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THE WAVE ANNIHILATION TECHNIQUE AND THE DESIGN OF NONREFLECTIVE COATINGS*

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Abstract. We develop theory and algorithms for the design of coatings which either eliminate or enhance reflection of waves from surfaces. For steady-state harmonic waves with continuous frequency spectrum that covers an arbitrarily prescribed frequency band, coatings are designed that essentially eliminate reflections of all frequencies within the band. Although we focus on acoustic waves in elastic media, the methods developed here can be adapted to electromagnetism or other phenomena governed by variants of the linear wave equation.

To create a nonreflective coating which is to operate in a frequency band $[\Omega_0, \Omega_1]$, we select n frequencies, uniformly distributed in the band, and design an n-layer coating of given thickness that *completely* eliminates reflections of waves at these frequencies. We show that if n is large, then the reflectivity of the coating designed by our method is small for *all* frequencies in the band. More precisely, the reflectivity at an arbitrary frequency $\omega \in (\Omega_0, \Omega_1)$ is $O(1/n)$ if $\Omega_0 = 0$, while it is $O(\alpha^n)$ if $\Omega_0 > 0$, where $0 < \alpha < 1$. Furthermore, extensive numerical studies show that when this discrete n-layer coating is smoothed out by spline interpolation, the reflectivities remain small not only for frequencies in the original band but also for all larger frequencies.

We also describe a procedure for designing coatings that *maximizes* reflectivity (or, equivalently, minimizes transmissivity). We show that through a proper layering technique, it is possible to obtain transmissivity of $O(\alpha^n)$, $0 < \alpha < 1$, in an n-layer design.

Key words. optimal design, absorbing coating, nonreflective coating, reflective coating, anechoic coating, stratified coating, layered coating

AMS subject classifications. 73D15, 73B10, 73C02

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1. Introduction.

This paper deals with the design and analysis of coatings which either eliminate or enhance the reflection of waves from surfaces. Coatings that are either reflective or absorptive have applications in diverse fields of science and technology including acoustics (concert halls and antireflective lids for acoustic transducers), optics (filters, mirrors, and coatings on binoculars and on glass-based telescopes), electromagnetics (antireflective radomes for radar antennas and antisurveillance technology), and seismology (insulation of buildings from tremors). Although we concentrate on sound waves propagating in elastic media, the methods developed here can be adapted to electromagnetism or other phenomena governed by variants of the linear wave equation. We restrict our study to steady-state time-harmonic waves; however, by use of a Fourier expansion, our results are applicable to more general waves.

For an interface between two homogeneous half-spaces, a single-layer homogeneous coating can be designed, whose impedance is the geometric mean of the impedances of the surrounding media, that totally eliminates the reflection of an incident plane longitudinal wave of a given frequency. The density and elastic modulus of this coating depend on the width of the layer and the frequency of the incident

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wave. One of the main objectives of this paper is to construct *multiple coating layers* that eliminate reflections of waves within an arbitrarily prescribed frequency band, even when the half-space impedances on either side of the coating are mismatched. Our approach, which we call the wave annihilation technique, is the following: Given a frequency band $[\Omega_0, \Omega_1]$ and an overall coating thickness T, we show that for any number n, we can find a coating material consisting of n distinct homogeneous layers of combined thickness T, such that reflections from the composite coating are completely eliminated for waves of uniformly distributed frequencies $\omega_1, \omega_2, \ldots, \omega_n$ within this band. We show that as the number of layers increases, the reflectivity of the composite coating at any frequency ω in the specified band (Ω_0, Ω_1) is $O(1/n)$ if $\Omega_0 = 0$, while it is $O(\alpha^n)$ if $\Omega_0 > 0$, where $0 < \alpha < 1$. As Ω_0 approaches 0, with Ω_1 fixed, α tends to 1. As Ω_0 approaches Ω_1, α tends to 0. The exponential decay of reflectivity when $\Omega_0 > 0$ indicates that a small number of layers should suffice to build an essentially nonreflective coating.

In the limit, as the number of layers increases, we obtain a continuously varying coating which eliminates reflections for all frequencies inside the frequency band $[\Omega_0, \Omega_1]$. Although the layered coating has small reflectivity within the design band, the frequency response can have large values at $\Omega_0 + \Omega_1$ and at all integer multiples of this frequency. On the other hand, we observe numerically that if this layered coating is replaced by a smoothly varying coating obtained by spline interpolation, then the reflectivity is small both in the design frequency band and at all frequencies $\omega > \Omega_1$ as well. Thus for the design of nonreflective coatings according to our scheme, it is sufficient to concentrate the effort in the low frequency range. For the frequency band $[0, \Omega_1]$, the frequency response of the coating for large ω was relatively insensitive to the method of smoothing since linear and cubic splines produced a similar frequency response, while for the frequency band $[\Omega_0, \Omega_1]$ with $\Omega_0 > 0$, cubic spline smoothing yielded a much smaller frequency response than linear spline smoothing.

It is observed numerically (for example, see [21, pp. 214–215]) that the effectiveness of a nonreflective coating can degrade as the incident angle of the incoming wave increases. Moreover, the initial angle where the effectiveness degrades decreases as the number of layers increases. In this paper, we restrict our analysis to normal incidence and do not consider the effect of the incident angle on the design of nonreflective coatings.

Our wave propagation model is nondissipative, and the mechanical energy is conserved. Hence, the reduction or elimination of the reflected waves is accomplished by channeling an incoming wave downstream rather than by dissipating its energy. From the viewpoint of an observer on the upstream side, a nonreflective coating appears to "absorb" the incident wave. For this reason, we refer to the effect achieved by a nonreflective coating as *absorption.* We emphasize that no energy conversion is implied by our use of the term.

A key observation in carrying out the steps of the procedure outlined above is a reciprocity relationship proved in Lemma 4.3 for the impedances of the n optimally designed layers which constitute the nonreflective coating: *The product of impedances of pairs of layers symmetrically situated about the midpoint of the coating is constant.* We fully exploit this reciprocity in our numerical scheme for computing the nonreflective coating. It is interesting to note that Konstanty and Santosa [27] have observed this reciprocity emerge from their numerical study of optimal coatings when the incident wave is a pulse of small width (see [27, Remark 5, p. 304]). For a given incident wave of finite duration and energy, Anderson and Lundberg [4] prove an analogue of the reciprocity property, which they call antisymmetry. It is important to take into account the structure derived in Lemma 4.3 when computing the material that absorbs the given frequencies. If one simply tries to vary the material parameters to minimize the reflectivity for the given frequencies, the iterates invariably converge to a local minimum where the reflectivity does not vanish. Moreover, since there are many values for the material parameters in an n-layer coating that make the reflectivity vanish at n given frequencies, the process of choosing frequencies and adjusting the coating layers to absorb these frequencies may not converge unless the optimal coating's structure is taken into account.

The mathematical framework for the formulation of wave propagation phenomena in layered materials can be found in many treatises on wave propagation. Closest to the spirit of our treatment are the books of Achenbach [1], Auld [5], Brekhovskikh [9], Brekhovskikh and Goncharov [10], and Hudson [24]. For expositions on applications of wave propagation in layered media, see Ben-Menahem and Singh [8], Burdic [11], Kennett [25], Kinsler and Frey [26], and Stumpf [30]. The systematic mathematical study of the design of nonreflective coatings seems to have begun in the field of optics in the 1940s. Mooney [28] presents a brief overview of the work before the 1940s and then proceeds to obtain a formula for the reflectivity of an optical coating with one or two layers. The formula for the reflectivity of a two-layer coating is rather complicated, and he suggests that in order to minimize reflectivity, one can take partial derivatives with respect to design parameters and set them to zero. Weinstein [33] extends Mooney's work to any number of layers, giving a procedure for computing the reflectivity and transmissivity of a stack of homogeneous layers, allowing for both oblique incidence and dissipation. He remarks, "It is obvious that the expressions for the reflected and transmitted amplitudes . . . become very complicated when more than two or three layers are considered." Brekhovskikh in his classic work [9] presents expressions for the reflection coefficients associated with one-, two-, and three-layered coatings. He notes that with more than one layer, there is more flexibility in how the material parameters can be chosen in order to absorb any given wave. In particular, for a two layer coating, he obtains a two-parameter family of materials that absorb a given wave. Chen and Bridges [13] formulate the problem of absorbing a given wave in terms of geometric optics, treating both plane and spherical wave fronts. In contrast to the single frequency works described above, our paper designs n-layer coatings that absorb n frequencies, obtaining for large n a material that absorbs essentially *any* wave.

Both this paper and the literature cited above deal almost exclusively with waves in the frequency domain. Anderson and Lundberg [4], Konstanty and Santosa [27], Hellberg [22], and Hellberg and Karlsson [23], on the other hand, look at the coating design problem in the time domain. Anderson and Lundberg show that for any given wave of finite duration and energy, there exists an antisymmetric impedance that minimizes reflectivity. Hellberg and Karlsson formulate the optimal coating problem in terms of Green's function. The general nature of the incident wave profile requires the imposition of a special boundary condition at the far end of the coating to eliminate the reflections. Konstanty and Santosa perform a detailed numerical study of the coating design problem. The optimization problem they arrive at is computationally delicate and ill-posed in general. They introduce a regularization scheme which increases the convexity of the cost function with a penalty term and stabilizes the computation. In any case, the optimal coating depends on the incident wave profile, as expected.

For the most part, the current work relies on the general formulation of wave propagation in elastic media in our previous work [20], although we have strived to make this paper self-contained to the extent possible. There are many other approaches to the study of wave propagation in layered media. The formulation in [20] is particularly suited for the purpose of the design of optimal coatings. See also the works by Eremin and Sveshnikov [17]; Bendali and Lemrabet [7]; Tenenbaum and Zindeluk [31], [32]; Babe and Gusev [6]; and Caviglia and Morro [12]. For practical applications to geophysics, see Aminzadeh and Mendel [2] and [3]. For applications to electromagnetism see Moses and Prosser [29]. Chopra [14, Chapter 7] has an extensive discussion and computational data for multilayered optical systems.

Although this introduction has focused on minimizing reflection, we also consider in this paper the problem of minimizing transmission. For normal incidence, conservation of energy yields the relation

$$|r|^2 + \frac{\Gamma_-}{\Gamma_+}|\tau|^2 = 1, \tag{1}$$

where r is the coating reflectivity, τ is the coating transmissivity, and Γ_+ and Γ_- are the impedances of the half-spaces surrounding the coating. Hence, minimum reflectivity corresponds to the maximum transmissivity, and the anechoic coating that we have designed in effect allows incident waves to be entirely transmitted without reflection. At the opposite extreme, a coating that minimizes transmissivity maximizes reflectivity. Thus a coating that achieves small transmissivity essentially reflects incident waves totally. Since transmissivity never vanishes, the problem of minimizing transmissivity must be approached differently from the problem of minimizing reflectivity; that is, we cannot make the transmissivity vanish at a discrete set of frequencies in the same way that we make the reflectivity vanish. On the other hand, we show that transmissivity can be made arbitrarily small for waves on a frequency band $[\Omega_0, \Omega_1]$ with $\Omega_0 > 0$ by a coating whose impedance oscillates between large and small values in successive layers. Using the same theory developed for the design and analysis of anechoic coatings, we show that the transmissivity of our design approaches zero exponentially fast in the number of layers of the coating.

An outline of the paper follows. In section 2 we consider the classic case of a single wave reflecting from a single homogeneous layer. There are two countable families of materials that absorb the given wave. In section 3, we consider inhomogeneous coatings and we develop a new formula for the reflection coefficient associated with a layered material. In section 4, we explore qualitative properties of materials that absorb frequencies distributed symmetrically about a central frequency, deriving the reciprocity relation mentioned earlier. In section 5, we consider the minimization of reflectivity over the frequency band $[0, \Omega_1]$, while in section 6 we analyze the band $[\Omega_0, \Omega_1]$ with $\Omega_0 > 0$. Section 7 examines the transmissivity of a coating and its minimization. Finally, section 8 provides numerical illustrations for the theory developed in the paper.

2. A homogeneous layer. In this section we consider a single harmonic wave of frequency ω and a homogeneous, isotropic elastic coating. The equation of motion for a one-dimensional elastic material with density ρ and stiffness κ is

$$\rho(x)\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial v}{\partial x}\right), \tag{2}$$

where $v = v(x,t)$ is the displacement at position x and at time t. Assuming harmonic time dependence and a unit amplitude for the incident wave, the general solution of the equation of motion has the form $v(x,t) = u(x)e^{\mathrm{i}\omega t}$, where ω is the wave frequency, and where

$$\begin{aligned} u(x) &= e^{\mathrm{i}\omega s_+(x-T)} + re^{-\mathrm{i}\omega s_+(x-T)} && \text{for} \quad x > T, \\ u(x) &= \tau^- e^{\mathrm{i}\omega s x} + \tau^+ e^{-\mathrm{i}\omega s x} && \text{for} \quad 0 \le x \le T, \\ u(x) &= \tau e^{\mathrm{i}\omega s_- x} && \text{for} \quad x < 0. \end{aligned}$$

Here the slowness parameters (reciprocal of wave speed) s_+, s, and s_- are defined by

$$s_+ = \sqrt{\rho_+/\kappa_+}, \quad s = \sqrt{\rho/\kappa}, \quad \text{and} \quad s_- = \sqrt{\rho_-/\kappa_-}.$$

Thus τ^+ corresponds to a right propagating wave and τ^- corresponds to a left propagating wave, while s_+ is the slowness of the right half-space and s_- is the slowness of the left half-space.

The amplitudes r, τ, τ_-, and τ_+ can be determined from the continuity of displacement v and stress $\kappa \partial v/\partial x$ at the interfaces $x = 0$ and $x = T$. Altogether, there are four equations of continuity:

$$\left\{ \begin{aligned} \tau &= \tau^- + \tau^+, \\ 1 + r &= \tau^- e^{\mathrm{i}\omega s T} + \tau^+ e^{-\mathrm{i}\omega s T}, \\ s_-\kappa_-\tau &= \kappa s(\tau^- - \tau^+), \\ s_+\kappa_+(1-r) &= \kappa s(\tau^- e^{\mathrm{i}\omega s T} - \tau^+ e^{-\mathrm{i}\omega s T}). \end{aligned} \right. \tag{3}$$

Solving these equations for r and setting $r = 0$ yields the relation

$$\frac{\gamma_+ - \gamma}{\gamma_+ + \gamma} = \frac{\gamma_- - \gamma}{\gamma_- + \gamma} e^{-2\mathrm{i}\omega s T}, \tag{4}$$

where the impedances are defined by

$$\gamma_+ = \sqrt{\kappa_+\rho_+}, \quad \gamma = \sqrt{\kappa\rho}, \quad \text{and} \quad \gamma_- = \sqrt{\kappa_-\rho_-}.$$

Since the mechanical parameters are all real, equation (4) only holds when the exponential term is $+1$ or -1. Hence, there are two cases to consider.

