta \leqslant a. \end{aligned}$$

Hence, the x_i, $i = 1, 2, \ldots, k$, lie in $B_a(x^*)$, and by (2)

$$\rho(g(x_i), y^*) \leqslant m \leqslant b, \quad i = 1, 2, \ldots, k.$$

By Aubin continuity of G^{-1}, there exists x_{k+1} such that $g(x_k) \in G(x_{k+1})$ and

$$\rho(x_{k+1}, x_k) \leqslant M\rho(g(x_k), g(x_{k-1})).$$

It follows from (5) and the Lipschitz continuity of g that

$$\rho(x_{k+1}, x_k) \leqslant M\lambda\rho(x_k, x_{k-1}) \leqslant (\lambda M)^k \rho(x_1, x_0).$$

This completes the induction step for (5) and (6).

From (5) and the condition $\lambda M < 1$ we conclude that $\{x_k\}$ is a Cauchy sequence, hence it has a limit, denoted x''. Since all $x_k \in B_a(x^*)$, the limit point $x'' \in B_a(x^*)$. Passing to the limit in (6), we obtain $g(x'') \in G(x'')$.

To prove (3) observe that for any choice of $k > 1$,

$$\begin{aligned}\rho(x', x'') &\leqslant \rho(x_0, x_k) + \rho(x_k, x'') \leqslant \sum_{i=0}^{k-1} \rho(x_{i+1}, x_i) + \rho(x_k, x'') \\ &\leqslant \sum_{i=0}^{k-1} (\lambda M)^i \rho(x_1, x_0) + \rho(x_k, x'') \leqslant \frac{\bar{M}}{M} \rho(x_1, x_0) + \rho(x_k, x'').\end{aligned}$$

Letting k tend to infinity in the last inequality and utilizing (4), we obtain (3). □

COROLLARY 2.1. *Let X be a complete metric space, U be a metric space, and Y be a linear normed space. Let F be a set-valued map from X to the subsets of Y, with closed graph. If the map F^{-1} is Aubin continuous at (v^*, x^*) with constants a, b and M, then for every $L \geqslant 0$ and every $\lambda \in (0, 1/M)$ there exist positive numbers $\hat{a}$ and $\hat{b}$ such that the following holds: For every function φ: $X \times U \mapsto Y$ which is Lipschitz continuous on $B_a(x^*) \times U$ with Lipschitz constants λ for x and L for u, the map*

$$(u, v) \mapsto \mathcal{X}(u, v) = \left\{x \in X \mid v \in \varphi(x, u) + F(x)\right\} \tag{7}$$

is Aubin continuous at $((u^*, v^* + \varphi(x^*, u^*)), x^*)$ with constants $\hat{a}$, $\hat{b}$, $\hat{M} = (L + 1)M/(1 - \lambda M)$ (the constant $\hat{M}$ corresponds to the metric $\rho((u', v'), (u'', v'')) =$ $\max\{\rho(u', u''), \|v' - v''\|\}$ in $U \times Y$).

Proof. Let us fix $L \geqslant 0$ and $\lambda < 1/M$. According to Property 2.1, there is no loss of generality in assuming that $\lambda a < b$. Choose positive numbers $\hat{a}$ and $\hat{b}$ such that

$$3(L + 1)\hat{b} \leqslant b - \lambda a, \qquad \hat{a} + \frac{2M}{1 - \lambda M}(L + 1)\hat{b} \leqslant a. \tag{8}$$

Let φ be a function which satisfies the conditions of the theorem, and define $y^* = v^* + \varphi(x^*, u^*)$. Let $(u', v'), (u'', v'') \in B_{\hat{b}}((u^*, y^*))$, and let $x' \in \mathcal{X}(u', v') \cap B_{\hat{a}}(x^*)$. We shall prove that there exists $x'' \in \mathcal{X}(u'', v'')$ satisfying $\rho(x', x'') \leqslant \hat{M} \max\{\rho(u', u''), \|v' - v''\|\}$.

The inclusion $x \in \mathcal{X}(u'', v'')$ is equivalent to $v'' - \varphi(u'', x) \in F(x)$. We apply Theorem 2.1 with the following specifications: $G = F$, $g(x) = v'' - \varphi(x, u'')$, $\bar{M} = M/(1 - \lambda M)$, $\bar{a} = \hat{a}$, $\bar{b} = \hat{b}$, $m = (L + 1)\hat{b} + \lambda a$ and $\delta = 2(L + 1)\hat{b}$. The inequalities in (1) are satisfied because of (8).

Defining $y' = v' - \varphi(x', u')$, $(x', y') \in \operatorname{graph} G$ since $x' \in \mathcal{X}(u', v')$. By definition, the function g is Lipschitz continuous on $B_a(x^*)$ with a constant λ. Further, for $x \in B_a(x^*)$,

$$\begin{aligned}\|g(x) - v^*\| &= \|v'' - \varphi(x, u'') - v^*\| \\ &\leqslant \|v'' - y^*\| + \|\varphi(x, u'') - \varphi(x^*, u^*)\| \leqslant \hat{b} + \lambda a + L\hat{b} = m\end{aligned}$$

and

$$\begin{aligned}\|y' - g(x')\| &= \|v' - \varphi(x', u') - v'' + \varphi(x', u''))\| \\ &\leqslant \|v' - v''\| + L\|u' - u''\| \leqslant 2(1+L)\hat{b} = \delta.\end{aligned}$$

Hence, by Theorem 2.1 there exists $x'' \in B_a(x^*)$ such that

$$v'' - \varphi(x'', u'') \in F(x'')$$

and

$$\begin{aligned}\rho(x', x'') &\leqslant \bar{M}\|v' - \varphi(x', u') - v'' + \varphi(x', u''))\| \\ &\leqslant \bar{M}(1+L)\max\{\rho(u', u''), \|v' - v''\|\}. \qquad \square\end{aligned}$$

COROLLARY 2.2. *The claim in Corollary* 2.1 *holds, under the same conditions, with* $\mathcal{X}(u, v)$ *replaced by the map*

$$(u, v) \mapsto \mathcal{X}_\varepsilon(u, v) = \big\{x \in X \mid v \in \varphi(x, u) + F(x) + B_\varepsilon(0)\big\},$$

for any ε *in the interval* $[0, \hat{b})$.

