DISSERTATION PROPOSAL – SIMPLICIAL METRIC THICKENINGS OF SPHERES

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Contents

1. Introduction

This dissertation will investigate certain connections between the following three topics of interest:

(1) metric thickenings of (infinite) metric spaces at large scales,
(2) theorems of Borsuk–Ulam type, and
(3) (real) convex geometry.

Motivating this research is a desire to understand, broadly speaking, the topology of certain simplicial complexes defined on spheres at large scales. In turn, a better understanding of the topology of these complexes is relevant to applications of persistent homology, where the "correct" scale parameter of a filtration of simplicial complexes (i.e., the scale at which one recovers the homotopy type of the underlying unknown space) is *a priori* unknown.

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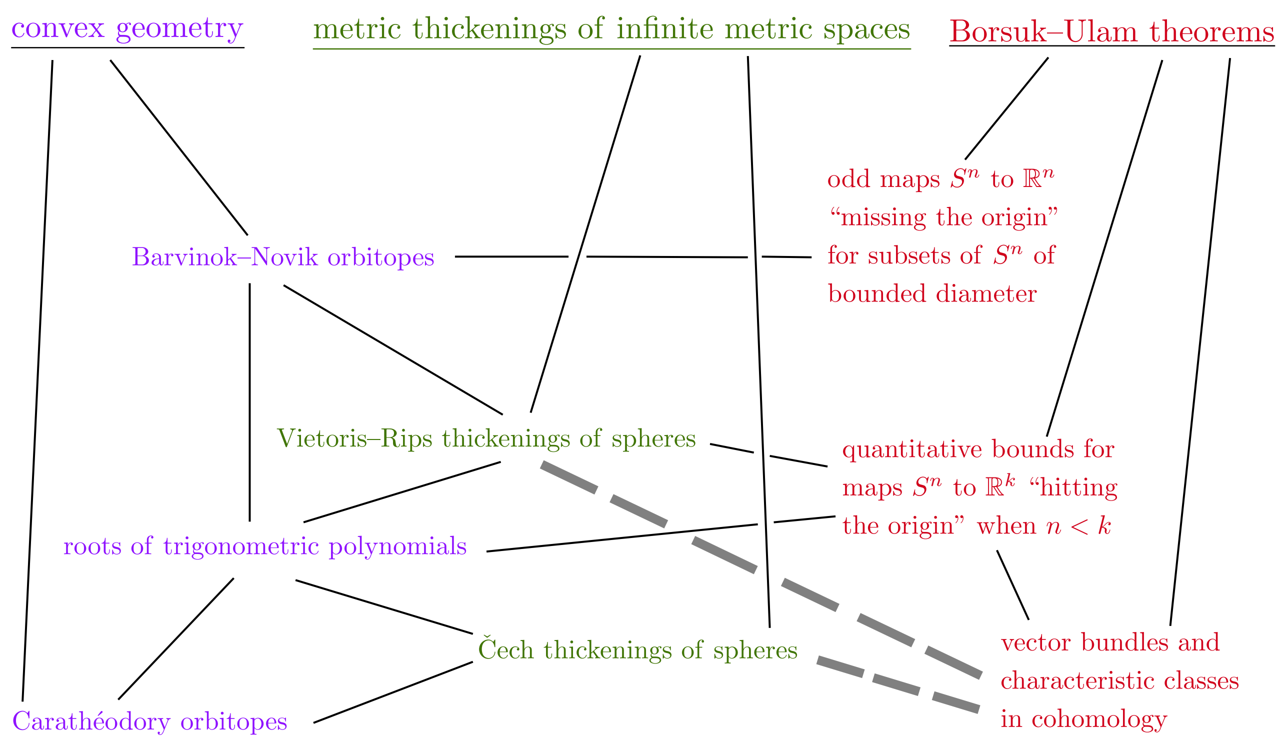

FIGURE 1. A schematic representation of connections between different topics of interest. Dashed thicker gray lines indicate newly formed connections which may provide deeper insight into the relationship between metric thickenings and Borsuk–Ulam type theorems.