Case 1. $e^{-2\mathrm{i}\omega s T} = -1$.

In this case, the exponent $2\omega s T$ is an odd multiple of π. In other words, $\omega s T = (m + \frac{1}{2})\pi$ for some integer m, or, equivalently,

$$\omega T \sqrt{\rho/\kappa} = \left(m + \frac{1}{2}\right)\pi. \tag{5}$$

Substituting -1 for the exponential term in (4) gives $\gamma^2 = \gamma_-\gamma_+$, or, equivalently,

$$\kappa\rho = \sqrt{\kappa_-\rho_-}\,\sqrt{\kappa_+\rho_+}. \tag{6}$$

Thus the impedance $\sqrt{\kappa\rho}$ of the coating is the geometric mean of the impedances of the surrounding half-spaces. Together, equations (5) and (6) determine the ratio ρ/κ and the product $\rho\kappa$. Therefore, they determine a unique ρ and κ for each choice of the integer m in (5). This family of coatings is often called the "quarter wavelength" family in the literature (for example, see [9, p. 136]). The generalization of (6) to multilayer coatings appears in Theorem 4.3.

Case 2. $e^{-2i\omega sT} = +1$.

In this case, the exponent $2\omega sT$ is an even multiple of π, which implies that

$$\omega sT = \omega T\sqrt{\rho/\kappa} = m\pi \tag{7}$$

for some integer m. Again, this equation restricts the slowness to a countable set of discrete values. However, when we substitute $+1$ for the exponential term in (4), we see that $\gamma_- = \gamma_+$. That is, this case can occur only when the materials in the two half-spaces have the same impedance. When this happens, there is a one-parameter family (for each integer m) of coating materials, with slowness given by (7), that totally absorbs the incoming wave. We call this family of coatings *degenerate* since they are only applicable in the special case where the materials on either side of the coating have the same impedance.

3. Reflection from multilayered coatings. Let us consider a harmonic wave with frequency ω propagating along the x-axis, perpendicular to an elastic coating that occupies the regions $0 \le x \le T$. We assume that all media are nondissipative, isotropic elastic materials, and the half-spaces $x < 0$ and $x > T$ are homogeneous. Let $\kappa(x)$ and $\rho(x)$ denote the stiffness and density in the region $0 \le x \le T$, and let (κ_+, ρ_+) and (κ_-, ρ_-) denote the corresponding mechanical parameters in the half-spaces $x > T$ and $x < 0$, respectively. By the theory developed in [19], the reflectivity r (the ratio of the amplitudes of the reflected and the incident waves) can be expressed[1] as

$$r = \frac{\Gamma_+ - G(T)}{\Gamma_+ + G(T)}, \quad \Gamma_+ = \sqrt{\kappa_+\rho_+}, \tag{8}$$

where G is the solution to the differential equation

$$G' = \mathrm{i}\frac{\omega}{\kappa}(\gamma^2 - G^2), \quad G(0) = \Gamma_- = \sqrt{\kappa_-\rho_-}, \quad \text{and} \quad \gamma = \sqrt{\kappa\rho}. \tag{9}$$

In section 2 we saw that for any given frequency ω, there exist infinitely many choices for a homogeneous absorbent coating. In order to absorb waves of more than one frequency, we must employ an inhomogeneous coating. It is impossible to absorb waves of all frequencies since the reflectivity corresponding to $\omega = 0$ is $(\Gamma_+ - \Gamma_-)/(\Gamma_+ + \Gamma_-)$, which does not vanish except for the trivial case $\Gamma_+ = \Gamma_-$. Nonetheless, we will see that the reflectivity can be made arbitrarily small for frequencies in an interval $[\Omega_0, \infty)$ by an appropriate choice of the elastic parameters in the coating.

To begin, we derive a new formula for the reflection coefficient associated with homogeneous layers.

PROPOSITION 3.1. *Suppose that the coating* $0 \le x \le T$ *is composed of* n *homogeneous layers, each layer of thickness* $\Delta x = T/n$. *If* κ_j *and* ρ_j *are the stiffness and density in the* j*th layer* $(j-1)\Delta x \le x \le j\Delta x$, *and* $\gamma_j = \sqrt{\kappa_j\rho_j}$ *is the associated impedance, then the reflectivity of the coating can be expressed as*

$$r = \frac{[\Gamma_- \;\; \gamma_1]\prod_{j=1}^{n} A_j \; \binom{-1}{1}}{[\Gamma_- \;\; \gamma_1]\prod_{j=1}^{n} A_j \; \binom{1}{1}}, \tag{10}$$

[1] In [19] "impedance" is the reciprocal of the impedance used in this paper. Hence, the formulas in [19] and in this paper are related by appropriate inversions.

where $[\Gamma_- \quad \gamma_1]$ *is a 2-component row vector,* $\prod_{j=1}^{n} A_j = A_1 A_2 \cdots A_n$,

$$A_j = \begin{pmatrix} \gamma_j e_j^+ & \gamma_{j+1} e_j^- \\ \gamma_j e_j^- & \gamma_{j+1} e_j^+ \end{pmatrix}, \quad \gamma_{n+1} = \Gamma_+, \quad \textit{and} \quad e_j^{\pm} = \exp\left(\frac{2\gamma_j \Delta x \omega \mathrm{i}}{\kappa_j}\right) \pm 1. \tag{11}$$

Proof. For a fixed given frequency ω, let G_j denote the solution to the differential equation (9) evaluated at $x = j\Delta x$. We make repeated application of the following formula (see [19, equation (16)]) for the solution of (9) across a homogeneous layer:

$$G_k = \gamma_k \frac{G_{k-1} e_k^+ + \gamma_k e_k^-}{G_{k-1} e_k^- + \gamma_k e_k^+}. \tag{12}$$

In particular, substituting the $k = 1$ case of (12) into the $k = 2$ case of (12) we get

$$G_2 = \gamma_2 \frac{\gamma_1 (G_0 e_1^+ + \gamma_1 e_1^-) e_2^+ + \gamma_2 e_2^- (G_0 e_1^- + \gamma_1 e_1^+)}{\gamma_1 (G_0 e_1^+ + \gamma_1 e_1^-) e_2^- + \gamma_2 e_2^+ (G_0 e_1^- + \gamma_1 e_1^+)}.$$

Referring to the definition of A_1, we see that G_2 can be expressed as

$$G_2 = \frac{\gamma_2 [G_0 \quad \gamma_1] A_1 \begin{pmatrix} e_2^+ \\ e_2^- \end{pmatrix}}{[G_0 \quad \gamma_1] A_1 \begin{pmatrix} e_2^- \\ e_2^+ \end{pmatrix}}. \tag{13}$$

In (13) we relate the value of G_2 to the value of G_0 and to the material properties in the region $0 \le x \le x_2$. Of course, there is an analogous formula relating G_k to G_{k-2} and to the material in the region $x_{k-2} \le x \le x_k$. Hence, the following formula holds for $j = k - 2$:

$$G_k = \frac{\gamma_k [G_j \quad \gamma_{j+1}] \prod_{l=j+1}^{k-1} A_l \begin{pmatrix} e_k^+ \\ e_k^- \end{pmatrix}}{[G_j \quad \gamma_{j+1}] \prod_{l=j+1}^{k-1} A_l \begin{pmatrix} e_k^- \\ e_k^+ \end{pmatrix}}. \tag{14}$$

Proceeding by induction, suppose that for some $\Delta \ge 2$, (14) holds for all k and j such that $k - \Delta \le j \le k - 2$. From (12) it follows that

$$\begin{aligned} \begin{pmatrix} G_j \\ \gamma_{j+1} \end{pmatrix} &= \frac{1}{G_{j-1} e_j^- + \gamma_j e_j^+} \begin{pmatrix} \gamma_j (G_{j-1} e_j^+ + \gamma_j e_j^-) \\ \gamma_{j+1} (G_{j-1} e_j^- + \gamma_j e_j^+) \end{pmatrix} \\ &= \frac{1}{G_{j-1} e_j^- + \gamma_j e_j^+} A_j^T \begin{pmatrix} G_{j-1} \\ \gamma_j \end{pmatrix}. \end{aligned}$$

With this substitution in (14), we have

$$G_k = \frac{\gamma_k [G_{j-1} \quad \gamma_j] \prod_{l=j}^{k-1} A_l \begin{pmatrix} e_k^+ \\ e_k^- \end{pmatrix}}{[G_{j-1} \quad \gamma_j] \prod_{l=j}^{k-1} A_l \begin{pmatrix} e_k^- \\ e_k^+ \end{pmatrix}}.$$

Since (14) holds with j decreased by 1, the induction step is complete, and (14) holds for all $j \leq k-2$.

Putting $j = 0$ in (14), we have that

$$G_n = \frac{\gamma_n [G_0 \quad \gamma_1] \prod_{j=1}^{n-1} A_j \begin{pmatrix} e_n^+ \\ e_n^- \end{pmatrix}}{[G_0 \quad \gamma_1] \prod_{j=1}^{n-1} A_j \begin{pmatrix} e_n^- \\ e_n^+ \end{pmatrix}}.$$

Since $G_0 = \Gamma_-$, the reflectivity can be expressed

$$r = \frac{\Gamma_+ - G_n}{\Gamma_+ + G_n} = \frac{[\Gamma_- \quad \gamma_1] \prod_{j=1}^{n-1} A_j \left[\Gamma_+ \begin{pmatrix} e_n^- \\ e_n^+ \end{pmatrix} - \gamma_n \begin{pmatrix} e_n^+ \\ e_n^- \end{pmatrix} \right]}{[\Gamma_- \quad \gamma_1] \prod_{j=1}^{n-1} A_j \left[\Gamma_+ \begin{pmatrix} e_n^- \\ e_n^+ \end{pmatrix} + \gamma_n \begin{pmatrix} e_n^+ \\ e_n^- \end{pmatrix} \right]}.$$

Since $\Gamma_+ = \gamma_{n+1}$ by (11), this expression for the reflectivity reduces to (10). ∎

Remark 3.2. When the layers do not have uniform thickness, simply replace Δx in (11) with the thickness of layer j.

4. A family of absorbers. Recall that our approach to the total absorption problem is to make the reflectivity vanish at a fixed collection of frequencies, and then to let the number of frequencies tend to infinity. To simplify the analysis, we assume that the frequencies are symmetrically distributed about a point. That is, we assume that

$$\frac{\omega_j + \omega_{n+1-j}}{2} = \overline{\omega} \tag{15}$$

for $j = 1, 2, \ldots, n$, where $\overline{\omega} > 0$ is some fixed frequency. When the frequencies are chosen in this way, they are symmetrically distributed about $\overline{\omega}$ for both even and odd values of n. In particular, when n is odd, the middle frequency is precisely $\overline{\omega}$. As mentioned earlier, there are typically many different choices of the material parameters that annihilate waves of the n given frequencies. And, in particular, we saw already that for a single homogeneous layer, there are a countable number of choices for the material parameters that totally absorb an incident wave of any given frequency. We will focus on a specific family of materials for which the stiffness and impedance in each layer satisfy the following condition:

$$\kappa_j = \frac{2\gamma_j \Delta x \overline{\omega}}{\pi}. \tag{16}$$

In the single layer/single frequency case, this corresponds to Case 1 of section 2 and $m = 0$. That is, for a single layer, $\Delta x = T$ and (16) implies that

$$\frac{\gamma_j T \overline{\omega}}{\kappa_j} = \sqrt{\rho_j / \kappa_j}\, T \overline{\omega} = \frac{\pi}{2},$$

which is (5) with $m = 0$. Note that the family of materials that we focus on excludes the degenerate materials of Case 2.

When κ_j is chosen in accordance with (16), the quantity $e_j^{\pm}$ of (11) reduces to

$$e_j^{\pm} = \exp\left(\frac{\pi\omega \mathrm{i}}{\overline{\omega}}\right) \pm 1, \tag{17}$$

which is independent of j. By Proposition 3.1, the reflectivity vanishes at the given frequencies if and only if

$$[\Gamma_- \quad \gamma_1] \prod_{j=1}^{n} A_j(z) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \quad \text{for} \quad z = \exp\left(\frac{\pi\omega_k \mathrm{i}}{\overline{\omega}}\right), \quad k = 1, 2, \ldots, n, \tag{18}$$

where

$$A_j(z) = z \begin{pmatrix} \gamma_j & \gamma_{j+1} \\ \gamma_j & \gamma_{j+1} \end{pmatrix} + \begin{pmatrix} \gamma_j & -\gamma_{j+1} \\ -\gamma_j & \gamma_{j+1} \end{pmatrix}. \tag{19}$$

Observe that the two terms forming $A_j(z)$ have the form

$$z^{\beta_j} \begin{pmatrix} \gamma_j & (-1)^{\beta_j+1}\gamma_{j+1} \\ (-1)^{\beta_j+1}\gamma_j & \gamma_{j+1} \end{pmatrix},$$

where $\beta_j = 1$ and $\beta_j = 0$ correspond to the first and second terms, respectively, in (19). Hence, the product in (18) has the following equivalent representation:

$$\prod_{j=1}^{n} A_j(z) = \sum_{|\beta| \le n} z^{|\beta|} \prod_{j=1}^{n} \begin{pmatrix} \gamma_j & (-1)^{\beta_j+1}\gamma_{j+1} \\ (-1)^{\beta_j+1}\gamma_j & \gamma_{j+1} \end{pmatrix}, \tag{20}$$

where β is an n component binary vector (each component either 0 or 1) and

$$|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n.$$

After grouping together the terms for which the exponents of z are the same, (18) takes the form

$$\sum_{j=0}^{n} c_j z^j = 0 \quad \text{for } z = \exp\left(\frac{\pi\omega_k \mathrm{i}}{\overline{\omega}}\right), \qquad k = 1, 2, \ldots, n, \tag{21}$$

where

$$c_j = \sum_{|\beta|=j} [\Gamma_- \quad \gamma_1] \prod_{k=1}^{n} \begin{pmatrix} \gamma_k & (-1)^{\beta_k+1}\gamma_{k+1} \\ (-1)^{\beta_k+1}\gamma_k & \gamma_{k+1} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{22}$$

In summary, we have Lemma 4.1.

LEMMA 4.1. *If the stiffness κ_j and the impedance γ_j in each layer satisfy* (16), *then the reflectivity vanishes for incident waves of frequencies $\omega_1, \omega_2, \ldots, \omega_n$ if and only if the polynomial* (21) *with coefficients given by* (22) *vanishes at*

$$z = \exp\left(\frac{\pi\omega_k \mathrm{i}}{\overline{\omega}}\right), \qquad k = 1, 2, \ldots, n.$$

Moreover, c_j can be expressed as

$$c_j = \sum_{|\beta|=j} - \prod_{k=0}^{n} (\gamma_k + (-1)^{\beta_k+\beta_{k+1}} \gamma_{k+1}), \tag{23}$$

where $\gamma_0 = \Gamma_-$, $\gamma_{n+1} = \Gamma_+$, $\beta_0 = 1$, $\beta_{n+1} = 0$ and β_k, $1 \le k \le n$, is binary.