Proof. Apply Corollary 2.1 to the map $F + B_\varepsilon(0)$, the inverse of which is Aubin continuous according to Property 2.3. □

COROLLARY 2.3. *Let* X, Y *and* U *be as in Corollary* 2.1, *let* F *be a set-valued map from* X *to the subsets of* Y, *and let the map* Φ *be a closed graph selection of* F^{-1}, *that is,* $\Phi(v) \subset F^{-1}(v)$ *for all* $x \in X$ *and* graph Φ *is closed. Let* Φ *be Aubin continuous at* (v^*, x^*) *with constants* a, b *and* M. *Then the claim in Corollary* 2.1 *holds with 'the map* $\mathcal{X}$ [*defined in* (7)]*' replaced by 'the map* $\mathcal{X}$ [*defined in* (7)] *has a selection'.*

Proof. Apply Corollary 2.1 with $G = \Phi^{-1}$ and observe that

$$\big\{x \in X \mid v \in \varphi(x, u) + \Phi^{-1}(x)\big\} \subset \mathcal{X}(u, v). \qquad \square$$

From Corollary 2.1 we also obtain

COROLLARY 2.4. *Let* X *be a Banach space,* U *be a metric space, and* Y *be a linear normed space. Let* F *be a set-valued map from* X *from to subsets of a* Y *with closed graph, let* $y^* \in F(x^*)$, *and let* g: $X \times U \to Y$ *be a* (*single-valued*) *function with the following properties:* (a) g *is Fréchet differentiable with respect to* x *and its derivative* $\nabla_x g$ *is continuous in a neighborhood of* (x^*, u^*)*;* (b) g *is Lipschitz continuous in* u *on* U *uniformly in* x *in a neighborhood of* x^*. *Then the following are equivalent:*

(i) *The map* $(u, y) \mapsto \{x \in X \mid y \in g(x, u) + F(x)\}$ *is Aubin continuous* [*resp., has a closed graph Aubin continuous selection*] *at* $((u^*, y^*), x^*)$*;*

(ii) *The map* $(g(x^*, u^*)+\nabla_x g(x^*, u^*)(\cdot - x^*)+F(\cdot))^{-1}$ *is Aubin continuous* [*resp., has a closed graph Aubin continuous selection*] *at* (y^*, x^*).

3. Lipschitzian Localization

We use the notation from the previous section.

DEFINITION 3.1. Let Γ maps Y to the subsets of X and let $(y^*, x^*) \in \operatorname{graph}\Gamma$. We say that Γ has a Lipschitzian localization at (y^*, x^*) with constants a, b and M if the map $y \mapsto \Gamma(y) \cap B_a(x^*)$ is single-valued (a function) and Lipschitz continuous in $B_b(y^*)$ with a Lipschitz constant M.

PROPERTY 3.1. If Γ has a Lipschitzian localization at (y^*, x^*), then it is Aubin continuous at (y^*, x^*) with the same constants. Conversely, if Γ is Aubin continuous at (y^*, x^*) with constants a, b and M and in addition, for some positive constants α and β, $\Gamma(y) \cap B_\alpha(x^*)$ consists of at most one point for every $y \in B_\beta(y^*)$, then Γ has a Lipschitzian localization at (y^*, x^*) with constants a', b', M provided that

$$0 < a' < \min\{a, \alpha\} \quad \text{and} \quad 0 < b' \leqslant \min\{b, \beta, a'/M, (\alpha - a')/M\}.$$

The following theorem is an analog of Theorem 2.1 for maps with the Lipschitzian localization property. It was first published in [7]; here we supply it with a shorter proof based on Theorem 2.1.

THEOREM 3.1. *Suppose that G maps X into the subsets of Y, $(x^*, y^*) \in \operatorname{graph} G$, G^{-1} has a Lipschitzian localization at (y^*, x^*) with constants a, b, and M, the set* $(\operatorname{graph} G) \cap (B_a(x^*) \times B_b(y^*))$ *is closed, and $B_a(x^*)$ is complete. Let the real numbers λ, $\bar{M}$, $\bar{a}$, m and β satisfy the relations*

$$\lambda M < 1, \quad \bar{M} \geqslant \frac{M}{1-\lambda M}, \quad m + \beta < b \quad \text{and} \quad \bar{a} + \bar{M}\beta < a,$$

let $g\colon X \to Y$ be a Lipschitz continuous function in the ball $B_a(x^*)$ with Lipschitz constant λ and such that

$$\sup_{x \in B_a(x^*)} \rho(g(x), y^*) \leqslant m,$$

and let the set $\Delta = \{x \in B_{\bar{a}} \mid \operatorname{dist}(g(x), G(x)) \leqslant \beta\}$ *be nonempty. Then the set*

$$\hat{X} = \{x \in B_a \mid g(x) \in G(x)\}$$

consists of exactly one point, $\hat{x}$, and for every $x' \in \Delta$, we have

$$\rho(x', \hat{x}) \leqslant \bar{M} \operatorname{dist}(g(x'), G(x')).$$

Proof. Let $x' \in \Delta$. Let $\varepsilon > 0$ be such that

$$m + \beta + \varepsilon \leqslant b \quad \text{and} \quad \bar{a} + \bar{M}(\beta + \varepsilon) \leqslant a.$$

Choose $y' \in G(x')$ such that

$$\rho(y', g(x')) \leqslant \operatorname{dist}(g(x'), G(x')) + \varepsilon.$$

Let $\delta = \beta + \varepsilon$. Then $\rho(y', g(x')) \leqslant \beta + \varepsilon = \delta$. Since G^{-1} has a Lipschitzian localization at (y^*, x^*), G^{-1} is Aubin continuous at the same point with the same constants. Applying Theorem 2.1 with the constants λ, $\bar{M}$, $\bar{a}$, m and δ we obtain that there exists $x'' \in B_a(x^*)$ with $g(x'') \in G(x'')$ and such that

$$\rho(x', x'') \leqslant \bar{M}\rho(y', g(x')) \leqslant \bar{M}\operatorname{dist}(g(x'), G(x')) + \varepsilon. \tag{9}$$

Suppose that the set $\hat{X}$ is not a singleton; that is, there exist $x, \bar{x} \in \hat{X}$ with $\rho(x, \bar{x}) > 0$. Then $\rho(g(\tilde{x}), y^*) \leqslant m \leqslant b$ for both $\tilde{x} = x$ and $\tilde{x} = \bar{x}$. From the Lipschitzian localization property of G^{-1} we obtain

$$\rho(x, \bar{x}) \leqslant M\rho(g(x), g(\bar{x})) \leqslant M\lambda\rho(x, \bar{x}) < \rho(x, \bar{x}),$$

which is a contradiction. Thus $\hat{X}$ consists of exactly one point, say $\hat{x}$, and the point $x'' = \hat{x}$ does not depend on the choice of ε. Passing to zero with ε in (9) we complete the proof. □

As a corollary of Theorem 3.1, we obtain an analog of Corollary 2.4 which is a version of Robinson's implicit function theorem [12]:

COROLLARY 3.1. *Under the assumptions of Corollary* 2.4, *the following are equivalent:*

(i) *The map* $(u, y) \mapsto \{x \in X \mid y \in g(x, u) + F(x)\}$ *has a Lipschitzian localization at* $((u^*, y^*), x^*)$*;*
(ii) *The map* $(g(x^*, u^*) + \nabla_x g(x^*, u^*)(\cdot - x^*) + F(\cdot))^{-1}$ *has a Lipschitzian localization at* (y^*, x^*).