Mounting evidence suggests that an effective approach to the study of the topology of metric thickenings at large scales may be indirect — specifically, through the study of certain objects in convex geometry and Borsuk–Ulam type theorems. In this proposal, we provide evidence for the validity and effectiveness of this indirect approach in the form of preliminary results and conjectures. For example, we explain how knowledge of the homotopy connectivity of metric thickenings of spheres implies generalizations of the Borsuk–Ulam theorem for maps to higher dimensional codomains, and how certain convex bodies may inform our understanding of the topology of these metric thickenings. We also describe new directions and methods for this research, including the potential to apply techniques of representation theory (specifically, through the study of a family of Schur polynomials) and characteristic classes of certain principal G-bundles.

2. Background and literature review

In this section we review notation and related work on Vietoris–Rips simplicial complexes, metric thickenings, convex geometry, moment curves, orbitopes, and Borsuk–Ulam type theorems.

2.1. Conventions regarding S^n. We equip the sphere S^n with the geodesic metric in which great circles have circumference 2π, although all stated results also hold (with easy modifications) when S^n is instead equipped with the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$.

2.2. Vietoris–Rips and Čech simplicial complexes. In applications of topological data analysis, one requires a method of turning a finite metric space (M, d), often a collection of discrete points in $\mathbb{R}^k$, into a useful topological space. Commonly, this topological space is taken to be the geometric realization of a simplicial complex with vertex set M. Further, it is often useful to define a parametrized family of these simplicial complexes in such a way as to incorporate the metric of the underlying vertex set. Two common families of parametrized simplicial complexes that are defined in terms of the metric d are the Vietoris–Rips and the Čech simplicial complexes.

Definition 1. Let X be a metric space and fix $r \geq 0$. The *Vietoris–Rips simplicial complex of X with scale parameter r*, denoted $\mathrm{VR}(X;r)$, has X as its vertex set and a finite subset $\sigma \subseteq X$ as a simplex whenever $\mathrm{diam}(\sigma) \leq r$.

Definition 2. Let X be a metric space, and fix $r \geq 0$. The *Čech simplicial complex of X with scale parameter r*, denoted $\check{\mathrm{C}}(X;r)$, has X as its vertex set and a finite subset $\sigma \subseteq X$ as a simplex whenever $\cap_{v\in\sigma} B(v;r) \neq \varnothing$, where $B(v;r)$ denotes the closed ball of radius r centered at v.

Note that we use the $\leq$ convention for Vietoris–Rips and Čech complexes throughout this document, rather than the $<$ convention. Additionally, we identify an abstract simplicial complex with its geometric realization, which is a topological space.

While the theorems of Hausmann and Latschev [21, 27] describe conditions under which the homotopy type of a manifold is recoverable from a Vietoris–Rips complex for sufficiently small $r \geq 0$, much less is known about the topological behavior of these constructions for large values of r, even though large values of r commonly arise in applications of persistent homology [15]. However, more is known in the specific case when the underlying manifold is the circle. The following theorem from [2] is based on [1, 4].

Theorem 3. *Let $0 \leq r < \pi$. There are homotopy equivalences*

$$\mathrm{VR}(S^1;r) \simeq \begin{cases} S^{2k-1} & \text{if } \frac{2\pi(k-1)}{2k-1} < r < \frac{2\pi k}{2k+1} \\ \bigvee^{\mathfrak{c}} S^{2k} & \text{if } r = \frac{2\pi k}{2k+1}, \end{cases}$$

where $k = 0, 1, 2, \ldots,$ and where $\mathfrak{c}$ denotes the cardinality of the continuum.

Related papers include [18], which studies the 1-dimensional persistence of Čech and Vietoris–Rips complexes of metric graphs, [40] which extends this to geodesic spaces, [41] which studies approximations of Vietoris–Rips complexes by finite samples even at higher scale parameters, and [43] which applies Bestvina–Brady discrete Morse theory to Vietoris–Rips complexes.