Proof. The only part of the lemma that remains to be verified is the formula (23). The following identity holds for any integer choices of β_k and β_{k+1}:

$$\begin{aligned} &\begin{pmatrix} \gamma_k & (-1)^{1+\beta_k}\gamma_{k+1} \\ (-1)^{\beta_k+1}\gamma_k & \gamma_{k+1} \end{pmatrix} \begin{pmatrix} 1 \\ (-1)^{1+\beta_{k+1}} \end{pmatrix} \\ &= (\gamma_k + (-1)^{\beta_k+\beta_{k+1}}\gamma_{k+1}) \begin{pmatrix} 1 \\ (-1)^{1+\beta_k} \end{pmatrix}. \end{aligned} \tag{24}$$

Also, by the definition of β_{n+1}, we have

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ (-1)^{1+\beta_{n+1}} \end{pmatrix}. \tag{25}$$

Combining (24) and (25) yields

$$\begin{aligned} &\left[\prod_{k=1}^{n} \begin{pmatrix} \gamma_k & (-1)^{1+\beta_k}\gamma_{k+1} \\ (-1)^{\beta_k+1}\gamma_k & \gamma_{k+1} \end{pmatrix}\right] \begin{pmatrix} 1 \\ (-1)^{1+\beta_{n+1}} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ (-1)^{1+\beta_1} \end{pmatrix} \prod_{k=1}^{n} (\gamma_k + (-1)^{\beta_k+\beta_{k+1}}\gamma_{k+1}). \end{aligned} \tag{26}$$

By the definition of γ_0 and β_0, we have

$$[\Gamma_- \quad \gamma_1] \begin{pmatrix} 1 \\ (-1)^{1+\beta_1} \end{pmatrix} = [\gamma_0 \quad \gamma_1] \begin{pmatrix} 1 \\ (-1)^{\beta_0+\beta_1} \end{pmatrix} = \gamma_0 + \gamma_1(-1)^{\beta_0+\beta_1}. \tag{27}$$

When we combine (22), (26), and (27), the proof is complete. $\square$

Observe that the coefficients c_j in (22) are real functions of the impedances $\gamma_1 \ldots, \gamma_n$. On the surface, (21) represents an overdetermined system of equations since there are n real unknowns, the impedances, and n complex equations, or $2n$ real equations, that must be satisfied. In other words, there are twice as many equations as unknowns.

We now observe that if (21) holds for $1 \le k \le n/2$, then it holds automatically for $n/2 < k \le n$. Consequently, (21) represents exactly the same number of equations as unknowns. Let $z(\omega)$ be defined by

$$z(\omega) = \exp\left(\frac{\pi\omega \mathrm{i}}{\overline{\omega}}\right). \tag{28}$$

By (15), we have

$$z(\omega_{n+1-k}) = \exp\left(2\pi\mathrm{i} - \frac{\pi\omega_k \mathrm{i}}{\overline{\omega}}\right) = \exp\left(\frac{-\pi\omega_k \mathrm{i}}{\overline{\omega}}\right) = z(-\omega_k). \tag{29}$$

Suppose that (21) is satisfied for $z = z(\omega_k)$. Taking the conjugate of (21) gives

$$\sum_{j=0}^{n} c_j z(-\omega_k)^j = 0,$$

since the coefficients are real. And combining this with (29), we have

$$\sum_{j=0}^{n} c_j z(\omega_{n+1-k})^j = 0. \tag{30}$$

Hence, when n is even, we only need to impose (21) at $k = 1, 2, \ldots, n/2$ since it automatically holds for $n/2 < k \le n$ by (30).

When n is odd, it follows from (15) that the middle frequency is $\overline{\omega}$, and by (28),

$$z(\overline{\omega}) = -1.$$

Thus at the middle frequency, (21) reduces to the real equation

$$\sum_{j=0}^{n} c_j (-1)^j = 0.$$

Since the complex part of this equation is trivially zero, there are again n equations in the n unknowns $\gamma_1, \gamma_2, \ldots, \gamma_n$—the last $(n-1)/2$ complex equations hold when the first $(n-1)/2$ equations hold while the middle equation is real.

Under the assumption that a solution to (21) exists, we now develop some of its properties. By the definition of $z(\omega)$ in (28), $z(\omega_{n+1-k}) = z(\omega_k)^{-1}$. Consequently both $z(\omega_k)$ and its reciprocal are zeros of the polynomial in (21). This leads us to the following observation about polynomials whose zeros occur in reciprocal pairs.

LEMMA 4.2. *Let $p(z)$ be a polynomial of degree n:*

$$p(z) = \sum_{j=0}^{n} c_j z^j.$$

Suppose that p satisfies the following conditions: $p(1) \neq 0$ and all the zeros of p occur in reciprocal pairs. That is, the zeros of p consist of either -1 or a collection of pairs of zeros of the form $(w, 1/w)$, $w \neq 0,\ 1,\ -1$. Then we have

$$c_j = c_{n-j} \text{ for } j = 0, 1, \ldots, n.$$

Proof. For $n = 1$ or 2, the result is obvious. Proceeding by induction, suppose the proposition holds for all $n \le m$ where $m \ge 2$. If p is any polynomial of degree $m+1$ that satisfies the hypotheses of the proposition, then either we can write $p(z) = q(z)(z+1)$ where q has degree m, or $p(z) = q(z)(z+w)(z+w^{-1})$ where q has degree $m-1$ and $w \neq 0$, 1. The polynomial q can be written

$$q(z) = \sum_{j=1}^{d} a_j z^j,$$

where d is the degree (either m or $m-1$) of q and the coefficients a_j satisfy the condition $a_j = a_{d-j}$ by the induction hypothesis. Defining $a_j = 0$ for $j < 0$ or for $j > d$, the condition $a_j = a_{d-j}$ holds for all choices of j. In the case that $p(z) = (z+1)q(z)$ and $d = m$, we have

$$c_j = a_j + a_{j-1},$$

which implies that

$$c_{m+1-j} = a_{m+1-j} + a_{m-j} = a_{j-1} + a_j = c_j.$$

In the case that $p(z) = (z+w)(z+w^{-1})q(z)$, we have

$$c_j = a_j + (w + w^{-1})a_{j-1} + a_{j-2},$$

which implies that

$$c_{m+1-j} = a_{m+1-j} + (w + w^{-1})a_{m-j} + a_{m-j-1} = a_{j-2} + (w + w^{-1})a_{j-1} + a_j = c_j.$$

This completes the induction step. $\square$

Lemma 4.2 is the basis for the following result.

THEOREM 4.3. *Suppose that impedances $\gamma_1, \gamma_2, \ldots, \gamma_n$, $n \geq 1$, can be found satisfying* (21) *and* (22)*, where the frequencies ω_j satisfy the following conditions for some $\overline{\omega} > 0$:*

$$\begin{aligned} &\frac{\omega_j + \omega_{n+1-j}}{2} = \overline{\omega}, \qquad 0 < \omega_j < 2\overline{\omega} \quad \text{for } j = 1, 2, \ldots, n, \\ &\omega_i \neq \omega_j \quad \text{for all } i \neq j. \end{aligned} \tag{31}$$

If the stiffness and impedance in each layer satisfy (16)*, then the impedances satisfy the following additional relation:*

$$\gamma_j \gamma_{n+1-j} = \Gamma_+ \Gamma_-, \quad j = 1, 2, \ldots, n, \tag{32}$$

where Γ_+ and Γ_- are the impedances of the half-spaces $x \geq T$ and $x \leq 0$, respectively.

Proof. By (31) the zeros of the polynomial in (21) occur in reciprocal pairs and $z = 1$ is not a zero. By Lemma 4.2, we know that $c_j = c_{n-j}$ for each j. By (23), c_j is expressed as

$$c_j = \sum_{\substack{|\beta|=j \\ \beta_0=1, \beta_{n+1}=0}} -\prod_{k=0}^{n} (\gamma_k + (-1)^{\beta_k + \beta_{k+1}} \gamma_{k+1}). \tag{33}$$

Separating out the factors containing β_0 and β_{n+1} in (33), we obtain

$$c_j = \sum_{|\beta|=j} ((-1)^{\beta_1}\gamma_1 - \Gamma_-)(\gamma_n + (-1)^{\beta_n}\Gamma_+) \prod_{k=1}^{n-1} (\gamma_k + (-1)^{\beta_k + \beta_{k+1}} \gamma_{k+1}). \tag{34}$$

When $j = 0$ or n, the sum in (33) contains precisely one term; that is, if $|\beta| = 0$, then $\beta_k = 0$ for $1 \leq k \leq n$, which implies that

$$c_0 = (\gamma_1 - \Gamma_-)(\gamma_n + \Gamma_+) \prod_{k=1}^{n-1} (\gamma_k + \gamma_{k+1}).$$

If $|\beta| = n$, then $\beta_k = 1$ for $1 \le k \le n$, which implies that

$$c_n = (\Gamma_- + \gamma_1)(\Gamma_+ - \gamma_n) \prod_{k=1}^{n-1} (\gamma_k + \gamma_{k+1}).$$

Equating c_0 and c_n gives

$$(\gamma_1 - \Gamma_-)(\gamma_n + \Gamma_+) = (\Gamma_- + \gamma_1)(\Gamma_+ - \gamma_n),$$

which implies that

$$\gamma_1 \gamma_n = \Gamma_+ \Gamma_-.$$

Proceeding by induction, suppose that

$$\gamma_1 \gamma_n = \gamma_2 \gamma_{n-1} = \cdots = \gamma_j \gamma_{n+1-j} = \Gamma_- \Gamma_+. \tag{35}$$

To complete the induction step, we need to show that $\gamma_{j+1}\gamma_{n-j} = \Gamma_-\Gamma_+$. Replacing j by $n-j$ in (34) gives

$$c_{n-j} = \sum_{|\beta|=n-j} ((-1)^{\beta_1}\gamma_1 - \Gamma_-)(\gamma_n + (-1)^{\beta_n}\Gamma_+) \prod_{k=1}^{n-1} (\gamma_k + (-1)^{\beta_k+\beta_{k+1}}\gamma_{k+1}). \tag{36}$$

The substitution $\beta_k = 1 - \overline{\beta}_k$ converts the binary vector β into a binary vector $\overline{\beta}$ where each 0 and 1 in β is replaced by a 1 and 0 in $\overline{\beta}$, respectively. Hence, $|\overline{\beta}| = n - |\beta|$, and we have

$$c_{n-j} = \sum_{|\overline{\beta}|=j} (\Gamma_- + (-1)^{\overline{\beta}_1}\gamma_1)((-1)^{\overline{\beta}_n}\Gamma_+ - \gamma_n) \prod_{k=1}^{n-1} (\gamma_k + (-1)^{\overline{\beta}_k+\overline{\beta}_{k+1}}\gamma_{k+1}). \tag{37}$$

Replacing the dummy index $\overline{\beta}$ in (37) by β, we obtain

$$c_{n-j} = \sum_{|\beta|=j} (\Gamma_- + (-1)^{\beta_1}\gamma_1)((-1)^{\beta_n}\Gamma_+ - \gamma_n) \prod_{k=1}^{n-1} (\gamma_k + (-1)^{\beta_k+\beta_{k+1}}\gamma_{k+1}). \tag{38}$$

Subtracting (38) from (34) and utilizing the identity $\gamma_1\gamma_n = \Gamma_-\Gamma_+$ already established, we obtain

$$0 = c_j - c_{n-j} = 2(\Gamma_-\Gamma_+) \sum_{|\beta|=j} ((-1)^{\beta_1} - (-1)^{\beta_n}) \prod_{k=1}^{n-1} (\gamma_k + (-1)^{\beta_k+\beta_{k+1}}\gamma_{k+1}). \tag{39}$$

This shows that the following relation holds in the case $m = 1$:

$$\sum_{|\alpha|=j+1-m} ((-1)^{\alpha_m} - (-1)^{\alpha_l}) \prod_{k=m}^{n-m} (\gamma_k + (-1)^{\alpha_k+\alpha_{k+1}}\gamma_{k+1}) = 0, \tag{40}$$

where $l = n+1-m$ and $\alpha = [\alpha_m, \alpha_{m+1}, \ldots, \alpha_l]$ is a binary vector.

Proceeding by induction, suppose that (40) holds for $m = 1, 2, \ldots, M$, where $1 \le M \le j-1$, and define $L = n+1-M$. Observe that $(-1)^{\alpha_L} - (-1)^{\alpha_M} = 0$ if either $\alpha_L = \alpha_M = 0$ or $\alpha_L = \alpha_M = 1$. Hence, when we sum over α in (40), we can restrict the summation to those L such that

$$(\alpha_M = 0, \alpha_L = 1) \text{ or } (\alpha_M = 1, \alpha_L = 0). \tag{41}$$

With these restrictions on α, (40) implies that

$$\sum_{\substack{|\alpha|=j+1-M \\ \alpha_M=1,\alpha_L=0}} \prod_{k=M}^{n-M} (\gamma_k + (-1)^{\alpha_k+\alpha_{k+1}}\gamma_{k+1}) = \sum_{\substack{|\alpha|=j+1-M \\ \alpha_M=0,\alpha_L=1}} \prod_{k=M}^{n-M} (\gamma_k + (-1)^{\alpha_k+\alpha_{k+1}}\gamma_{k+1}). \tag{42}$$

Let $\overline{\alpha}$ be the vector obtained from α by deleting the first and last components. If $|\alpha| = j+1-M$ and (41) holds, then $|\overline{\alpha}| = j - M$. Hence, the left side of (42) can be written

$$\sum_{|\overline{\alpha}|=j-M} (\gamma_M - (-1)^{\overline{\alpha}_{M+1}}\gamma_{M+1})(\gamma_{L-1} + (-1)^{\alpha_{L-1}}\gamma_L) \prod_{k=M+1}^{n-M-1} (\gamma_k + (-1)^{\alpha_k+\alpha_{k+1}}\gamma_{k+1}). \tag{43}$$

Similarly, the right side of (42) takes the form

$$\sum_{|\overline{\alpha}|=j-M} (\gamma_M + (-1)^{\alpha_{M+1}}\gamma_{M+1})(\gamma_{L-1} - (-1)^{\alpha_{L-1}}\gamma_L) \prod_{k=M+1}^{n-M-1} (\gamma_k + (-1)^{\alpha_k+\alpha_{k+1}}\gamma_{k+1}). \tag{44}$$

By the induction assumption (35) and the fact that $M \le j-1$, we have

$$\Gamma_-\Gamma_+ = \gamma_M\gamma_L = \gamma_{M+1}\gamma_{L-1}. \tag{45}$$

Substituting (43) and (44) in (42), utilizing (45), and rearranging, we have

$$0 = 2(\Gamma_-\Gamma_+) \sum_{|\overline{\alpha}|=j-M} ((-1)^{\overline{\alpha}_{M+1}} - (-1)^{\overline{\alpha}_{L-1}}) \prod_{k=M+1}^{n-M-1} (\gamma_k + (-1)^{\overline{\alpha}_k\overline{\alpha}_{k+1}}\gamma_{k+1}). \tag{46}$$

This shows that (40) holds for $m = M+1$ provided $M \le j-1$. This completes the induction step on m, and (40) holds for all m such that $1 \le m \le j$.