4. Mathematical Programming

In this section we consider the following infinite-dimensional nonlinear program with a variable x in a Hilbert space H and parameters $p \in H$, $q \in \mathbf{R}^m$ and $u \in \mathbf{R}^l$:

$$\text{minimize } g_0(x, u) + \langle p, x \rangle \text{ subject to } x \in Q(u, q), \tag{10}$$

where

$$Q(u, q) = \big\{x \in H \mid g_i(x, u) \leqslant q_i,\ i = 1, \ldots, m\big\}. \tag{11}$$

We study the properties of the solution to (10) in a neighborhood of a fixed reference point (x_0, u_0, p_0, q_0). We assume that the functions $g_i\colon H \times \mathbf{R}^l \to \mathbf{R}$, $i = 0, 1, \ldots, m$ are twice continuously Fréchet differentiable with respect to x and their first and second derivatives are continuous functions in both x and u in a neighborhood of (x_0, u_0); moreover, the first derivatives $\nabla_x g_i$, $i = 0, 1, \ldots, m$ are Lipschitz continuous with respect to u uniformly in x around (x_0, u_0). In the sequel $\langle \cdot, \cdot \rangle$ denotes the inner product. Denoting $g = (g_1, \ldots, g_m)$ and the positive orthant in $\mathbf{R}^m$ by $\mathbf{R}^m_+$, we also assume that

$$0 \in \operatorname{int}\bigl\{-q_0 + g(x_0, u_0) + \nabla_x g(x_0, u_0)(H - x_0) + \mathbf{R}^m_+\bigr\}, \tag{12}$$

the latter being equivalent to the Aubin continuity of the map

$$\bigl(g(\cdot, u_0) + \mathbf{R}^m_+\bigr)^{-1}$$

at the point (q_0, x_0), via the Robinson–Ursescu theorem and Corollary 2.4. In finite dimensions the regularity condition (12) becomes the well-known Mangasarian–Fromovitz constraint qualification.

Under (12), the local optimality of a point x_0 for (p_0, q_0, u_0) implies the existence of a (normal) Lagrange multiplier y_0 such that (x_0, y_0) solves the Karush–Kuhn–Tucker (KKT) system associated with (10) for (p_0, q_0, u_0). The KKT system has the form

$$\begin{aligned} &p + \nabla_x g_0(x, u) + \nabla_x g(x, u)^* y = 0,\\ &-q + g(x, u) \in \mathcal{N}_{\mathbf{R}^m_+}(y). \end{aligned} \tag{13}$$

Here $\mathcal{N}_{\mathbf{R}^m_+}(x)$ denotes the normal cone to $\mathbf{R}^m_+$ at the point x and * denotes the adjoint operator.

Let v denote the parameter triple (u, p, q) and set $v_0 = (u_0, p_0, q_0)$. For a given v, let $S_{\mathrm{KKT}}(v)$ be the set of solutions (x, y) of the KKT system (13). We study the continuity properties of the KKT map $v \mapsto S_{\mathrm{KKT}}(v)$.

We associate with a point $(u_0, p_0, q_0, x_0, y_0) \in \operatorname{graph} S_{\mathrm{KKT}}$ the index sets I_1, I_2, I_3 in $\{1, 2, \ldots, m\}$ defined as

$$\begin{aligned} I_1 &= \bigl\{i \in \{1, \ldots, m\} \bigm| g_i(x_0, u_0) - q_{0i} = 0,\ y_{0i} > 0\bigr\},\\ I_2 &= \bigl\{i \in \{1, \ldots, m\} \bigm| g_i(x_0, u_0) - q_{0i} = 0,\ y_{0i} = 0\bigr\},\\ I_3 &= \bigl\{i \in \{1, \ldots, m\} \bigm| g_i(x_0, u_0) - q_{0i} < 0,\ y_{0i} = 0\bigr\}. \end{aligned}$$

We introduce the Lagrangian

$$L(x, y, u) = g_0(x, u) + \langle y, g(x, u)\rangle,$$

and the bounded linear operators

$$\begin{aligned} &B_0 = (\nabla_x g_i(x_0, u_0))_{i \in I_1 \cup I_2}, \quad B_+ = (\nabla_x g_i(x_0, u_0))_{i \in I_1},\\ &A = \nabla^2_{xx} L(x_0, y_0, u_0); \end{aligned}$$

that is, B_0 maps H into the space $\mathbf{R}^{I_1 \cup I_2}$ of vectors with components corresponding to the active constraints at the reference point (p_0, q_0, x_0, y_0), while B_+ maps H into the space $\mathbf{R}^{I_1}$ of vectors whose components correspond to the active constraints associated with positive components y_{0i} of the Lagrange multiplier y_0.

We say that the *surjectivity condition* (S) holds at $(u_0, p_0, q_0, x_0, y_0) \in$ graph S_{KKT} if *the operator* B_0 *is surjective*. We say that the *strong second-order sufficient condition* (SSOSC) holds at $(u_0, p_0, q_0, x_0, y_0) \in$ graph S_{KKT} if

$$\langle x, \nabla^2_{xx} L(x_0, y_0, u_0)x \rangle \geqslant \alpha \|x\|^2 \quad \textit{for all } x \in \ \ker \ B_+.$$

In the proof of the following theorem we use the solution set $L_{\text{KKT}}(p, q)$ of the following linear variational inequality obtained by a linearization of (13) with respect to (x, y) at the point $(u_0, p_0, q_0, x_0, y_0)$:

$$\begin{aligned} &p - p_0 + A(x - x_0) + B^*(y - y_0) = 0, \\ &-(q - q_0) + g^0 + B(x - x_0) \in \mathcal{N}_{\mathbf{R}^m_+}(y), \end{aligned} \tag{14}$$

where $B = \nabla_x g(x_0, u_0)$ and $g^0 = g(x_0, u_0)$. Note that

$$(p_0, q_0, x_0, y_0) \in \text{graph } L_{\text{KKT}}.$$

THEOREM 4.1. *The following are equivalent*:

(i) *The map* S_{KKT} *is Aubin continuous at* $(u_0, p_0, q_0, x_0, y_0)$ *and has the property that for all* $(u, p, q, x, y) \in$ graph S_{KKT} *in some neighborhood of* $(u_0, p_0, q_0, x_0, y_0)$, x *is a local solution of* (10) *for* (u, p, q);

(ii) *The map* S_{KKT} *has a Lipschitzian localization at* $(u_0, p_0, q_0, x_0, y_0)$ *and has the property that for all* $(u, p, q, x, y) \in$ graph S_{KKT} *in some neighborhood of* $(u_0, p_0, q_0, x_0, y_0)$, x *is a local solution of* (10) *for* (u, p, q);

(iii) *Both* (S) *and* (SSOSC) *hold at* $(u_0, p_0, q_0, x_0, y_0)$.

We note that the implication (iii) ⇒ (ii) is a known result; in finite dimensions, it is due to Robinson [12], while for infinite-dimensional programs, see, e.g., [6]. The equivalence (ii) of (iii) has been established in [9] for finite-dimensional programs (see [4] for a simpler proof) and is based on the following general result ([9], Theorem 3): *for the solution map of a linear variational inequality over a polyhedral convex set, Aubin continuity and Lipschitzian localization properties coincide*. It is an open problem whether this result can be extended to variational inequalities over nonpolyhedral sets or to more general maps with a polyhedral structure.