2.3. **Metric thickenings and optimal transport.** When a metric space X is not finite, it is often impossible[1] to equip $\mathrm{VR}(X;r)$ with a metric without changing the homeomorphism type. In such instances the simplicial complex $\mathrm{VR}(X;r)$ destroys the metric information about the underlying space X. This motivates the consideration of the *Vietoris–Rips metric thickening*, $\mathrm{VR}^m(X;r)$, which preserves metric information (the superscript m denotes "metric").

Let δ_x denote the Dirac delta mass at a point $x \in X$.

Definition 4 ([3])**.** Let X be a metric space and let $r \geq 0$. The *Vietoris–Rips thickening* is the set

$$\mathrm{VR}^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \;\middle|\; k \in \mathbb{N},\ x_i \in X,\ \mathrm{diam}(\{x_0,\ldots,x_k\}) \leq r,\ \lambda_i \geq 0,\ \sum \lambda_i = 1 \right\},$$

equipped with the 1-Wasserstein metric.

[1]A simplicial complex (for example, $\mathrm{VR}(X;r)$) is metrizable if and only if it is locally finite [32, Proposition 4.2.16(2)].

This metric is also called the Kantorovich, optimal transport, or earth mover's metric [36, 37, 38]; it provides a notion of distance between probability measures defined on a metric space. Although it exists much more generally [16, 24, 25], the 1-Wasserstein metric on $\mathrm{VR}^m(X;r)$ can be defined as follows. Given $\mu, \mu' \in \mathrm{VR}^m(X;r)$ with $\mu = \sum_{i=0}^{k} \lambda_i \delta_{x_i}$ and $\mu' = \sum_{j=0}^{k'} \lambda'_j \delta_{x'_j}$, define a *matching* p between μ and μ' to be any collection of non-negative real numbers $\{p_{i,j}\}_{i,j}$ such that $\sum_{j=0}^{k'} p_{i,j} = \lambda_i$ and $\sum_{i=0}^{k} p_{i,j} = \lambda'_j$. Define the *cost* of the matching p to be $\sum_{i,j} p_{i,j} d(x_i, x'_j)$. The 1-Wasserstein distance between $\mu, \mu' \in \mathrm{VR}^m(X;r)$ is then the infimum, varying over all matchings p between μ and μ', of the cost of p.

Note that $\mathrm{VR}^m(X;0)$ is isometric to X. Contrary to the situation for an arbitrary Vietoris–Rips complex, the embedding $X \hookrightarrow \mathrm{VR}^m(X;r)$ into the Vietoris–Rips metric thickening given by $x \mapsto \delta_x$ is continuous. In fact, more is true: $\mathrm{VR}^m(X;r)$ is an *r-thickening* of X [20, 3]. For this reason, we identify $x \in X$ with the measure $\delta_x \in \mathrm{VR}^m(X;r)$ in the image of this embedding. Given a measure $\mu = \sum_{i=0}^{k} \lambda_i \delta_{x_i}$ with $\lambda_i > 0$ for all i, we denote the *support* of μ by $\mathrm{supp}(\mu) = \{x_0, \ldots, x_k\}$.

If M is a complete Riemannian manifold with curvature bounded from above and below, then $\mathrm{VR}^m(M;r)$ is homotopy equivalent to M for r sufficiently small [3, 7]. This property provides an analogue of Hausmann's theorem [21] for metric thickenings.

In the obvious way, there is an analogous definition of the Čech metric thickening of X at scale r, denoted $\check{\mathrm{C}}^m(X;r)$. Similar results hold for Čech metric thickenings, including continuity of the inclusion (which is an isometry onto its image) and an analogue of Hausmann's theorem. Throughout, we simply write *metric thickening* when the distinction is unimportant, or if the underlying simplicial complex is clear through context. For convenience, we make the following definition.

Definition 5. Given a complete Riemannian manifold M, define *the first critical scale* of the metric thickening $\mathrm{VR}^m(M;t)$ of M to be

$$r_0 = \sup\{t \in [0,\infty] \mid \mathrm{VR}^m(M;t) \simeq M \text{ for all } r < t\}.$$

This supremum is well-defined because $K^m(M;0)$ is isometric (and hence homeomorphic) to M. We say a scale parameter r is *large* if $r_0 \leq r$.