Substituting $m = j$ in (40), we obtain the relation

$$\sum_{|\alpha|=1} ((-1)^{\alpha_j} - (-1)^{\alpha_l}) \prod_{k=j}^{n-j} (\gamma_k + (-1)^{\alpha_k+\alpha_{k+1}}\gamma_{k+1}) = 0, \tag{47}$$

where $l = n+1-j$. Once again, (47) vanishes when $a_j = \alpha_l = 0$. But in the case that $\alpha_j = 1$ and $\alpha_l = 0$, or $\alpha_j = 0$ and $\alpha_l = 1$, the remaining components of α must vanish since $|\alpha| = 1$. Hence, after substituting for α_j and α_l in (47), we obtain the relation

$$(\gamma_j - \gamma_{j+1})(\gamma_{n-j} + \gamma_{n-j+1}) \prod_{k=j+1}^{n-j-1} (\gamma_k + \gamma_{k+1}) = (\gamma_j + \gamma_{j+1})(\gamma_{n-j} - \gamma_{n-j+1}) \prod_{k=j+1}^{n-j-1} (\gamma_k + \gamma_{k+1}).$$

This simplifies to

$$\gamma_j\gamma_{n+1-j} = \gamma_{j+1}\gamma_{n-j}. \tag{48}$$

By (35), $\gamma_j\gamma_{n+1-j} = \Gamma_-\Gamma_+$. Consequently, (48) implies that $\gamma_{j+1}\gamma_{n-j} = \Gamma_-\Gamma_+$. This completes the induction step on j, and the proof is complete. $\square$

5. Limiting coating structure for frequency band $[0, \Omega_1]$. In this section, we study the structure of coatings that are constructed to make the reflectivity vanish at a discrete set of frequencies on the interval $[0, \Omega_1]$, where $\Omega_1 = 2\overline{\omega}$ with $\overline{\omega}$ defined in (15). We focus in particular on the limiting behavior as the number of layers in the coating increases. Observe that when $\omega = 0$ in (9), the reflectivity in (8) is

$$\frac{\Gamma_+ - \Gamma_-}{\Gamma_+ + \Gamma_-},$$

which does not vanish except for the special case $\Gamma_+ = \Gamma_-$. Hence, it is generally not possible to absorb waves of all frequencies since the reflectivity is near $(\Gamma_+ - \Gamma_-)/(\Gamma_+ + \Gamma_-)$ when ω is near 0. In addition, for the family of coatings of section 4, we have the following periodicity property.

LEMMA 5.1. *Let $r(\omega)$ denote the reflectivity* (8) *associated with the incident wave frequency ω. For a coating composed of homogeneous layers where for some $\overline{\omega} > 0$ the elastic constant of layer j satisfies* (16) *for each j, we have $r(\omega) = r(\omega + 2\overline{\omega})$ for each choice of ω.*

Proof. When (16) holds, the exponential terms of (11) have the form

$$e_j^{\pm} = \exp\left(\frac{\pi\omega \mathrm{i}}{\overline{\omega}}\right) \pm 1 = \exp\left(\frac{\pi(\omega + 2\overline{\omega})\mathrm{i}}{\overline{\omega}}\right) \pm 1.$$

Since the A_j factors of r in (10) are periodic with period $2\overline{\omega}$, r is periodic with period $2\overline{\omega}$. ◻

As a consequence of Lemma 5.1,

$$r(2j\overline{\omega}) = r(0) = (\Gamma_+ - \Gamma_-)/(\Gamma_+ + \Gamma_-) \tag{49}$$

for any integer j. Hence, even though $r(\omega_k) = 0$ for $k = 1, 2, \ldots, n$, the reflectivity cannot approach zero everywhere, as the number of layers increases, since r is a fixed constant at integer multiples of $2\overline{\omega}$. Nonetheless, using the smoothing technique described later, we will design coatings that absorb waves both in the design frequency band and at larger frequencies.

Let p and q denote the numerator and denominator of r in (10). In other words,

$$p = [\Gamma_- \quad \gamma_1] \prod_{j=1}^{n} A_j \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad q = [\Gamma_- \quad \gamma_1] \prod_{j=1}^{n} A_j \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{50}$$

Both p and q are functions of ω, as can be seen from the formula (11) for A_j, which are denoted $p(\omega)$ and $q(\omega)$, respectively. We begin with a result concerning the denominator q.

LEMMA 5.2. *For any choice of ω, we have*

$$|q(\omega)|^2 \geq 2^{2n+2}\Gamma_+\Gamma_- \prod_{j=1}^{n} \gamma_i^2.$$

Furthermore, if $|r(\omega)|^2 \leq \varepsilon < 1$, then

$$|q(\omega)|^2 \leq \frac{2^{2n+2}\Gamma_+\Gamma_-}{1-\varepsilon} \prod_{j=1}^{n} \gamma_i^2.$$

Proof. From the definition of A_j in (11), we have

$$e^{-\alpha_j} A_j = 2B_j, \qquad \text{where } B_j = \begin{pmatrix} \gamma_j \cos\alpha_j & \mathrm{i}\gamma_{j+1}\sin\alpha_j \\ \mathrm{i}\gamma_j \sin\alpha_j & \gamma_{j+1}\cos\alpha_j \end{pmatrix}, \alpha_j = \frac{\gamma_j \Delta x \omega \mathrm{i}}{\kappa_j}.$$

Hence, we have

$$p = 2^n [\Gamma_- \quad \gamma_1] \prod_{j=1}^{n} (e^{\alpha_j} B_j) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad q = 2^n [\Gamma_- \quad \gamma_1] \prod_{j=1}^{n} (e^{\alpha_j} B_j) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and r can be expressed

$$r = \frac{p}{q} = \frac{[\Gamma_- \quad \gamma_1] \prod_{j=1}^{n} B_j \begin{pmatrix} -1 \\ 1 \end{pmatrix}}{[\Gamma_- \quad \gamma_1] \prod_{j=1}^{n} B_j \begin{pmatrix} 1 \\ 1 \end{pmatrix}}. \tag{51}$$

Observe that for any real numbers a_{ij} and b_{ij}, $i = 1$ or 2, $j = 1$ or 2, we have

$$\begin{bmatrix} a_{11} & \mathrm{i}a_{12} \\ \mathrm{i}a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & \mathrm{i}b_{12} \\ \mathrm{i}b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} - a_{12}b_{21} & \mathrm{i}(a_{11}b_{12} + a_{12}b_{22}) \\ \mathrm{i}(a_{21}b_{11} + a_{22}b_{21}) & a_{22}b_{22} - a_{21}b_{12} \end{bmatrix}.$$

It follows that the matrix product appearing in (51) has the form

$$\prod_{j=1}^{n} B_j = \prod_{j=1}^{n} \begin{pmatrix} \gamma_j \cos\alpha_j & \mathrm{i}\gamma_{j+1}\sin\alpha_j \\ \mathrm{i}\gamma_j \sin\alpha_j & \gamma_{j+1}\cos\alpha_j \end{pmatrix} = \begin{pmatrix} a & \mathrm{i}b \\ \mathrm{i}c & d \end{pmatrix},$$

where a, b, c, and d are all real. Hence.

$$r = \frac{\gamma_1 d - \Gamma_- a + \mathrm{i}(\Gamma_- b - \gamma_1 c)}{\Gamma_- a + \gamma_1 d + \mathrm{i}(\gamma_1 c + \Gamma_- b)}.$$

Consequently, the square magnitude of r can be written

$$|r|^2 = \frac{C - D}{C + D}, \quad \text{where} \tag{52}$$

$$C = (\Gamma_- a)^2 + (\gamma_1 d)^2 + (\gamma_1 c)^2 + (\Gamma_- b)^2 \quad \text{and} \quad D = 2\gamma_1 \Gamma_- (ad + bc).$$

Note that $C - D$ represents the square magnitude of the numerator in (51), while $C + D$ represents the square magnitude of the denominator in (51). Observe that

$$ad + bc = \det \begin{pmatrix} a & \mathrm{i}b \\ \mathrm{i}c & d \end{pmatrix} = \det \prod_{i=1}^{n} \begin{pmatrix} \gamma_j \cos\alpha_j & \mathrm{i}\gamma_{j+1}\sin\alpha_j \\ \mathrm{i}\gamma_j \sin\alpha_j & \gamma_{j+1}\cos\alpha_j \end{pmatrix} = \prod_{j=1}^{n} (\gamma_j \gamma_{j+1}).$$

After substituting in the definition of D, we have

$$D = 2\Gamma_- \Gamma_+ \prod_{j=1}^{n} \gamma_j^2 \geq 0. \tag{53}$$

Since $|r|^2 \geq 0$, we conclude that $C \geq D$, and $C + D \geq 2D$. Since q is 2^n times the denominator of r in (51), we have

$$2^{-2n}|q(\omega)|^2 = C + D \geq 2D = 4\Gamma_-\Gamma_+ \prod_{j=1}^{n} \gamma_j^2.$$

If $|r| \leq \varepsilon$, then after rearranging (52),

$$C \leq \frac{1+\varepsilon}{1-\varepsilon} D.$$

Hence, we have

$$2^{-2n}|q(\omega)|^2 = C + D \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right) D + D = \frac{2D}{1-\varepsilon}.$$

Finally, substituting for D using (53), the proof is complete. ☐

Remark 5.3. Since $C \geq 0$ and $D \geq 0$, (52) implies that $|r| \leq 1$. This unit bound for the reflectivity can be obtained from energy conservation as well (see Lemma 7.3).

In the case where the stiffness of layer j is chosen to satisfy (16), the matrices A_j can be expressed

$$A_j = z \begin{pmatrix} \gamma_j & \gamma_{j+1} \\ \gamma_j & \gamma_{j+1} \end{pmatrix} + \begin{pmatrix} \gamma_j & -\gamma_{j+1} \\ -\gamma_j & \gamma_{j+1} \end{pmatrix}, \quad z = \exp\left(\frac{\pi\omega\mathrm{i}}{\overline{\omega}}\right).$$

Thus there exist polynomials P and Q of degree at most n with the property that

$$p(\omega) = P(z(\omega)) \quad \text{and} \quad q(\omega) = Q(z(\omega)), \quad \text{where} \quad z(\omega) = \exp\left(\frac{\pi\omega\mathrm{i}}{\overline{\omega}}\right). \tag{54}$$

Let us consider the case where the coating is designed so that

$$r(\omega_k) = 0 \quad \text{for } \omega_k = \frac{2k\overline{\omega}}{n+1}, \qquad k = 1, 2, \ldots, n. \tag{55}$$

In this case, the design frequencies lie in the interior of the interval $[0, \Omega_1]$ where $\Omega_1 = 2\overline{\omega}$. It follows that P vanishes at

$$z = \exp\left(\frac{\pi\omega_k\mathrm{i}}{\overline{\omega}}\right) = \exp\left(\frac{2k\pi\mathrm{i}}{n+1}\right), \qquad k = 1, 2, \ldots, n. \tag{56}$$

These zeros are precisely those of the polynomial $1 - z^{n+1}$ with the zero $z = 1$ removed. Therefore, $P(z)$ is a multiple of the polynomial

$$\frac{1 - z^{n+1}}{1 - z}. \tag{57}$$

By the definition of P in (54), we see that $P(z = 1) = p(\omega = 0)$. Putting $\omega = 0$ in (11) and substituting for A_j in (50), we see that

$$P(1) = 2^n(\Gamma_+ - \Gamma_-) \prod_{j=1}^{n} \gamma_j. \tag{58}$$

Since (57) has the limit $n+1$ at $z=1$, we conclude that

$$P(z) = \frac{2^n(\Gamma_+ - \Gamma_-)\prod_{j=1}^n \gamma_j}{n+1}\left(\frac{1-z^{n+1}}{1-z}\right).$$

Combining this representation for P with Lemma 5.2 yields Theorem 5.4.

THEOREM 5.4. *For a coating composed of n homogeneous layers where for some $\overline{\omega} > 0$, the elastic constant of layer j satisfies* (16) *for each j and the reflectivity vanishes in accordance with* (55), *we have for any choice of $\omega \neq 2k\overline{\omega}$, k an integer,*

(59)
$$|r(\omega)|^2 = \frac{|p(\omega)|^2}{|q(\omega)|^2} \leq \frac{(\Gamma_+ - \Gamma_-)^2}{4\Gamma_+\Gamma_-(n+1)^2}\left(\frac{|1-z(\omega)^{n+1}|^2}{|1-z(\omega)|^2}\right), \qquad z(\omega) = \exp\left(\frac{\pi\omega\mathrm{i}}{\overline{\omega}}\right).$$

Moreover, if $|r(\omega)|^2 \leq \varepsilon \leq 1$, then

(60)
$$|r(\omega)|^2 \geq \frac{(1-\varepsilon)(\Gamma_+ - \Gamma_-)^2}{4\Gamma_+\Gamma_-(n+1)^2}\left(\frac{|1-z(\omega)^{n+1}|^2}{|1-z(\omega)|^2}\right).$$

Theorem 5.4 provides very precise information concerning the reflectivity of a coating that absorbs the n frequencies in (55). In particular, we make the following observations:

(O1) The upper bound (59) and the lower bound (60) only differ by the factor $1-\varepsilon$. If the upper bound is $\frac{1}{2}$ (for example), the lower bound is $\frac{1}{2}$ times the upper bound. When the upper bound is small, then the lower bound is essentially equal to the upper bound.

(O2) The reflectivity only vanishes at the n given frequencies ω_k, $k=1,2,\ldots,n$, nowhere else on the interval $(0, 2\overline{\omega}]$.

(O3) Since $|1-z^{n+1}| \leq 2$ when z has unit magnitude, we have

(61)
$$|r(\omega)|^2 \leq \frac{(\Gamma_+ - \Gamma_-)^2}{\Gamma_+\Gamma_-(n+1)^2}\frac{1}{\left|1-\exp\left(\frac{\pi\omega\mathrm{i}}{\overline{\omega}}\right)\right|^2} = \frac{(\Gamma_+ - \Gamma_-)^2}{4\Gamma_+\Gamma_-(n+1)^2\sin^2\left(\frac{\pi\omega}{2\overline{\omega}}\right)}.$$

(O4) Focusing on small ω, note that

$$\frac{1}{\sin^2\theta} = 1 + \cot^2\theta \leq 1 + \frac{1}{\theta^2}, \qquad 0 < \theta \leq \frac{\pi}{2}.$$

Applying this estimate to (61) with $\theta = \pi\omega/(2\overline{\omega})$, we have

(62)
$$|r(\omega)|^2 \leq \frac{(\Gamma_+ - \Gamma_-)^2}{4\Gamma_+\Gamma_-(n+1)^2}\left(1 + \left(\frac{2\overline{\omega}}{\pi\omega}\right)^2\right).$$

Combining (49), (O3), and (O4), we obtain a bound for the reflection magnitude that starts from $|\Gamma_+ - \Gamma_-|/(\Gamma_+ + \Gamma_-)$ at $\omega = 0$ and drops to

(63)
$$\frac{|\Gamma_+ - \Gamma_-|}{2(n+1)\sqrt{\Gamma_+\Gamma_-}}$$

as ω approaches $\overline{\omega}$, where the denominator in (61) is largest.