Note that the Lipschitzian localization property of S_{KKT} in (ii) alone is not sufficient to obtain (iii); one needs information about optimal solutions. For instance, one can require the map '$v \mapsto$ set of optimal solutions for v' be lower semicontinuous at (v_0, x_0, y_0). In finite dimensions this requirement is automatically satisfied if x_0 is merely an isolated local minimizer of (10) for v_0, see [11].

Remark 4.1. From Corollary 2.2, the Aubin continuity of the map S_{KKT} is equivalent to the Aubin continuity of the map associated with the appropriately defined ε-KKT points.

Proof of Theorem 4.1. For a proof that (iii) implies the Lipschitzian localization property of S_{KKT} at $(p_0, q_0, u_0, x_0, y_0)$, see Lemma 5 in [6]. Further, (SSOSC) is a sufficient condition for local optimality of x for (u, p, q, x, y) in a neighborhood of $(u_0, p_0, q_0, x_0, y_0)$; hence for points (u, p, q, x, y) in the graph of S_{KKT} around $(u_0, p_0, q_0, x_0, y_0)$, x is a local solution to (10). Thus (iii) $\Rightarrow$ (ii). Of course, (ii) $\Rightarrow$ (i); it remains to show that (i) $\Rightarrow$ (iii).

Let (i) hold. For a real $\delta > 0$, let $\hat{\delta} \in \mathbf{R}^m$ have components

$$\hat{\delta}_i = \begin{cases} 0 & \text{for } i \in I_1 \cup I_3, \\ \delta & \text{for } i \in I_2 \end{cases}$$

and let

$$\hat{y} = y_0 + \hat{\delta}, \qquad \hat{p} = p_0 - B^* \hat{\delta}.$$

Observe that for all sufficiently small $\delta \geqslant 0$, $(u_0, \hat{p}, q_0, x_0, \hat{y}) \in \text{graph}\, S_{\text{KKT}}$. From Property 2.2, S_{KKT} is Aubin continuous at $(u_0, \hat{p}, q_0, x_0, \hat{y})$. By applying Corollary 2.4, thus passing to the linearization, we obtain that the map $\hat{L}_{\text{KKT}}$, defined by the relations (14) with p_0 and y_0 replaced by $\hat{p}$ and $\hat{y}$ respectively, is Aubin continuous at $(\hat{p}, q_0, x_0, \hat{y})$.

On the other hand, for a sufficiently small neighborhood $\mathcal{W}$ of $(\hat{p}, q_0, x_0, \hat{y})$ such that $\hat{y}_i > 0$ for $i \in I_1 \cup I_2$, if $(p, q, x, y) \in \text{graph}\, \hat{L}_{\text{KKT}} \cap \mathcal{W}$, then

$$B_0 x = B_0 x_0 - [g^0]_{1,2} + [q_0]_{1,2} + [q]_{1,2}, \tag{15}$$

where $[q]_{1,2}$ denotes the subvector of q containing components with indices from the set $I_1 \cup I_2$. In particular, from the Aubin continuity of $\hat{L}_{\text{KKT}}$, the equation $B_0 x = r$ must have a solution for any r in the neighborhood of the origin, hence for every $r \in \mathbf{R}^{I_1 \cup I_2}$. Thus, B_0 must be surjective; that is, (S) holds.

We proceed with the proof of (SSOSC). For $\delta > 0$, let

$$\tilde{q}_i = \begin{cases} q_{0i} & \text{for } i \in I_1 \cup I_3, \\ q_{0i} + \delta & \text{for } i \in I_2. \end{cases} \tag{16}$$

Observe that $(u_0, p_0, \tilde{q}, x_0, y_0) \in \text{graph}\, S_{\text{KKT}}$. Let $\tilde{L}_{\text{KKT}}$ be the map obtained by linearization of the KKT system around $(u_0, p_0, \tilde{q}, x_0, y_0)$; that is $\tilde{L}_{\text{KKT}}$ is defined by (14) with q_0 replaced by $\tilde{q}$. From Property 2.2 and Corollary 2.4, for every sufficiently small and fixed $\delta > 0$, the map $\tilde{L}_{\text{KKT}}$ is Aubin continuous at $(p_0, \tilde{q}, x_0, y_0)$. In particular, for any (p, q, x, y) close to $(p_0, \tilde{q}, x_0, y_0)$, the equation

$$\begin{aligned} Ax + B_+^* y &= Ax_0 + B_+^* y_0 + p_0 - p, \\ B_+ x &= B_+ x_0 - [g_0]_1 - [q_0]_1 + [q]_1 \end{aligned} \tag{17}$$

has a solution (x, y). This means that for any $(a, b) \in H \times \mathbf{R}^{I_1}$ the equation

$$\begin{bmatrix} A & B_+^* \\ B_+ & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{18}$$

has a solution. Thus the operator

$$\mathcal{A} := \begin{bmatrix} A & B_+^* \\ B_+ & 0 \end{bmatrix}$$

is surjective; since it is self-adjoint, it is invertible. Thus there exists a constant $\lambda > 0$ such that $\|\mathcal{A}z\| \geqslant \lambda\|z\|$ for all $z \in H \times \mathbf{R}^{I_1}$. The last inequality implies

$$\|Ax\| \geqslant \lambda\|x\| \quad \text{for all } x \in \ker B_+. \tag{19}$$

Hence the interval $(-\lambda, \lambda)$ does not belong to the spectrum σ of the operator $\pi A\pi$, where π is the projection from H into the closed subspace $\ker B_+$. Recall that for self-adjoint operators in Hilbert spaces we have

$$\min\{\mu \in \mathbf{R} \mid \mu \in \sigma\} = \inf\{\langle x, Ax\rangle \mid x \in \ker B_+,\ \|x\| = 1\}. \tag{20}$$

Thus it is sufficient to show that $\sigma \subset [\lambda, \infty]$.

For any y near y_0 the normal cone $\mathcal{N}_{\mathbf{R}_+^m}(y)$ is a subset of the subspace $L := \{0\}^{I_1} \times \mathbf{R}^{I_2 \cup I_3}$. Hence, for any (p, q) near (p_0, q_0), every element of $L_{\mathrm{KKT}}(p, q)$ is a solution of the equation (17). We established that the operator $\mathcal{A}$ is invertible, hence L_{KKT} is single-valued, that is, it has a Lipschitzian localization at (v_0, x_0, y_0). Hence, S_{KKT} has a Lipschitzian localization at (v_0, x_0, y_0), by Corollary 3.1.