To date, the homotopy type of the Vietoris–Rips metric thickening of a sphere S^n is known only up to and including the first critical scale. In fact, for the sphere S^n, we have $r_0 = \pi - \arccos\left(\frac{1}{n+1}\right)$, which is the diameter of an inscribed $(n+1)$-simplex in S^n.

Theorem 6 ([3, Proposition 5.3 and Theorem 5.4])**.** *There are homotopy equivalences*

$$\mathrm{VR}^m(S^n;r) \simeq \begin{cases} S^n & \text{if } 0 \leq r < \pi - \arccos\left(\frac{1}{n+1}\right) \\ \sum^{n+1} \frac{\mathrm{SO}(n+1)}{A_{n+2}} & \text{if } r = \pi - \arccos\left(\frac{1}{n+1}\right), \end{cases}$$

where $\frac{\mathrm{SO}(n+1)}{A_{n+2}}$ *is a finite quotient of a special orthogonal group, as described in* [3].

2.4. **Convex geometry.** Convex geometry is the study of convex sets, especially polytopes and their facial structures [44]. Given an arbitrary subset $Y \subseteq \mathbb{R}^n$, we let

$$\mathrm{conv}(Y) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; k \in \mathbb{N},\ v_i \in Y,\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

denote the *convex hull* of Y. For example, Figure 2 shows the convex hull of the image of the map $f\colon S^1 \to \mathbb{R}^3$ defined by $f(t) = (\cos(t), \sin(t), \cos(3t))$. Given any finite set $\{\lambda_1, \ldots, \lambda_k\} \subset \mathbb{R}^n$ such that $\lambda_i \geq 0$ for all i and $\sum \lambda_i = 1$, we say $\lambda_1, \ldots, \lambda_k$ are a collection of *convex coefficients*.

Let $Y \subseteq \mathbb{R}^n$ be convex. Define a *face* of Y to be any convex set $F \subseteq Y$ such that, given $x \in F$, if $x = \lambda y + (1-\lambda)z$ for some $0 < \lambda < 1$ and $y, z \in Y$, then $y, z \in F$. If F is a face of Y and $F \neq \varnothing$ and $F \neq Y$, we say F is a *proper* face of Y. Let $Y \subseteq \mathbb{R}^k$ be a set in Euclidean space. Carathéodory's theorem states that if the convex hull of Y contains the origin, then there is a subset of Y of at most $k+1$ points whose convex hull also contains the origin.

Definition 7. Given $Y \subseteq \mathbb{R}^n$, we say $Y' \subseteq Y$ is a *Carathéodory subset of* Y if the convex hull of Y' contains the origin.

2.5. **Trigonometric polynomials.** A *trigonometric polynomial* is an expression of the form

$$p(t) = c + \sum_{k=1}^{n} \left(a_k \cos(kt) + b_k \sin(kt)\right),$$

inducing a map $S^1 \to \mathbb{R}$. Throughout, we assume all coefficients are real. In the case that $c = 0$, we call p a *homogeneous* trigonometric polynomial. The set $S \subseteq \{1, \ldots, n\}$ of integers k with $a_k \neq 0$ or $b_k \neq 0$ is called the *spectrum* of p, and the largest integer in S is the *degree* of p. The spectrum of p constrains the set of roots of p; for example, if p is homogeneous of degree n then it has a root on any closed circular arc of length $\frac{2\pi n}{n+1}$; see [8, 19]. Kozma and Oravecz in [26] give upper bounds on the length of an arc where a trigonometric polynomial with spectrum bounded away from zero (that is, $S \subseteq [k, n]$) is non-zero. If the spectrum of p consists only of odd integers, then p is called a *raked* trigonometric polynomial.