6. Limiting coating structure for frequency band $[\Omega_0, \Omega_1]$. Suppose that we wish to make the reflectivity small over a given frequency band $[\Omega_0, \Omega_1]$, where $\Omega_0 > 0$, by making the reflectivity vanish at the following evenly spaced frequencies between Ω_0 and Ω_1:

$$\omega_k = \Omega_0 + \left(\frac{k-1}{n-1}\right)(\Omega_1 - \Omega_0), \qquad k = 1, 2, \ldots, n. \tag{64}$$

For this choice of the frequencies, the parameter $\overline{\omega}$ of section 4 is given by $\overline{\omega} = (\Omega_1 + \Omega_0)/2$. The main difference between these coatings and those investigated in section 5 is that it is possible to make the reflectivity approach zero in the design interval $[\Omega_0, \Omega_1]$ exponentially fast in the number of layers, while for the coating of section 5, the reflectivity magnitude approaches zero like $1/n$ (see (61)) in the interior of the frequency band $[0, \Omega_0 + \Omega_1]$. As a result, when $\Omega_0 > 0$, we can achieve small reflectivity over the frequency band using a small number of layers in the coating.

Our method for analyzing the uniformly distributed frequencies (64) on $[\Omega_0, \Omega_1]$ is analogous to the method developed in section 5 to analyze the frequencies (55) on $[0, \Omega_1]$. Again, the reflectivity is expressed $r(\omega) = p(\omega)/q(\omega)$ where p and q are defined in (50) and q has the upper and lower bounds given in Lemma 5.2. However, the numerator p now has a different form. Assuming the stiffness of layer j is chosen to satisfy (16) we express

$$p(\omega) = P(z(\omega)), \quad \text{where } z(\omega) = \exp\left(\frac{\pi\omega\mathrm{i}}{\overline{\omega}}\right),$$

where P is the polynomial of degree n that vanishes at the points $z(\omega_k)$, with ω_k given by (64), and which satisfies the normalization condition (58). In particular, we have

$$P(z) = 2^n(\Gamma_+ - \Gamma_-)\prod_{j=1}^{n}\gamma_j\frac{(z - e^{\mathrm{i}\theta_j})}{(1 - e^{\mathrm{i}\theta_j})}, \qquad \theta_j = \pi\omega_j/\overline{\omega}.$$

Based on the estimates for q in Lemma 5.2, we have Lemma 6.1.

LEMMA 6.1. *For a coating composed of n homogeneous layers where the reflectivity vanishes at each of the frequencies* (64) *and where the elastic constant of layer j satisfies* (16) *for each j with $\overline{\omega} = (\Omega_0 + \Omega_1)/2$, we have for any choice of ω*

$$|r(\omega)|^2 \le \frac{(\Gamma_+ - \Gamma_-)^2}{4\Gamma_+\Gamma_-}\prod_{j=1}^{n}\frac{|e^{\mathrm{i}\theta(\omega)} - e^{\mathrm{i}\theta_j}|^2}{|1 - e^{\mathrm{i}\theta_j}|^2}, \qquad \theta(\omega) = \pi\omega/\overline{\omega}. \tag{65}$$

Moreover, if $|r(\omega)|^2 \le \varepsilon \le 1$, then

$$|r(\omega)|^2 \ge \frac{(1-\varepsilon)(\Gamma_+ - \Gamma_-)^2}{4\Gamma_+\Gamma_-}\prod_{j=1}^{n}\frac{|e^{\mathrm{i}\theta(\omega)} - e^{\mathrm{i}\theta_j}|^2}{|1 - e^{\mathrm{i}\theta_j}|^2}.$$

As ω increases from Ω_0 to Ω_1, $z(\omega)$ travels around the unit circle from the angle θ_1 to the angle θ_n. Based on (65), an upper bound for the reflectivity on the interval $[\Omega_0, \Omega_1]$ can be expressed as a product between a constant (involving Γ_+ and Γ_-) and the quantity

$$\max_{\theta_1 \le \theta \le \theta_n}\prod_{j=1}^{n}\frac{|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta_j}|}{|1 - e^{\mathrm{i}\theta_j}|}. \tag{66}$$

In the numerator of (66), we compute the product of the distances between a given point $e^{i\theta}$ on the unit circle and each of the points $e^{i\theta_j}$ on the unit circle. Since the $e^{i\theta_j}$ are uniformly spaced on the unit circle, the maximum in (66) is attained at a value of θ between θ_1 and θ_2. Hence, we have the following upper bound:

$$\max_{\theta_1 \le \theta \le \theta_n} \prod_{j=1}^{n} |e^{i\theta} - e^{i\theta_j}| \le |e^{i\theta_1} - e^{i\theta_2}| \prod_{j=2}^{n} |e^{i\theta_1} - e^{i\theta_j}|. \tag{67}$$

We consider the following three cases.

Case 1. $\theta_1 \ge 2\pi/3$. Since $\theta_1 \le \theta_j \le \theta_n = 2\pi - \theta_1$ for all j, we have

$$\prod_{j=1}^{n} |1 - e^{i\theta_j}| \ge \prod_{j=1}^{n} |1 - e^{i\theta_1}| = 2^n \sin^n(\theta_1/2), \tag{68}$$

where $\theta_1 = \pi\omega_1/\overline{\omega} = 2\pi\Omega_0/(\Omega_0 + \Omega_1)$. For the factors in (67), we note that the distance between two points on the unit circle is bounded by the angle between them:

$$\begin{gathered} |e^{i\theta_1} - e^{i\theta_j}| \le |\theta_1 - \theta_j| = (j-1)\Delta\theta, \\ \text{where } \Delta\theta = (\theta_n - \theta_1)/(n-1) = 2(\pi - \theta_1)/(n-1) \end{gathered} \tag{69}$$

since $\theta_n = 2\pi - \theta_1$. Combining (69) with (67) and (68) gives

$$\max_{\theta_1 \le \theta \le \theta_n} \prod_{j=1}^{n} \frac{|e^{i\theta} - e^{i\theta_j}|}{|1 - e^{i\theta_j}|} \le \frac{(n-1)!(\Delta\theta)^n}{2^n \sin^n \theta_1/2} = (n-1)! \left(\frac{\pi - \theta_1}{(n-1)\sin\theta_1/2} \right)^n. \tag{70}$$

Recall Stirling's upper bound for a factorial:

$$n! \le \sqrt{2\pi n}(n/e)^n \left(1 + \frac{1}{12n-1} \right).$$

After using this expression to estimate $(n-1)!$ in (70) and after rearranging the result, we obtain

$$\max_{\theta_1 \le \theta \le \theta_n} \prod_{j=1}^{n} \frac{|e^{i\theta} - e^{i\theta_j}|}{|1 - e^{i\theta_j}|} \le e\sqrt{\frac{2\pi}{n-1}} \left(1 + \frac{1}{12n-13} \right) \left(\frac{\pi - \theta_1}{e\sin\theta_1/2} \right)^n.$$

Noting that

$$\frac{\pi - \theta_1}{e\sin\theta_1/2} \le \frac{\pi/3}{e\sqrt{3}/2} = \frac{2\pi}{3\sqrt{3}e} \le .44484 \quad \text{when } \pi \ge \theta_1 \ge 2\pi/3,$$

we conclude that the maximum reflectivity magnitude for ω on the interval $[\Omega_0, \Omega_1]$ approaches zero at least as fast as $.44484^n$ when

$$\theta_1 = \frac{2\pi\Omega_0}{\Omega_0 + \Omega_1} \ge 2\pi/3 \quad \text{or, equivalently,} \quad \Omega_1 \le 2\Omega_0.$$

Moreover, as θ_1 approaches π, we have

$$\lim_{\theta_1 \to \pi} \frac{\pi - \theta_1}{e\sin\theta_1/2} = 0.$$

Hence, the reflectivity magnitude on $[\Omega_0, \Omega_1]$ approaches zero like ε^n where ε tends to zero as Ω_0 approaches Ω_1.

Case 2. $\pi/2 \le \theta_1 < 2\pi/3$. The factors in the product (67) are the distance between the points $e^{\mathrm{i}\theta_1}$ and $e^{\mathrm{i}\theta_j}$ on the unit circle, while the factors in the denominator of (66) are the distance between the point $1+0\mathrm{i}$ in the complex plane and each of the points $e^{\mathrm{i}\theta_j}$ on the unit circle. Thinking geometrically, we can pair together a factor in the denominator of (66) with a slightly smaller factor in the product (67) in such a way that the ratio of these paired terms is less than 1. In particular, if k is the first integer $\le \theta_1/\Delta\theta$ (often denoted $\lfloor \theta_1/\Delta\theta \rfloor$), then we have

$$|e^{\mathrm{i}\theta_{k+1}} - e^{\mathrm{i}\theta_1}| = |e^{\mathrm{i}(\theta_1 + k\Delta\theta)} - e^{\mathrm{i}\theta_1}| = |e^{\mathrm{i}k\Delta\theta} - 1| \le |1 - e^{\mathrm{i}\theta_1}|.$$

In general, if $j > 0$ with $j + k \le n$, then

$$\begin{aligned} |e^{\mathrm{i}\theta_{j+k}} - e^{\mathrm{i}\theta_1}| &= |e^{\mathrm{i}[\theta_1 + (j+k-1)\Delta\theta]} - e^{\mathrm{i}\theta_1}| = |e^{\mathrm{i}(j+k-1)\Delta\theta} - 1| \\ &\le |e^{\mathrm{i}[\theta_1 + (j-1)\Delta\theta]} - 1| = |1 - e^{\mathrm{i}\theta_j}|. \end{aligned} \tag{71}$$

Hence, the ratio $|e^{\mathrm{i}\theta_{j+k}} - e^{\mathrm{i}\theta_1}|/|1 - e^{\mathrm{i}\theta_j}|$ is at most one for $j \ge 1$. Combining (66) and (67), and removing these paired factors, we obtain the following upper bound:

$$\max_{\theta_1 \le \theta \le \theta_n} \prod_{j=1}^{n} \frac{|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta_j}|}{|1 - e^{\mathrm{i}\theta_j}|} \le \frac{|e^{\mathrm{i}\theta_2} - e^{\mathrm{i}\theta_1}| \prod_{j=2}^{k} |e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_j}|}{\prod_{j=1}^{k} |1 - e^{\mathrm{i}\theta_j}|}. \tag{72}$$

(To obtain this bound, we make use of the relation $\theta_{n+1-j} = 2\pi - \theta_j$ to simplify the denominator.) Estimating the denominator as in (68) and the numerator as in (69), we have the following analogue of (70):

$$\max_{\theta_1 \le \theta \le \theta_n} \prod_{j=1}^{n} \frac{|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta_j}|}{|1 - e^{\mathrm{i}\theta_j}|} \le \frac{(k-1)!(\Delta\theta)^k}{2^k \sin^k \theta_1/2} = \frac{k!}{k} \left(\frac{\pi - \theta_1}{(n-1)\sin\theta_1/2} \right)^k. \tag{73}$$

Again, utilizing Stirling's bound yields

$$\max_{\theta_1 \le \theta \le \theta_n} \prod_{j=1}^{n} \frac{|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta_j}|}{|1 - e^{\mathrm{i}\theta_j}|} \le \sqrt{\frac{2\pi}{k}} \left(1 + \frac{1}{12n-1}\right) \left(\frac{k(\pi - \theta_1)}{(n-1)e\sin\theta_1/2} \right)^k. \tag{74}$$

Since

$$k \le \frac{\theta_1}{\Delta\theta} = \frac{\theta_1(n-1)}{\theta_n - \theta_1} = \frac{\theta_1(n-1)}{2(\pi - \theta_1)}, \tag{75}$$

(74) implies that

$$\max_{\theta_1 \le \theta \le \theta_n} \prod_{j=1}^{n} \frac{|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta_j}|}{|1 - e^{\mathrm{i}\theta_j}|} \le \sqrt{\frac{2\pi}{k}} \left(1 + \frac{1}{12n-1}\right) \left(\frac{\theta_1}{2e\sin\theta_1/2} \right)^k. \tag{76}$$

Since k differs from its upper bound (75) by at most 1, we conclude that the maximum in (76) approaches zero like $\alpha(\theta_1)^n$, where

$$\alpha(\theta_1) = \left(\frac{\theta_1}{2e\sin\theta_1/2} \right)^{\theta_1/2(\pi - \theta_1)}. \tag{77}$$

For θ_1 between $2\pi/3$ and $\pi/2$, the maximum value for $\alpha(\theta_1)$ is attained at $\theta_1 = \pi/2$ and

$$\alpha(\pi/2) = \left(\frac{\pi}{2e\sqrt{2}}\right)^{1/2} < .63923.$$

Case 3. $0 < \theta_1 < \pi/2$. This case can be treated using the strategy of Case 2. The only difference is that (71) does not hold when $\theta_j > \pi$. On the other hand, examining the geometry of points on the unit circle, we see that when $\theta_j > \pi$, the following holds:

$$|e^{\mathrm{i}\theta_{j+k+1}} - e^{\mathrm{i}\theta_1}| \leq |1 - e^{\mathrm{i}\theta_j}|.$$

This leads to the following adjustment in (72):

$$\max_{\theta_1 \leq \theta \leq \theta_n} \prod_{j=1}^{n} \frac{|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta_j}|}{1 - e^{\mathrm{i}\theta_j}|} \leq \frac{2|e^{\mathrm{i}\theta_2} - e^{\mathrm{i}\theta_1}| \prod_{j=2}^{k} |e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_j}|}{\prod_{j=1}^{k+1} |1 - e^{\mathrm{i}\theta_j}|}.$$

The 2 in the numerator corresponds to the uncanceled numerator factor for which $\theta_j - \theta_1 \approx \pi$, while k in the denominator of (72) is changed to $k+1$ since one fewer pair of factors is removed. Continuing the analysis as in Case 2, using the estimate $|1 - e^{\mathrm{i}\theta_j}| \geq 2 \sin \theta_1/2$ in the denominator and the estimate $|e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_j}| \leq (j-1)\Delta\theta$ in the numerator, yields

$$\max_{\theta_1 \leq \theta \leq \theta_n} \prod_{j=1}^{n} \frac{|e^{\mathrm{i}\theta} - e^{\mathrm{i}\theta_j}|}{|1 - e^{\mathrm{i}\theta_j}|} \leq \frac{2(k-1)!(\Delta\theta)^k}{2^{k+1}\sin^{k+1}\theta_1/2} = \left(\frac{k!}{k \sin \theta_1/2}\right)\left(\frac{\pi - \theta_1}{(n-1)\sin \theta_1/2}\right)^k.$$

Since the right side above only differs from the right side of (73) by the factor $\sin \theta_1/2$ in the denominator, we can simply divide the upper bound (76) by $\sin \theta_1/2$ to obtain the corresponding estimate for Case 3.