By (i), for any $v = (p, q, u)$ near $v_0 = (p_0, q_0, u_0)$ there exists a local solution $x(v)$ of (10), which is close to x_0 (Property 2.1). This solution must satisfy the KKT system (13) with some multiplier $y(v)$ which, because of the already proved surjectivity property (S), is unique and close to y_0. Since S_{KKT} is locally single-valued, for some neighborhood $\mathcal{U}$ of (x_0, y_0), $(x(v), y(v))$ is the unique element of $S_{\mathrm{KKT}}(v) \cap \mathcal{U}$. Observe that, for $q = \tilde{q}$, $\tilde{q}$ as in (16), with $\delta > 0$ sufficiently small, the point $(x_0, y_0) \in S_{\mathrm{KKT}}(\tilde{v}) \cap \mathcal{U}$ for $\tilde{v} = (p_0, \tilde{q}, u_0)$. Since x_0 is locally optimal for $\tilde{v}$, the second-order necessary optimality condition holds at $(\tilde{v}, x_0, y_0)$. In this particular case the critical cone associated with this condition is just the subspace $\ker B_+$, hence

$$\langle x, Ax\rangle \geqslant 0 \quad \text{for all } x \in \ker B_+.$$

Then the spectrum σ of the operator $\pi A\pi$ is a subset of the nonnegative reals, hence, by (19), $\sigma \subset [\lambda, \infty]$. This implies (SSOSC) and the proof is complete. □

5. Optimal Control

In this section we show that, with appropriate definitions of the surjectivity and the strong second-order sufficient condition, we can obtain a result with exactly the same statement as Theorem 4.1 for an optimal control problem with pointwise inequality control constraints. This result extends the characterization of the Lipschitzian stability in optimal control obtained in [8].

In this section we use the standard notation in optimal control which is not consistent with the notation in Section 4.

We consider the following problem:

$$
\begin{aligned}
(\mathrm{O})_p \quad & \text{Minimize } F(x,u,p) = \int_0^1 [\varphi(x(t),u(t),h) + \vartheta(t)^{\mathsf T}u(t) + \psi(t)^{\mathsf T}x(t)]\,\mathrm{d}t \\
& \text{subject to} \\
& \dot{x}(t) - f(x(t),u(t),h) + \xi(t) = 0 \text{ for a.e. } t \in [0,1], \quad x(0) = 0, \\
& g(u(t),h) + \chi(t) \leqslant 0 \text{ for a.e. } t \in [0,1], \\
& x \in W^{1,\infty}(0,1;\mathbf{R}^n), \quad u \in L^\infty(0,1;\mathbf{R}^m),
\end{aligned}
$$

where $^{\mathsf T}$ denotes transposition,

$$x(t) \in \mathbf{R}^n, \qquad u(t) \in \mathbf{R}^m, \qquad h \in \mathbf{R}^d, \qquad \varphi\colon \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^d \mapsto \mathbf{R},$$

$$f\colon \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^d \mapsto \mathbf{R}^n, \qquad g\colon \mathbf{R}^m \times \mathbf{R}^d \mapsto \mathbf{R}^k,$$

$(h,\xi,\chi,\psi,\vartheta) := p$ is the vector of parameters, and

$$p \in P := \mathbf{R}^d \times L^\infty(0,1;\mathbf{R}^n) \times L^\infty(0,1;\mathbf{R}^k) \times L^\infty(0,1;\mathbf{R}^n) \times L^\infty(0,1;\mathbf{R}^m).$$

We assume that the functions φ, f, and g are twice continuously differentiable.

Let $p_0 = (h_0,\xi_0,\chi_0,\psi_0,\vartheta_0)$ be a given reference value of the parameter and let (x_0,u_0) be a local solution of $(\mathrm{O})_{p_0}$. Without going into details when exactly necessary optimality conditions of the form below hold (there is a vast literature on that question), we assume that there exists an associated (normal) Lagrange multiplier $(q_0,\nu_0) \in W^{1,\infty} \times L^\infty$ such that the following first-order optimality system is satisfied at $w_0 := (x_0,u_0,q_0,\nu_0)$ for p_0:

$$
\begin{aligned}
(\mathrm{VI})_p \qquad & \dot{x} - f(x,u,h) + \xi = 0, \quad x(0) = 0, \\
& g(u,h) + \chi \in \mathcal{N}_{\mathbf{R}^k_+}(\nu), \\
& \dot{q} + \nabla_x H(x,u,q,h) + \psi = 0, \quad q(1) = 0, \\
& \nabla_u \tilde{H}(x,u,q,\nu,h) + \vartheta = 0,
\end{aligned}
$$

for a.e. $t \in [0,1]$, where

$$H(x,u,q,h) = \varphi(x,u,h) + q^{\mathsf T} f(x,u,h),$$

$$\tilde{H}(x,u,q,\nu,h) = H(x,u,q,h) + \nu^{\mathsf T} g(u,h).$$

We denote by $S_{\mathrm P}(p)$ the set of solutions to $(\mathrm{VI})_p$ (analogous to S_{KKT}) and by $L_{\mathrm P}(\delta)$ the set of solutions to the following linear variational inequality $(\mathrm{LVI})_\delta$, being the linearization of $(\mathrm{VI})_p$ at the reference point (p_0,w_0):

$$
\begin{aligned}
(\mathrm{LVI})_\delta \qquad & \dot{y}(t) - A(t)y(t) - B(t)v(t) + \delta_1(t) = 0, \quad y(0) = 0, \\
& \Theta(t)v(t) + \delta_2(t) \in \mathcal{N}_{\mathbf{R}^k_+}(\mu(t)), \\
& \dot{r}(t) + A^{\mathsf T}(t)r(t) + \nabla^2_{xx}H_0(t)y(t) + \nabla^2_{xu}H_0(t)v(t) + \delta_3(t) = 0, \\
& r(1) = 0, \\
& B(t)^{\mathsf T}r(t) + \nabla^2_{uu}\tilde{H}_0(t)v(t) + \nabla^2_{ux}H_0(t)y(t) + \Theta(t)^{\mathsf T}\mu(t) + \delta_4(t) = 0,
\end{aligned}
$$

for a.e. $t \in [0, 1]$, where the subscript $_0$ indicates that a given function is evaluated at the reference point, $A = \nabla_x f_0, \quad B = \nabla_u f_0, \quad \Theta = \nabla_u g_0$, and the parameter $\delta := (\delta_1, \delta_2, \delta_3, \delta_4) \in \Delta := L^\infty(0, 1; \mathbf{R}^n) \times L^\infty(0, 1; \mathbf{R}^k) \times L^\infty(0, 1; \mathbf{R}^n) \times L^\infty(0, 1; \mathbf{R}^m)$. The reference value $\delta_0 = (\delta_{01}, \delta_{02}, \delta_{03}, \delta_{04})$ is:

$$\begin{aligned}
\delta_{01}(t) &= -(\dot{x}_0(t) - Ax_0(t) - Bu_0(t)),\\
\delta_{02}(t) &= g(u_0(t), h_0) - \Theta(t)u_0(t),\\
\delta_{03}(t) &= -(\dot{q}_0(t) + A^\mathsf{T}(t)q_0(t) + \nabla^2_{xx}H_0(t)x_0(t) + \nabla^2_{xu}H_0(t)u_0(t)),\\
\delta_{04}(t) &= -(B(t)^\mathsf{T}q_0(t) + \nabla^2_{uu}\tilde{H}_0(t)u_0(t) + \nabla^2_{ux}H_0(t)x_0(t) + \Theta(t)^\mathsf{T}\nu_0(t)).
\end{aligned}$$

Let $z_\delta = (y_\delta, v_\delta, r_\delta, \mu_\delta)$ denote a solution to $(\mathrm{LVI})_\delta$. Certainly, $z_{\delta_0} = w_0$.

Let $I := \{1, 2, \ldots, k\}$. For $t \in [0, 1]$ and $\alpha \geqslant 0$ denote

$$I_\alpha(t) := \{i \in I \mid g^i(u_0(t), h_0) \geqslant -\alpha\},$$

that is, $I_\alpha(t)$ is the index set of the α-active control constraints at t. Define the submatrix

$$\Theta_\alpha(t) = \left[\Theta^i(t)\right]_{i \in I_\alpha(t)}.$$

In a similar way, for a $\beta \geqslant 0$ we introduce the set

$$J_\beta(t) := \{i \in I_0(t) \mid \nu_0(t) > \beta\}$$

of the indices of those constraints active at t for which the strict complementarity is satisfied with a margin β. We also define the submatrix $\Theta^+_\beta(t) = [\Theta^i(t)]_{i \in J_\beta(t)}$.

Next, we introduce the condition which will play the role of the conditions (S) and (SSOSC) in Theorem 4.1.

(S) *There exist constants* $\alpha > 0$ *and* $\eta > 0$ *such that*

$$|\Theta_\alpha(t)^\mathsf{T}\nu| \geqslant \eta|\nu|$$

for a.e. $t \in [0, 1]$ *and for all* ν *of appropriate dimension.*

(SSOSC) *There exist constants* $\gamma > 0$ *and* $\beta > 0$ *such that the quadratic form*

$$\langle z, \mathcal{B}z\rangle := \frac{1}{2}\int_0^1 \begin{bmatrix} x(t)\\ u(t)\end{bmatrix}^\mathsf{T} \begin{bmatrix} \nabla^2_{xx}H_0(t) & \nabla^2_{xu}H_0(t)\\ \nabla^2_{ux}H_0(t) & \nabla^2_{uu}\tilde{H}_0(t)\end{bmatrix} \begin{bmatrix} x(t)\\ u(t)\end{bmatrix} \mathrm{d}t$$

satisfies

$$\langle z, \mathcal{B}z\rangle \geqslant \gamma\|u\|^2_{L^2} \quad \textit{for all } z = (x, u) \in \Upsilon^2_\beta,$$

where $\Upsilon^2_\beta = \big\{(x, u) \in W^{1,2}(0, 1; \mathbf{R}^n) \times L^2(0, 1; \mathbf{R}^m) \mid \dot{x} - Ax - Bu = 0, x(0) = 0$ *and* $\Theta^+_\beta u = 0\big\}$.

In the following lines we give a more compact reformulation of (SSOSC). Define the linear and continuous map $\mathcal{S}$ from L^2 to L^2:

$$(\mathcal{S}u)(t) = \int_0^\mathsf{T} \Psi(t, s)B(s)u(s)\mathrm{d}s,$$

where $\Psi(t, s)$ is the fundamental matrix solution of the equation $\dot{x} = Ax$. If we consider the matrices $\nabla^2_{xx} H_0$, $\nabla^2_{xu} H_0$ and $\nabla^2_{uu} H_0$ as linear and continuous maps from L^2 to L^2 (that is, formally, $(\nabla^2_{xx} H_0 w)(t) = \nabla^2_{xx} H_0(t) w(t)$), then (SSOSC) can be written as

$$\langle u, (\mathcal{M} + \nabla^2_{uu} \tilde{H}_0) u \rangle \geqslant \gamma \|u\|^2_{L^2} \quad \text{for all } u \in \mathcal{U}^2_\beta, \tag{21}$$

where

$$\mathcal{M} = \mathcal{S}^* \nabla^2_{xx} H_0 \mathcal{S} + \mathcal{S}^* \nabla^2_{xu} H_0 + \nabla^2_{ux} H_0 \mathcal{S} \tag{22}$$

and

$$\mathcal{U}^\kappa_\beta = \left\{ u \in L^\kappa(0, 1; \mathbf{R}^m) \mid \Theta^+_\beta(t) u(t) = 0 \text{ for a.e. } t \in [0, 1] \right\}, \quad \kappa \in [1, \infty].$$

If (S) holds, then the matrix $\Theta^+_\beta(t)(\Theta^+_\beta(t))^\mathsf{T}$ is nonsingular for a.e. $t \in [0, 1]$ and the projection map from $\mathbf{R}^m$ on the linear subspace $\{v \in \mathbf{R}^m \mid \Theta^+_\beta(t) v = 0\}$ has the form

$$\Xi(t) = I - (\Theta^+_\beta(t))^\mathsf{T} [\Theta^+_\beta(t)(\Theta^+_\beta(t))^\mathsf{T}]^{-1} \Theta^+_\beta(t).$$

Then the condition (21) can be further written as

$$\langle u, \mathcal{C} u \rangle \geqslant \gamma \|\Xi u\|^2_{L^2} \quad \text{for all } u \in L^2(0, 1; \mathbf{R}^m),$$

where, regarding again Ξ as a linear and bounded operator from L^2 to L^2 given by the matrix $\Xi(t)$, the operator $\mathcal{C}$ is defined as $\mathcal{C} = \Xi^*(\mathcal{M} + \nabla^2_{uu} \tilde{H}_0)\Xi$. Note that $\mathcal{C}$ is a linear, continuous and self-adjoint operator from L^2 to L^2.

The next theorem is analogous to Theorem 4.1, but it concerns the optimal control problem $(\mathrm{O})_p$ and the behavior of the set of extremal primal/dual solutions $S_P(p)$ considered as a mapping from P to the subsets of $W^{1,\infty}(0, 1; \mathbf{R}^n) \times L^\infty(0, 1; \mathbf{R}^m) \times W^{1,\infty}(0, 1; \mathbf{R}^n) \times L^\infty(0, 1; \mathbf{R}^k)$.

THEOREM 5.1. *The following are equivalent:*

(i) *The map* S_P *is Aubin continuous at* (p_0, w_0), *where* $w_0 = (x_0, u_0, q_0, \nu_0)$, *and has the property that for all* $(p, x, u, q, \nu) \in \operatorname{graph} S_\mathrm{P}$ *in some neighborhood of* (p_0, w_0), (x, u) *is a local solution of* $(\mathrm{O})_p$*;*

(ii) *The map* S_P *has a Lipschitzian localization at* (p_0, w_0) *and has the property that for all* $(p, x, u, q, \nu) \in \operatorname{graph} S_\mathrm{P}$ *in some neighborhood of* (p_0, w_0), (x, u) *is a local solution of* $(\mathrm{O})_p$*;*

(iii) *Both* (S) *and* (SSOSC) *hold at* (p_0, w_0).

Proof of Theorem 5.1. The proof goes along the lines of the proof of Theorem 4.1, with some important adjustments connected with the so called *two-norm discrepancy*. Namely, the Aubin continuity holds in the norm of the space L^∞, while (SSOSC) holds in L^2. We adapt the approach in [8]. The main steps of the proof of (i) $\Rightarrow$ (iii) are the following.