The quantity that is exponentiated in (76) satisfies

$$\frac{\theta_1}{2e \sin \theta_1/2} \leq \frac{\pi/2}{2e \sin \pi/4} < .40862 \quad \text{for } 0 < \theta_1 \leq \frac{\pi}{2}.$$

Hence, reflectivity is approaching zero exponentially fast on the frequency band $[\Omega_0, \Omega_1]$, but the decay factor $\alpha(\theta_1)$, defined in (77), approaches one as θ_1 approaches zero, and although the convergence is exponentially fast, the actual convergence could be slow when θ_1 is near zero. Note that letting θ_1 tend to zero is analogous to letting Ω_0 tend to zero with Ω_1 fixed. The case $\Omega_0 = 0$, analyzed in section 5, gave convergence of the form $1/n$, rather than exponential convergence. In summary, our results on exponential decay of the reflectivity are shown in Theorem 6.2.

THEOREM 6.2. *For a coating composed of n homogeneous layers where the reflectivity vanishes at each of the frequencies* (64) *on the interval* $[\Omega_0, \Omega_1]$, $\Omega_0 > 0$, *and where the elastic constant of layer j satisfies* (16) *for each j with* $\overline{\omega} = (\Omega_0 + \Omega_1)/2$, *the maximum magnitude of the reflectivity on the interval* $[\Omega_0, \Omega_1]$, *denoted* $r_{\max}$, *has the following upper bounds, depending on the ratio of* Ω_1 *to* Ω_0, *on* $\theta_1 = 2\pi\Omega_0/(\Omega_0 + \Omega_1)$, *and on* $k = \lfloor \theta_1(n-1)/2(\pi-\theta_1) \rfloor$*:*

$$\Omega_0 < \Omega_1 \le 2\Omega_0: \quad r_{\max} \le e\sqrt{\frac{\pi|\Gamma_+ - \Gamma_-|^2}{2\Gamma_+\Gamma_-(n-1)}}\left(1 + \frac{1}{12n-13}\right)\left(\frac{\pi-\theta_1}{e\sin\theta_1/2}\right)^n,$$

$$2\Omega_0 < \Omega_1 \le 3\Omega_0: \quad r_{\max} \le \sqrt{\frac{\pi|\Gamma_+ - \Gamma_-|^2}{2k\Gamma_+\Gamma_-}}\left(1 + \frac{1}{12n-1}\right)\left(\frac{\theta_1}{2e\sin\theta_1/2}\right)^k,$$

$$3\Omega_0 < \Omega_1: \quad r_{\max} \le \frac{1}{\sin\theta_1/2}\sqrt{\frac{\pi|\Gamma_+ - \Gamma_-|^2}{2k\Gamma_+\Gamma_-}}\left(1 + \frac{1}{12n-1}\right)\left(\frac{\theta_1}{2e\sin\theta_1/2}\right)^k.$$

7. Transmission in multilayered coatings. In this section, we consider the minimization of transmission. This minimization must be approached in a completely different way from the reflection problem since transmissivity does not vanish. That is, for the single homogeneous layer of section 2, $\tau = 0$ implies by (3) that $\tau_- + \tau_+ = 0$ and $\tau_- - \tau_+ = 0$. Hence, $\tau_- = \tau_+ = 0$. But by (3) this also forces $1 + r = 0$ and $1 - r = 0$. Since this is impossible, the transmission coefficient can never vanish. Our approach to the minimum reflection problem was to make the reflection coefficient vanish at a discrete set of frequencies. This same approach does not work with the transmission problem since transmissivity never vanishes. Nonetheless, the transmission coefficient can be made arbitrarily small. We begin by deriving a formula for the transmissivity analogous to (10).

PROPOSITION 7.1. *Suppose that the coating* $0 \le x \le T$ *is composed of* n *homogeneous layers, each layer of thickness* $\Delta x = T/n$*. If* κ_j *and* γ_j *are the stiffness and impedance of the* j*th layer* $(j-1)\Delta x \le x \le j\Delta x$*, then the transmissivity of the coating can be expressed as*

$$\tau = \frac{2\Gamma_+ \prod_{j=1}^n \gamma_j}{[\Gamma_- \quad \gamma_1]\prod_{j=1}^n B_j \begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \tag{78}$$

where

$$B_j = \begin{pmatrix} \gamma_j \cos\alpha_j & \mathrm{i}\gamma_{j+1}\sin\alpha_j \\ \mathrm{i}\gamma_j \sin\alpha_j & \gamma_{j+1}\cos\alpha_j \end{pmatrix}, \qquad \alpha_j = \frac{\gamma_j \Delta x \omega}{\kappa_j}.$$

Proof. The general solution of the equation of motion (2), assuming harmonic time dependence, is $v(x,t) = u(x)e^{\mathrm{i}\omega t}$ where u in layer j can be expressed

$$u(x) = \tau_j^- e^{\mathrm{i}\omega s_j(x-x_{j-1})} + \tau_j^+ e^{-\mathrm{i}\omega s_j(x-x_{j-1})}, \quad s_j = \gamma_j/\kappa_j, \quad \text{and } x_j = j\Delta x.$$

Since the displacement and stress are continuous at the interfaces $x = x_j$ for each j, we have

$$\begin{aligned} \tau_{j+1}^- + \tau_{j+1}^+ &= \tau_j^- e^{\mathrm{i}\omega s_j \Delta x} + \tau_j^+ e^{-\mathrm{i}\omega s_j\Delta x}, \\ \gamma_{j+1}(\tau_{j+1}^- - \tau_{j+1}^+) &= \gamma_j(\tau_j^- e^{\mathrm{i}\omega s_j\Delta x} - \tau_j^+ e^{-\mathrm{i}\omega s_j\Delta x}). \end{aligned} \tag{79}$$

Letting τ_j denote the vector $[\tau_j^+ \quad \tau_j^-]^T$, we solve the linear system (79) to obtain

$$\tau_j = C_j \tau_{j+1}, \quad \text{where} \quad C_j = \frac{1}{2\gamma_j}\begin{pmatrix} (\gamma_j + \gamma_{j+1})e^{\mathrm{i}\alpha_j} & (\gamma_j - \gamma_{j+1})e^{\mathrm{i}\alpha_j} \\ (\gamma_j - \gamma_{j+1})e^{-\mathrm{i}\alpha_j} & (\gamma_j + \gamma_{j+1})e^{-\mathrm{i}\alpha_j} \end{pmatrix}.$$

By (3), the transmission coefficient τ is given by $\tau = \tau_1^- + \tau_1^+$, while to the right of $x = T$ we have $\tau_{n+1}^- = 1$ and $\tau_{n+1}^+ = r$, where r is the reflection coefficient given in (10). Hence, we have

$$\tau = [1 \ \ 1]\tau_1 = [1 \ \ 1] \prod_{j=1}^{n} C_j \begin{pmatrix} r \\ 1 \end{pmatrix}.$$

Letting p and q denote the numerator and denominator of r defined in (50),

$$\tau = \frac{1}{q}[1 \ \ 1] \prod_{j=1}^{n} C_j \begin{pmatrix} p \\ q \end{pmatrix}. \tag{80}$$

Since p and q are scalars,

$$p = [\Gamma_- \ \ \gamma_1] \prod_{j=1}^{n} A_j \begin{pmatrix} -1 \\ 1 \end{pmatrix} = [-1 \ \ 1] \prod_{j=1}^{n} A_{n-j+1}^T \begin{pmatrix} \Gamma_- \\ \gamma_1 \end{pmatrix},$$

and

$$q = [\Gamma_- \ \ \gamma_1] \prod_{j=1}^{n} A_j \begin{pmatrix} 1 \\ 1 \end{pmatrix} = [1 \ \ 1] \prod_{j=1}^{n} A_{n-j+1}^T \begin{pmatrix} \Gamma_- \\ \gamma_1 \end{pmatrix}. \tag{81}$$

It follows that the numerator of τ can be written

$$[1 \ \ 1] \prod_{j=1}^{n} C_j \begin{pmatrix} p \\ q \end{pmatrix} = [1 \ \ 1] C_1 C_2 \cdots C_n \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} A_n^T \cdots A_2^T A_1^T \begin{pmatrix} \Gamma_- \\ \gamma_1 \end{pmatrix}.$$

Observe that

$$\begin{aligned} C_j \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} A_j^T &= \frac{1}{2\gamma_j} \begin{pmatrix} (\gamma_j + \gamma_{j+1}) e^{\mathrm{i}\alpha_j} & (\gamma_j - \gamma_{j+1}) e^{\mathrm{i}\alpha_j} \\ (\gamma_j - \gamma_{j+1}) e^{-\mathrm{i}\alpha_j} & (\gamma_j + \gamma_{j+1}) e^{-\mathrm{i}\alpha_j} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &\quad \begin{pmatrix} \gamma_j (e^{2\mathrm{i}\alpha_j} + 1) & \gamma_j (e^{2\mathrm{i}\alpha_j} - 1) \\ (\gamma_{j+1} (e^{2\mathrm{i}\alpha_j} - 1) & \gamma_{j+1} (e^{2\mathrm{i}\alpha_j} + 1) \end{pmatrix} \\ &= \tfrac{1}{\gamma_j} \begin{pmatrix} -\gamma_{j+1} e^{\mathrm{i}\alpha_j} & \gamma_j e^{\mathrm{i}\alpha_j} \\ \gamma_{j+1} e^{-\mathrm{i}\alpha_j} & \gamma_j e^{-\mathrm{i}\alpha_j} \end{pmatrix} \begin{pmatrix} \gamma_j (e^{2\mathrm{i}\alpha_j} + 1) & \gamma_j (e^{2\mathrm{i}\alpha_j} - 1) \\ \gamma_{j+1} (e^{2\mathrm{i}\alpha_j} - 1) & \gamma_{j+1} (e^{2\mathrm{i}\alpha_j} + 1) \end{pmatrix} \\ &= 2\gamma_{j+1} e^{\mathrm{i}\alpha_j} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

Hence, the numerator of τ reduces to the following:

$$[1 \ \ 1] \prod_{j=1}^{n} C_j \begin{pmatrix} p \\ q \end{pmatrix} = 2^n [1 \ \ 1] \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_- \\ \gamma_1 \end{pmatrix} \prod_{j=1}^{n} \gamma_{j+1} e^{\mathrm{i}\alpha_j} = 2^{n+1} \Gamma_+ \prod_{j=1}^{n} \gamma_j e^{\mathrm{i}\alpha_j}. \tag{82}$$

Combining (80), (81), and (82), along with the observation that $e^{-\mathrm{i}\alpha_j} A_j = 2B_j$, the proof is complete. $\square$

Remark 7.2. From the form of the transmissivity in Proposition 7.1, it is clear that it never vanishes.

We now give the relation between reflectivity and transmissivity.

LEMMA 7.3. *For a coating composed of homogeneous layers, transmissivity and reflectivity satisfy the following relation:*

$$|r|^2 + \frac{\Gamma_-}{\Gamma_+}|\tau|^2 = 1. \tag{83}$$

Minimizing reflectivity is equivalent to maximizing transmissivity, and minimizing transmissivity is equivalent to maximizing reflectivity. Moreover, the maximum magnitude for the transmissivity is $\sqrt{\Gamma_+/\Gamma_-}$*, and the maximum is attained if and only if the reflectivity vanishes. For any coating, the magnitude of the reflectivity is strictly less than* 1.

Proof. Using the notation from the proof of Lemma 5.2, we have

$$|r|^2 = \frac{C-D}{C+D}. \tag{84}$$

Also referring to the notation of Lemma 5.2 and to the expression for τ in (78), we have

$$|\tau|^2 = \frac{2D(\Gamma_+/\Gamma_-)}{C+D}. \tag{85}$$

Combining (84) and (85) yields (83). The remaining part of the lemma follows from (83) and the fact that $|\tau| > 0$. $\square$

Remark 7.4. An alternative derivation of (83) is obtained from the principle of energy conservation, and with this approach it is not necessary to consider a coating composed of homogeneous layers. Alternatively, Lemma 7.3 can be extended to a coating whose impedance and stiffness are piecewise continuous by approximating the material parameters by a sequence of materials composed of homogeneous layers and taking limits.

Although the transmissivity never vanishes, it can be made arbitrarily small using a coating whose impedance oscillates between large and small values.