(i) $\Rightarrow$ (S). As in the proof of Theorem 4.1, by applying Corollary 2.4 we conclude that the map L_{P} is Aubin continuous at (δ_0, z_0), say with constants a, b and M. Let α and ε be positive numbers such that

$$\alpha < b \quad \text{and} \quad \varepsilon < \min\left\{\frac{a}{2}, \alpha\left[\sum_{i=1}^{k} \|\Theta^i\|_{L^\infty}\right]^{-1}\right\}.$$

Let $M_\alpha^i = \{t \in [0,1] \mid i \in I_\alpha(t)\}$. Introduce the following variations of the parameter δ:

$$\Delta\delta_2^i(t) = \begin{cases} -g^i(u_0(t), h_0) & \text{for } t \in M_\alpha^i, \\ 0 & \text{otherwise,} \end{cases} \qquad \Delta\delta_4(t) = -\varepsilon \Sigma_{i \in I_\alpha(t)} \Theta^i(t).$$

Denote $\Delta\delta = (0, \Delta\delta_2, 0, \Delta\delta_4)$ and $\tilde{\delta} = \delta_0 + \Delta\delta$. Clearly, $\tilde{\delta} \in B_b(\delta_0)$ and we have

$$\Theta^i(t)v_0(t) + \tilde{\delta}_2^i(t) \begin{cases} = 0 & \text{for } t \in M_\alpha^i, \\ \leqslant -\varepsilon & \text{for } t \in [0,1] \setminus M_\alpha^i. \end{cases} \tag{23}$$

Let us define

$$\tilde{\mu}^i(t) = \begin{cases} \mu_0^i(t) + \varepsilon & \text{for } t \in M_\alpha^i, \\ \mu_0^i(t) = 0 & \text{otherwise,} \end{cases} \tag{24}$$

and let $\tilde{z} = (y_0, v_0, q_0, \tilde{\mu})$. It can be easily checked that $\tilde{z} \in L_P(\tilde{\delta}) \cap B_a(z_0)$, then the Aubin property holds at $(\tilde{\delta}, \tilde{z})$ (cf. Property 2.2). It follows from (23) and (24) that, in a small neighborhood of $(\tilde{\delta}, \tilde{z})$, the inclusion in $(\mathrm{LVI})_\delta$ reduces to the equality

$$\Theta^i(t)v(t) + \delta_2^i(t) = 0 \quad \text{for all } i \in I_\alpha(t), \text{ and a.a. } t \in [0,1].$$

By the Aubin continuity of L_{P}, this equation must have a solution $v \in L^\infty(0,1;\mathbf{R}^m)$ for any $\delta_2(t) = \tilde{\delta}_2(t) + \Delta\tilde{\delta}_2(t)$, with $\|\Delta\tilde{\delta}_2\|_{L^\infty}$ sufficiently small. This implies (S).

(i) $\Rightarrow$ (SSOSC). Let

$$\beta < a \quad \text{and} \quad \varepsilon < \min\left\{a\|\Theta\|_{L^\infty}^{-1},\ b\right\}, \tag{25}$$

and let $N_\beta^j = \{t \in [0,1] \mid j \in J_\beta(t)\}$. Introduce the following variations:

$$\begin{aligned} &\Delta\chi^i(t) = \begin{cases} 0 & \text{for } t \in N_\beta^i, \\ -\varepsilon & \text{for } t \in [0,T] \setminus N_\beta^i, \end{cases} \\ &\Delta\nu^i(t) = \begin{cases} 0 & \text{for } t \in N_\beta^i, \\ -\nu_0^i & \text{for } t \in [0,T] \setminus N_\beta^i, \end{cases} \\ &\Delta\vartheta(t) = -\Theta(t)\Delta\nu(t). \end{aligned} \tag{26}$$

Denote $\Delta p := (0, 0, \Delta\chi, 0, \Delta\vartheta)$, $\hat{p} := p_0 + \Delta p$, $\hat{\nu} := \nu_0 + \Delta\nu$ and $w_{\hat{p}} := (x_0, u_0, q_0, \hat{\nu})$. It is easy to check that $w_{\hat{p}}$ is a solution to $(\mathrm{VI})_{\hat{p}}$. By (25), $\hat{p} \in B_b(p_0)$

and $w_{\hat{p}} \in B_a(w_0)$. Hence, in particular, (x_0, u_0) is a solution to $(\mathrm{O})_{\hat{p}}$. From the choice of variations (26), it follows that:

$$g^i(u_0(t), h_0) + \hat{\chi}^i(t) \begin{cases} = 0 & \text{for } t \in N^i_\beta, \\ \leqslant -\varepsilon & \text{for } t \in [0, T] \setminus N^i_\beta, \end{cases} \tag{27}$$

$$\hat{\nu}^i(t) = \begin{cases} \nu^i_0(t) \geqslant \beta & \text{for } t \in N^i_\beta, \\ 0 & \text{for } t \in [0, T] \setminus N^i_\beta. \end{cases} \tag{28}$$

In view of (27) and (28), in a small neighborhood of $(\hat{p}, w_{\hat{p}})$, problem $(\mathrm{O})_p$ can be considered as the problem with *equality* constraints:

$$g^i(u(t), h) + \chi^i(t) \begin{cases} = 0 & \text{for } t \in N^i_\beta, \\ \text{free} & \text{for } t \in [0, T] \setminus N^i_\beta. \end{cases} \tag{29}$$

Applying the second-order necessary optimality condition, see [8], Section 4, we obtain

$$\langle u, \mathcal{C}_{\hat{p}} u \rangle \geqslant 0 \quad \text{for all } u \in \mathcal{U}^2_\beta, \tag{30}$$

where

$$\mathcal{C}_{\hat{p}} = \Xi^*(\mathcal{M} + \nabla^2_{uu}\tilde{H}_{\hat{p}})\Xi$$

with $\mathcal{M}$ defined in (22) and $\tilde{H}_{\hat{p}} = \tilde{H}(x_0, u_0, q_0, \hat{\nu}, h_0)$. By (30) we have

$$\sigma \subset [0, \infty), \tag{31}$$

where σ is the spectrum of the self-adjoint operator $\mathcal{C}_{\hat{p}}\colon\ \mathcal{U}^2_\beta \to \mathcal{U}^2_\beta$. We are going to show that

$$\langle u, \mathcal{C}_{\hat{p}} u \rangle \geqslant \frac{M^{-1}}{2}\|u\|^2_{L^2} \quad \text{for all } u \in \mathcal{U}^2_\beta. \tag{32}$$

Since

$$\min\{\lambda \in \mathbf{R} \mid \lambda \in \sigma\} = \inf\{(v, \mathcal{C}_{\hat{p}} v) \mid v \in \mathcal{U}^2_\beta,\ \|v\| = 1\},$$

then in view of (31), to prove (32), it is enough to show that

$$\left[0, \frac{M^{-1}}{2}\right) \not\subset \sigma. \tag{33}$$

Denote $\hat{\delta} = (0, \Delta\chi, 0, \Delta\vartheta)$. It can be easily seen that $w_{\hat{p}} = z_{\hat{\delta}}$ is a solution to the linear variational inequality $(\mathrm{LVI})_{\hat{\delta}}$. In the same way as in (29), we find that, for all (δ, z_δ), in a small neighborhood of $(\hat{\delta}, z_{\hat{\delta}})$, the inclusion in $(\mathrm{LVI})_{\hat{\delta}}$ becomes an equation of the form

$$\Theta^+_\beta(t) + \delta^+_2(t) = 0, \tag{34}$$

where $\delta_2^+(t)$ is the subvector of $\delta_2(t)$ containing the components belonging to $J_\beta(t)$. Choose $\Delta\delta = (0, 0, 0, \Delta\delta_4)$, with $\Delta\delta_4 \in \mathcal{U}_\beta^\infty$, so small that (34) holds for $\delta = \hat{\delta} + \Delta\delta$. Subtracting (LVI)$_\delta$ evaluated at $\delta = \hat{\delta} + \Delta\delta$ and at $\delta = \hat{\delta}$, respectively, and performing simple calculations, we find that the equation

$$\mathcal{C}\Delta u = \Delta\delta_4 \tag{35}$$

must have a solution for any $\Delta\delta_4 \in \mathcal{U}_\beta^\infty$, that is, $\mathcal{C}\colon\ \mathcal{U}_\beta^\infty \to \mathcal{U}_\beta^\infty$ is surjective. This implies that the homogeneous equation

$$\mathcal{C}u = 0 \quad \text{has a unique, in } \mathcal{U}_\beta^2\text{, solution } u = 0. \tag{36}$$

Indeed, let $u \in \mathcal{U}_\beta^2$ be such that $\mathcal{C}u = 0$, then for any $v \in \mathcal{U}_\beta^\infty$ we have $0 = (v, \mathcal{C}u) = (\mathcal{C}v, u)$. Since $\mathcal{C}$ is surjective and the embedding $\mathcal{U}_\beta^\infty \subset \mathcal{U}_\beta^2$ is dense, we get $u = 0$.

The property (36) together with the surjectivity implies that $\mathcal{C}\colon\ \mathcal{U}_\beta^\infty \to \mathcal{U}_\beta^\infty$ is invertible and (35) has a unique solution for any $\Delta\delta_4 \in \mathcal{U}_\beta^\infty$. Hence, by the Aubin continuity, (35) implies

$$\|\mathcal{C}u\|_{L^\infty} \geqslant M^{-1}\|u\|_{L^\infty} \quad \text{for all } u \in \mathcal{U}_\beta^\infty. \tag{37}$$

Choosing a sufficiently small $\varepsilon > 0$ in (25), $w_{\hat{p}}$ can be moved arbitrarily close to w_0. In particular, we can choose ε so small that

$$\|(\mathcal{C}_{\hat{p}} - \mathcal{C})u\|_{L^\infty} = \|\,\Xi^*(\nabla_{uu}^2\tilde{H}_{\hat{p}} - \nabla_{uu}^2\tilde{H}_0)\,\Xi u\|_{L^\infty} \leqslant \frac{M^{-1}}{2}\|u\|_{L^\infty} \tag{38}$$

for all $u \in \mathcal{U}_\beta^\infty$. Hence

$$\|\mathcal{C}_{\hat{p}}u\|_{L^\infty} \geqslant \frac{M^{-1}}{2}\|u\|_{L^\infty} \quad \text{for all } u \in \mathcal{U}_\beta^\infty. \tag{39}$$

Let J denote the identity in $\mathbf{R}^n$. It follows from Lemma 4.3 in [8] and from (39) that, for any $\mu \in [0, M^{-1}/2)$ the matrix

$$\Xi(t)^\mathsf{T}[\nabla_{uu}^2\tilde{H}_{\hat{p}}(t) - \mu J]\,\Xi(t)$$

is invertible for a.e. $t \in [0, T]$, with the norm of its inverse bounded by a constant independent of t. Hence the operator

$$\Xi^*(\nabla_{uu}^2\tilde{H}_{\hat{p}} - \mu\cdot\mathcal{J})\,\Xi\colon\ \mathcal{U}_\beta^\kappa \to \mathcal{U}_\beta^\kappa$$

is invertible for any $\kappa \in [1, \infty]$. Define the operator

$$\mathcal{N}_{\hat{p}} := \Big[\Xi^*(\nabla_{uu}^2\tilde{H}_{\hat{p}} - \mu\cdot\mathcal{J})\,\Xi\Big]^{-1}\,\Xi^*\mathcal{M}\,\Xi + \mathcal{J}\colon\ \mathcal{U}_\beta^\kappa \to \mathcal{U}_\beta^\kappa.$$

In view of (39), $\mathcal{N}_{\hat{p}}$ is invertible for $\kappa = \infty$. Observe that (33) will be satisfied if $\mathcal{N}_{\hat{p}}$ is invertible for $\kappa = 2$. Also, observe that the operator $\mathcal{S}\colon\ L^2(0, T; \mathbf{R}^m) \to$

$L^2(0,T;\mathbf{R}^n)$ is compact, hence $\mathcal{M}\colon L^2(0,T;\mathbf{R}^m) \to L^2(0,T;\mathbf{R}^m)$ is also compact. Then the operator

$$\mathcal{R}_{\hat{p}} := \left[\Xi^*(\nabla^2_{uu}\tilde{H}_{\hat{p}} - \mu \cdot \mathcal{J})\Xi\right]^{-1} \Xi^*\mathcal{M}\Xi\colon\ \mathcal{U}^2_\beta \to \mathcal{U}^2_\beta$$

is compact. Moreover, it follows from the definition of $\mathcal{R}_{\hat{p}}$ that

$$\mathcal{R}_{\hat{p}}\ \mathcal{U}^2_\beta \subset \mathcal{U}^\infty_\beta. \tag{40}$$

Consider the homogeneous equation

$$\mathcal{N}_{\hat{p}}\ u := (\mathcal{R}_{\hat{p}} + \mathcal{J})u = 0, \quad u \in \mathcal{U}^2_\beta. \tag{41}$$

From (40) we have $u = -\mathcal{R}_{\hat{p}}u \in \mathcal{U}^\infty_\beta$, then by (39), $u = 0$ is the only solution of (41). By a known property of the compact operators (see, e.g., Theorem 2, Chap. XIII, Sec. 1 in [10]), the uniqueness of the solution to the homogeneous equation (41) implies that the operator $\mathcal{N}_{\hat{p}} := (\mathcal{R}_{\hat{p}} + \mathcal{J})\colon\ \mathcal{U}^2_\beta \to \mathcal{U}^2_\beta$ has a bounded inverse. This implies that (33) holds, i.e., (32) is satisfied.

As in (38), moving $w_{\hat{p}}$ sufficiently close to w_0 and using (32), we obtain

$$\langle u, \mathcal{C}u\rangle = \langle u, [\mathcal{C} - \mathcal{C}_{\hat{p}}]u\rangle + \langle u, \mathcal{C}_{\hat{p}}u\rangle \geqslant \frac{M^{-1}}{4}\|u\|^2_{L^2} \quad \text{for all } u \in \mathcal{U}^2_\beta, \tag{42}$$

which completes the proof of (SSOSC). Note that, in view of (37), the estimate (42) can be strengthened to

$$\langle u, \mathcal{C}u\rangle \geqslant M^{-1}\|u\|^2_{L^2} \quad \text{for all } u \in \mathcal{U}^2_\beta. \qquad \square$$

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