THEOREM 7.5. *Given a frequency band* $[\Omega_0, \Omega_1]$ *where* $\Omega_0 > 0$*, and given a (small) scalar* $\varepsilon > 0$*, let us consider a coating composed of* n *homogeneous layers where*

$$\gamma_j = \varepsilon \text{ for } j \text{ even}, \quad \gamma_j = 1/\varepsilon \text{ for } j \text{ odd}, \quad \text{and } \kappa_j = \frac{(\Omega_0+\Omega_1)\gamma_j \Delta x}{\pi}. \tag{86}$$

Then for all $\omega \in [\Omega_0, \Omega_1]$ *we have*

$$|\tau| = \begin{cases} 2\Gamma_+\varepsilon\left(\dfrac{\varepsilon}{\sin\alpha(\omega)}\right)^n + \mathrm{O}(\varepsilon^{n+2}) & \text{for } n \text{ even}, \\[2ex] 2\Gamma_+\left(\dfrac{\varepsilon}{\sin\alpha(\omega)}\right)^n + \mathrm{O}(\varepsilon^{n+1}) & \text{for } n \text{ odd}, \end{cases} \tag{87}$$

where $\alpha(\omega) = \pi\omega/(\Omega_0+\Omega_1)$*. For all* $\omega \in [\Omega_0, \Omega_1]$*,* $\sin\alpha(\omega) \ge \sin \pi\Omega_0/(\Omega_0+\Omega_1)$*.*

Proof. For γ_j chosen according to (86), the numerator of τ in (78) is

$$2\Gamma_+ \prod_{j=1}^{n} \gamma_j = \begin{cases} 2\Gamma_+ & \text{if } n \text{ is even,} \\ 2\Gamma_+/\varepsilon & \text{if } n \text{ is odd.} \end{cases} \tag{88}$$

The denominator of τ is

$$\begin{aligned} & [\Gamma_- \quad \gamma_1] \prod_{j=1}^{n} B_j \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ = \ & \varepsilon^{-(n+1)} [\varepsilon\Gamma_- \quad 1] \begin{pmatrix} \cos\alpha(\omega) & \mathrm{i}\varepsilon^2 \sin\alpha(\omega) \\ \mathrm{i}\sin\alpha(\omega) & \varepsilon^2 \cos\alpha(\omega) \end{pmatrix} \begin{pmatrix} \varepsilon^2 \cos\alpha(\omega) & \mathrm{i}\sin\alpha(\omega) \\ \mathrm{i}\varepsilon^2 \sin\alpha(\omega) & \cos\alpha(\omega) \end{pmatrix} \\ & \times \begin{pmatrix} \cos\alpha(\omega) & \mathrm{i}\varepsilon^2 \sin\alpha(\omega) \\ \mathrm{i}\sin\alpha(\omega) & \varepsilon^2 \cos\alpha(\omega) \end{pmatrix} \cdots \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \tag{89}$$

The factor multiplying $\varepsilon^{-(n+1)}$ in (89) is a polynomial of degree $n+1$ in ε whose lowest degree (constant) term can be evaluated by putting $\varepsilon = 0$. In particular, the term of lowest degree is

$$[0 \quad 1] \begin{pmatrix} \cos\alpha(\omega) & 0 \\ \mathrm{i}\sin\alpha(\omega) & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i}\sin\alpha(\omega) \\ 0 & \cos\alpha(\omega) \end{pmatrix} \begin{pmatrix} \cos\alpha(\omega) & 0 \\ \mathrm{i}\sin\alpha(\omega) & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathrm{i}^n \sin^n \alpha(\omega). \tag{90}$$

Combining (88), (89), and (90) gives (87). For ω between Ω_0 and Ω_1, $\alpha(\omega)$ lies between 0 and π. Since $\alpha(\Omega_0) = \pi - \alpha(\Omega_1)$, it follows that

$$\sin\alpha(\omega) \geq \sin\alpha(\Omega_0) = \sin \pi\Omega_0/(\Omega_0 + \Omega_1)$$

for $\Omega_0 \leq \omega \leq \Omega_1$. This completes the proof. ◻

8. Numerical illustrations. Now we provide numerical illustrations for the theorems of the previous sections. In the first set of illustrations, we consider the nonreflective coatings discussed in section 5 using $[0,1]$ for the frequency band. According to (55), an n-layer coating should be designed such that the reflectivities vanish at

$$\omega_k = k/(n+1) \text{ for } k = 1, 2, \ldots, n.$$

For the impedances of the two half-spaces, we take $\Gamma_- = 28.14776$ and $\Gamma_+ = 1$. The ration Γ_-/Γ_+ corresponds to the ratio between the impedances of steel and water. In the absence of a coating, the reflectivity of a water/steel interface is $(\Gamma_- - \Gamma_+)/(\Gamma_- + \Gamma_+) = 0.93138$. To obtain a sense of the effectiveness of our optimal coatings, we should compare this value with that of the coated surface.

We use Newton's method with the Armijo step rule described in [18, p. 179] to solve the system of equations

$$r(\omega_k) = 0, \qquad k = 1, 2, \ldots, n,$$

with n as large as 2000. A brute-force approach to solving this system is not practical. It has infinitely many solutions for each n, and without additional guidance, a solution obtained for one n may not be related to a solution corresponding to another n. To

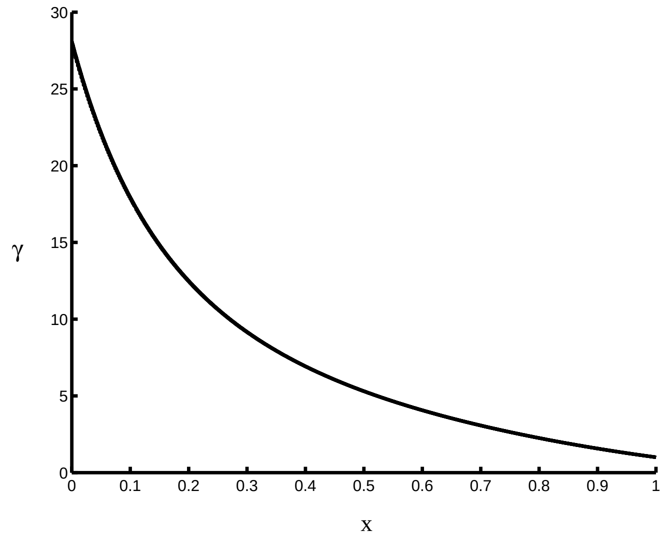


Fig. 1. *Impedance as a function of depth for* 500-*,* 1000-*, and* 2000-*layer coatings designed to absorb the frequency band* $[0, 1]$.

guide our computations, we take the stiffness of the special form (16), and we constrain the impedance to satisfying the relation (32) of Theorem 4.3.

The computed impedances of nonreflective coatings for $n = 500$, 1000, and 2000 layers are shown in Figure 1. The reason that the figure shows only one curve is that the coating impedances for the three choices did not differ by much more than the width of a line on a 600 dpi printer, indicating that further refinements of the layer structure is not necessary. Although the curve appears to be continuous, it is actually piecewise constant with between 500 and 2000 fine steps.

Figure 2 depicts the frequency response of the 500-layer coating. By construction, the response vanishes at exactly 500 points in the frequency interval $(0, 1)$. Between two consecutive zeros, the graph has a local maximum which is bounded above and below by the estimates in (59) and (60). This fine structure for the graph is not visible in Figure 2 since we have squeezed the 500 zeros on the horizontal axis. What we see in Figure 2 is the envelope of the local maxima. In [21, Figs. 7.3, 7.5] two- and three-layer coatings are shown which absorb waves of two and three frequencies and which have a saddle-like shape similar to that of our Figure 2.

Observe that the reflectivity near $\omega = 0$ is large, its value being 0.93138. As the frequency approaches zero, the wavelength grows large, and relative to the scale of the wavelength, the thickness of the coating is insignificant. In the limit as ω tends to zero, the coating loses its effect entirely, and the reflectivity approaches that of the uncoated interface. Also observe that the reflectivity at $\omega = 1$ is large as well. In fact, by Lemma 5.1 the frequency response is periodic with period 1. Hence, the peak value 0.93138 of the reflectivity must recur at frequencies $\omega = 1, 2, 3, \ldots$. The full frequency response for the 500 layer coating is obtained by periodically extending the graph of Figure 2 to $[0, \infty)$.

In Figure 3 we plot the frequency responses for the 500-, 1000-, and 2000-layer coatings together for comparison, magnifying the region near the origin. In accordance with our optimal design procedure, between two consecutive zeros of the frequency

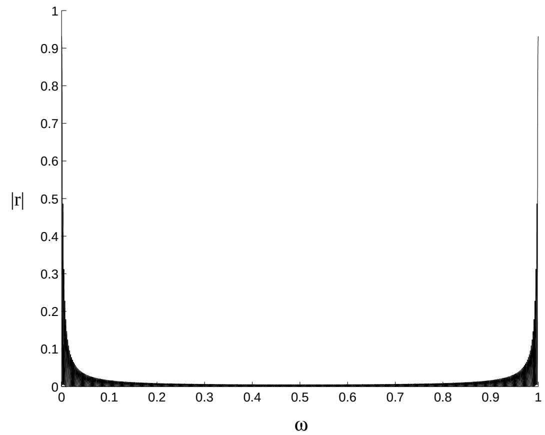


FIG. 2. *Frequency response (reflectivity versus frequency) on* $[0, 1]$ *for* 500-*layer coating.*

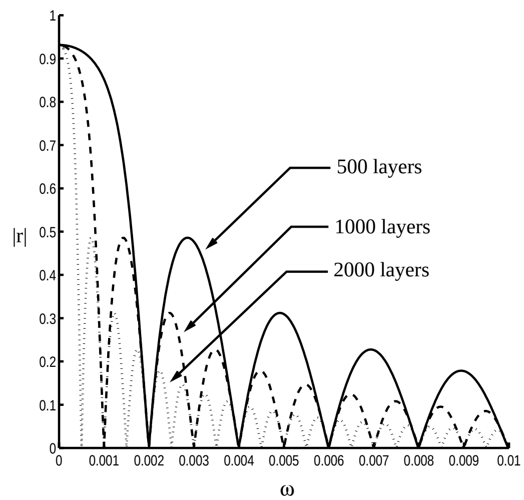


FIG. 3. *Frequency response near zero for* 500-*,* 1000-*, and* 2000-*layer coatings.*

response corresponding to the 500-layer coating, there is one zero of the frequency response for the 1000-layer coating and three zeros of the frequency response for the 2000-layer coating. At any fixed value of ω, the reflectivity is bounded by a constant over n by (61), and the envelopes of the peaks fall off like $1/\omega^2$ in accordance with (62). Figure 4 shows the same frequency response graphs in a region near $\omega = 0.5$. The decay in the peaks of the graphs proportional to $1/n$ is quite apparent.

The three coatings discussed so far are designed strictly according to the procedures described in the previous sections of this paper; each coating consists of a number of homogeneous layers sandwiched together. Because of the large number of layers, the impedance of the assembly is very nearly continuous throughout the coating, as seen in Figure 1. When this nearly continuous impedance function is replaced by a spline interpolant, we find that this almost imperceptible change drastically alters the coating's frequency response at larger frequencies. Figure 5 shows the frequency response on the interval $[0, 2]$ of the coating obtained by a linear spline interpolation of the 500-layer coating impedance. Comparing this with the frequency

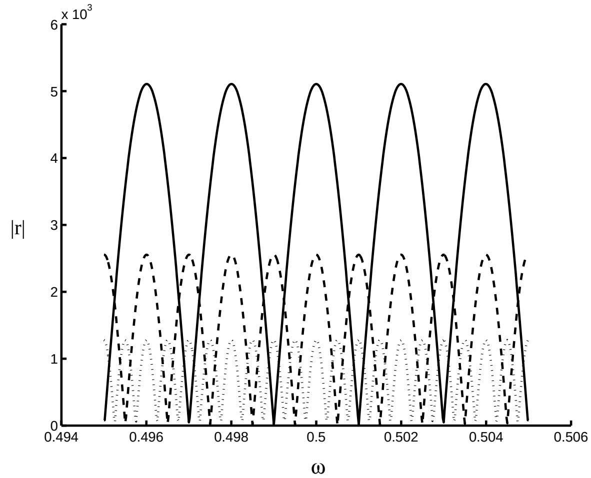


FIG. 4. *Frequency response near* .5 *for* 500-*,* 1000-*, and* 2000-*layer coatings.*

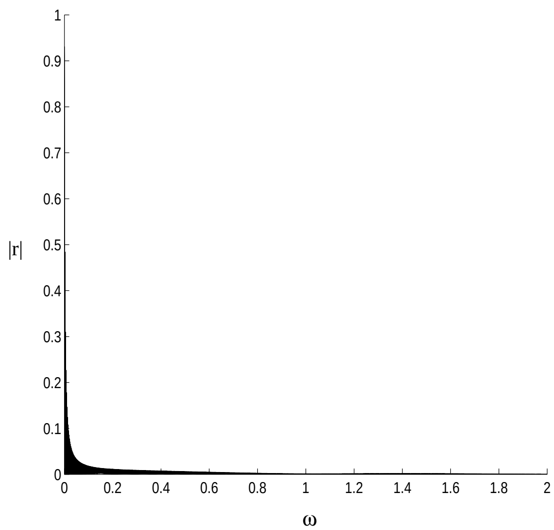


FIG. 5. *Frequency response on* $[0, 2]$ *for* 500-*layer coating and linear spline smoothing.*

response of the original layered coating shown in Figure 2, we see that the peak near $\omega = 1$ has disappeared. Since the frequency response values near $\omega = 0$ dwarf the values for larger ω, Figure 6 restricts the graph to the interval $[1, 2]$. Observe that the reflectivity oscillates between values near 0 and near 2×10^{-3} on this interval. In Figure 7, we expand the frequency response plot to the band $[1, 10]$.

For comparison, Figure 8 gives the frequency response on the interval $[1, 10]$ for a coating obtained by cubic spline interpolation of the 500-layer impedance. Observe that there are fewer wiggles in the profile of the frequency response, but the overall effect remains the same: Unlike the layered optimal coating for which the frequency response diagram is periodic and peak reflectivities of 0.93138 recur at all integer-valued frequencies, the smooth coating's frequency response remains small (and decreasing) outside the design interval.

For the layered coatings, the smallest frequency response occurs near $\omega = .5$, and in Figure 4 we see that its maximum value for the 500-layer coating is around

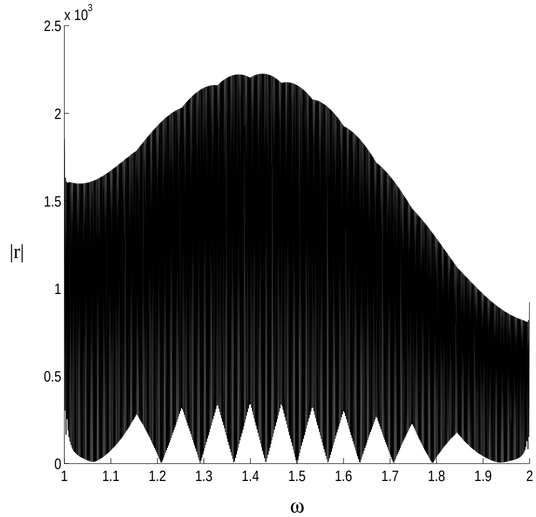


FIG. 6. *Frequency response on* $[1, 2]$ *for* 500*-layer coating and linear spline smoothing.*

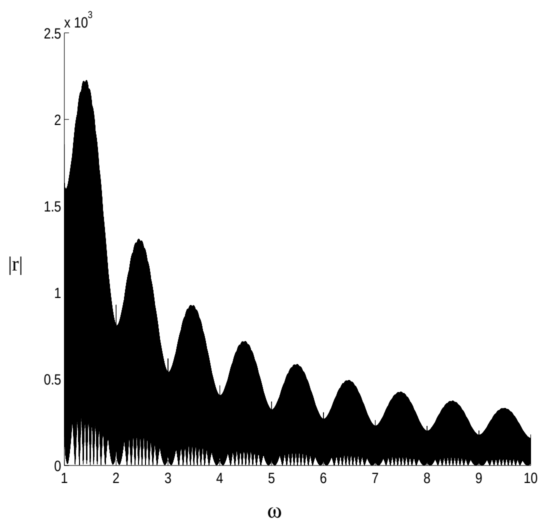


FIG. 7. *Frequency response on* $[1, 10]$ *for* 500*-layer coating and linear spline smoothing.*

5×10^{-3}. In Figures 7 and 8, we see that the maximum reflectivity of the smooth coatings for $\omega > 1$ is less than the best maximal values for the layered coating. The implications are significant for practical applications: An optimally designed smooth coating eliminates reflections not only within the design frequency band but also at all higher frequencies. The same effect described for the 500-layer coating occurs when smoothing the 1000- and 2000-layer coating designs.

For obtaining the graphs in this paper, the spline interpolants pass through the midpoints of the vertical impedance jumps in each layer, and at the end points the splines match the impedances of the adjoining half-spaces. For cubic spline interpolation, we employed de Boor's routines cubspl and ppvalu, obtained through Netlib [16] and described in [15]. The final two degrees of freedom in the cubic spline inter-

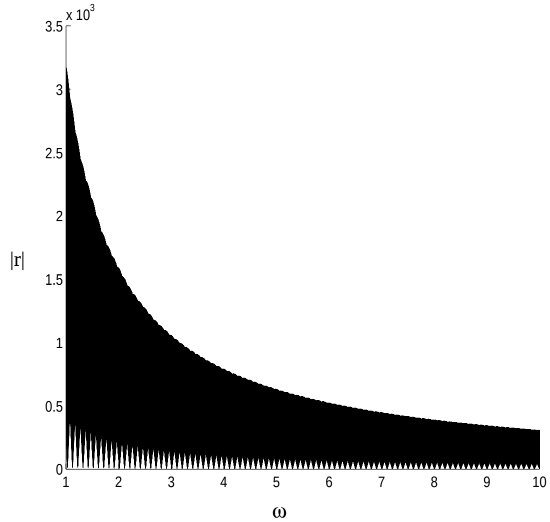


FIG. 8. *Frequency response on* $[1, 10]$ *for* 500*-layer coating and cubic spline smoothing.*

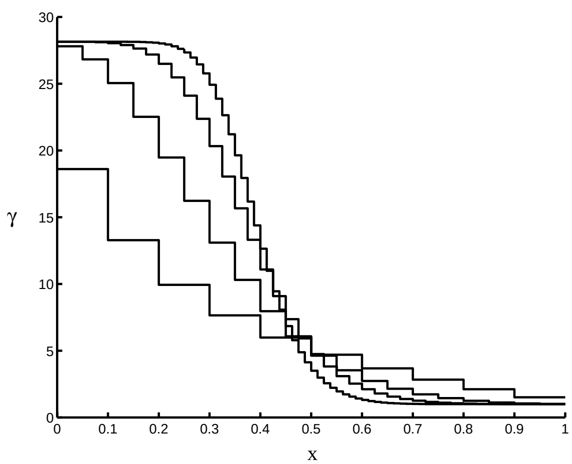


FIG. 9. *Impedance as a function of depth for* 10-*,* 20-*,* 40-*, and* 80*-layer coatings designed to absorb the band* $[1, 10]$*.*

polant were specified using the not-a-knot condition. That is, the jump in the third derivative at the ends of the first and the last interval was set to zero.

In the next set of illustrations, we consider the nonreflective coatings discussed in section 6 where the design frequency band is bounded away from zero. We chose the frequency band $[1, 10]$ for this set of illustrations. According to the theory developed in section 6, we expect the reflectivity of such coatings to be exponentially small as a function of the number of layers. Therefore, we consider designs with $n = 10$, 20, 40, and 80 layers only.

For each choice of n, the design frequencies in the interval $[1, 10]$ are given by the expression in (64). The impedances of the resulting optimal coatings are shown in Figure 9. Unlike the previous cases, the number of layers is small enough that the piecewise constant structure of the coatings is readily apparent. Figure 10 shows the frequency responses of these four coatings for the interval $[0, 11]$ along the ω-axis. By

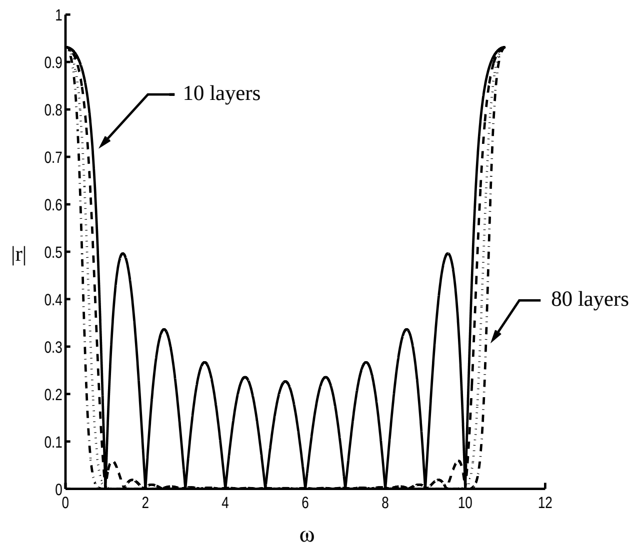


FIG. 10. *Frequency response on* $[0, 11]$ *for* 10-*,* 20-*,* 40-*, and* 80-*layer coatings.*

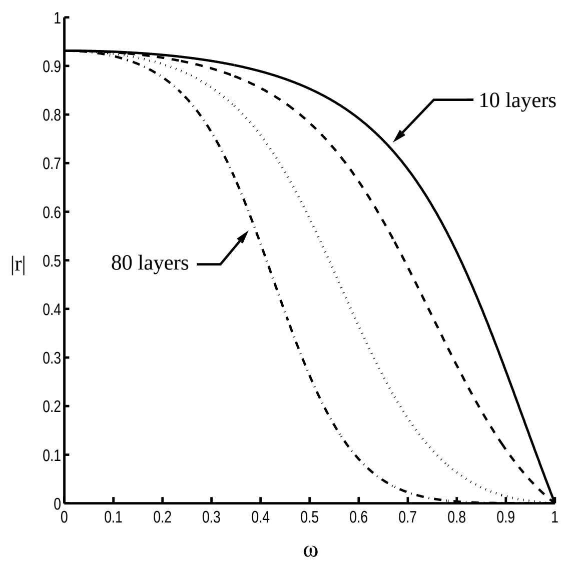


FIG. 11. *Frequency response on* $[0, 1]$ *for* 10-*,* 20-*,* 40-*, and* 80-*layer coatings.*

Lemma 5.1 the frequency responses on $[0, \infty)$ are obtained by periodic extension of the plots shown in Figure 10.

The reflectivities associated with the 40- and 80-layer coatings are so vanishingly small that they coincide with the ω-axis in Figure 10. Figure 11 shows the details of these graphs on the interval $[0, 1]$ where we can clearly see how the frequency response approaches zero as we approach the leading edge of the design frequency band at $\omega = 1$. Figure 12 shows further details of the frequency response for the 80-layer case on the design interval $[1, 10]$ along the ω-axis. Except for small spikes of magnitude 3×10^{-6} near the end points of the interval, the reflectivity is extremely small, with magnitude less than 10^{-9} on the interior interval $[2, 9]$.

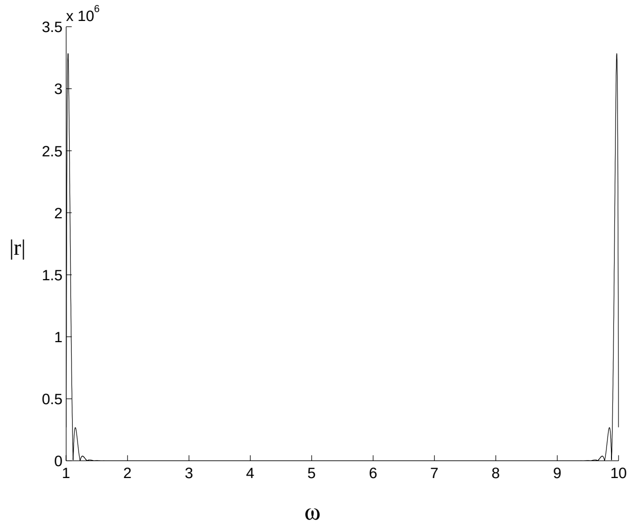


FIG. 12. *Frequency response on* $[1, 10]$ *for* 80*-layer coating.*

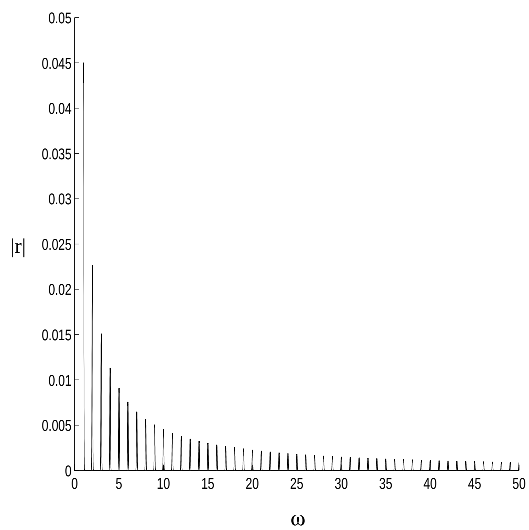
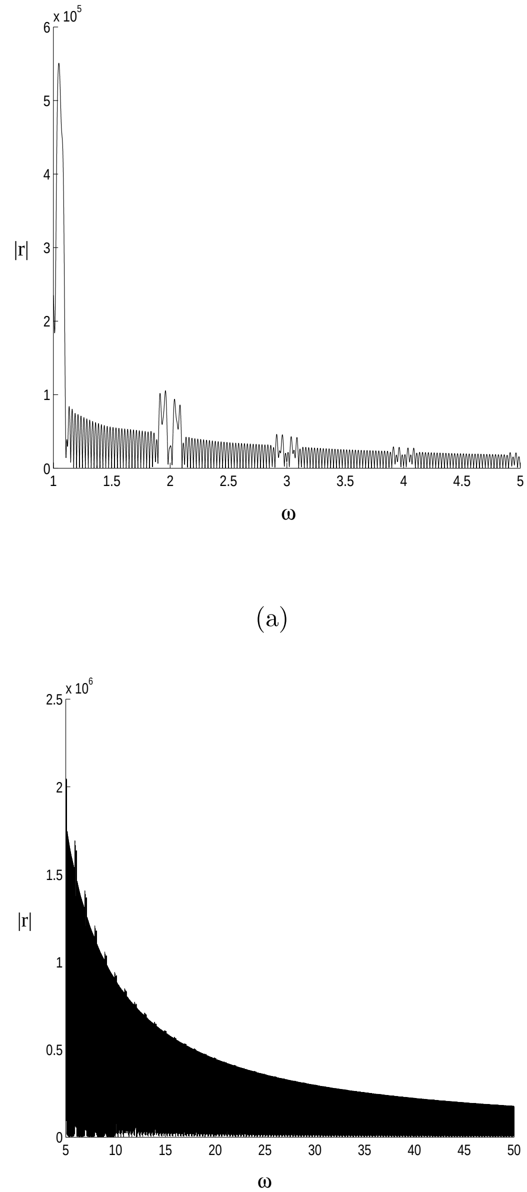


FIG. 13. *Frequency response on* $[1, 50]$ *for* 40*-layer coating with linear spline smoothing.*

As with the previous set of illustrations, smoothing the piecewise homogeneous optimally designed coatings has the effect of reducing the reflectivities for frequencies outside the design interval. Figure 13 shows the frequency response of the 40-layer coating in the frequency interval $[1, 50]$ for linear spline interpolation of the impedance. Observe that the peaks of amplitude 0.93138 at $\omega = 11, 22, 33, \ldots$ are gone; however, some new but smaller peaks emerge at all the integer values of the frequency. When the coating is smoothed using cubic spline interpolation, we obtain the frequency response on $[1, 5]$ shown in Figure 14(a) and on $[6, 50]$ shown in Figure 14(b). Since the vertical axis on the graph of Figure 14(b) is multiplied by 10^{-6}, the frequency



(a)

(b)

Fig. 14. (a) *Frequency response on* $[1, 5]$ *for* 40*-layer coating with cubic spline smoothing.* (b) *Frequency response on* $[5, 50]$ *for* 40*-layer coating with cubic spline smoothing.*

response is now small everywhere and the peaks at the integers in Figure 13 have been eliminated. The reflectivity for the 40-layer coating is on the order of 10^{-7} for much of the interval $[1, 10]$, and it rises to 0.93138 near $\omega = 0$ or 11. For the coating obtained by cubic spline interpolation, the reflectivity is on the order of 10^{-6} or smaller for

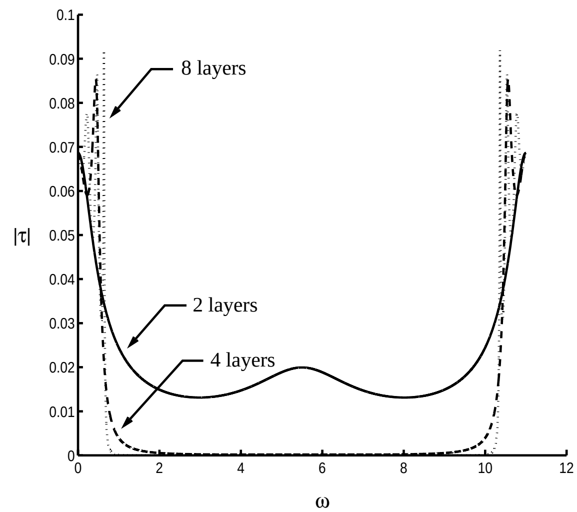


FIG. 15. *Transmissivity for* 2-, 4-, *and* 8-*layer coatings designed to reflect waves on the band* $[1, 10]$.

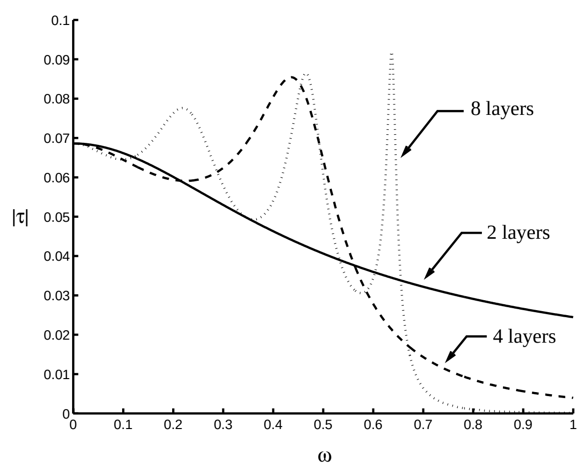


FIG. 16. *Transmissivity for* 2-, 4-, *and* 8-*layer coatings designed to reflect waves on the band* $[1, 10]$.

$\omega > 5$. Hence, the frequency response of the smooth coating is not quite as small on the design interval as that of the layered coating; however, the periodic large frequency response values associated with the layered coating have been eliminated.

Now we consider the coatings of Theorem 7.5 that lead to small transmissivity for the frequency band $[1, 10]$. Taking $\varepsilon = .1$, we consider coatings for which the impedance in successive layers oscillates between .1 and 10 and for which the stiffness is given by (86). In Figure 15 we plot the transmissivity on $[0, 11]$ for coatings with 2, 4, and 8 layers. The transmissivity on $[0, \infty)$ is obtained by periodic extension of these plots. Observe that with a small number of layers, the transmissivity is quite small on the design interval $[1, 10]$. In Figure 16, we magnify the part of Figure 15 associated with the frequency band $[0, 1]$, and we see that after a few oscillations the transmissivity for the 8-layer coating becomes very small as the frequency approaches the design interval. On the interval $[1, 10]$, the transmissivity for the 8-layer coating lies between 1.2×10^{-4} and 2×10^{-8}.

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