2.6. **The trigonometric moment curve.**

Definition 8. For $k \in \mathbb{N}$, the *trigonometric moment curve* $\mathrm{M}_{2k} \colon S^1 \to \mathbb{R}^{2k}$ is defined by

$$\mathrm{M}_{2k}(t) = (\cos(t), \sin(t), \cos(2t), \sin(2t), \ldots, \cos(kt), \sin(kt))^{\intercal}.$$

Here, we identify the domain S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. A related map is the moment curve $\gamma_k \colon \mathbb{R} \to \mathbb{R}^k$, which is defined by $\gamma_k(t) = (t, t^2, \ldots, t^k)^{\intercal}$. In [17], Gale shows that the facial lattices of the convex bodies $\mathrm{conv}(\gamma_{2k}(\mathbb{R}))$ and $\mathrm{conv}(\mathrm{M}_{2k}(S^1))$ are equivalent for all $k \geq 1$.

2.7. **Carathéodory orbitopes.** The *Carathéodory orbitope* is defined by $\mathcal{C}_{2k} = \mathrm{conv}(\mathrm{M}_{2k}(S^1)) \subseteq \mathbb{R}^{2k}$.

Remark 9. We note that $\{t_1, \ldots, t_n\} \subset S^1$ defines a proper face $\mathrm{conv}(\{\mathrm{M}_{2k}(t_1), \ldots, \mathrm{M}_{2k}(t_n)\})$ of $\mathcal{C}_{2k}$ if and only if there exists a trigonometric polynomial $p \not\equiv 0$ of degree at most k such that p is non-negative on S^1 and $p(t_i) = 0$ for all $1 \leq i \leq n$.

This convex body is not the convex hull of a finite set of points; it is an *orbitope* instead of a polytope [33].

Theorem 10 ([33, Corollary 5.4])**.** *The faces of $\mathcal{C}_{2k}$ are in inclusion-preserving bijection with sets of at most k points in S^1.*

In particular, any point in $\partial\mathcal{C}_{2k}$ can be expressed as a convex combination $\sum_{i=1}^{m} \lambda_i \mathrm{M}_{2k}(t_i)$ with $t_i \in S^1$ and $m \leq k$. Note that any $\{t_1, \ldots, t_m\} \subset S^1$ with $m \leq k$ must be disjoint from some open arc of length at least $\frac{2\pi}{k}$, and hence in some ball of S^1 of radius $r \geq \frac{(k-1)\pi}{k}$. In this way, the facial structure of Carathéodory orbitopes is related to Čech simplicial complexes defined on S^1 at scale $r \geq \frac{(k-1)\pi}{k}$.

2.8. **The centrally symmetric trigonometric moment curve.** The centrally symmetric moment curve is analogous to the trigonometric moment curve, with the additional property that it is symmetric under the involution $x \mapsto -x$.

Definition 11. For $k \in \mathbb{N}$, the *centrally symmetric moment curve* $\mathrm{SM}_{2k} \colon S^1 \to \mathbb{R}^{2k}$ is defined by

$$\mathrm{SM}_{2k}(t) = \big(\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(2k-1)t, \sin(2k-1)t\big)^{\intercal}.$$

Again, we identify the domain S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Since $\mathrm{SM}_{2k}(t+\pi) = -\mathrm{SM}_{2k}(t)$, we say that SM_{2k} is *centrally symmetric* about the origin. Interestingly, this curve is closely related to the multidimensional scaling (MDS) embedding $S^1 \hookrightarrow \mathbb{R}^{2k}$ of the geodesic circle [5, 11, 23, 42]; multidimensional scaling is a way to map a metric space into Euclidean space in a way that distorts the metric (in some sense) as little as possible.

Given a vector $z \in \mathbb{R}^{2k}$, observe that the inner product of z and $\mathrm{SM}_{2k}(t)$ is a raked homogeneous trigonometric polynomial of degree $2k-1$.

2.9. **Barvinok–Novik orbitopes.** The *Barvinok–Novik orbitope* is defined by $\mathcal{B}_{2k} = \mathrm{conv}(\mathrm{SM}_{2k}(S^1)) \subseteq \mathbb{R}^{2k}$ [10] for $k \geq 1$. Note that the boundary of $\mathcal{B}_{2k}$ is homeomorphic to the sphere S^{2k-1}.

Remark 12. We note that $\{t_1, \ldots, t_n\} \subset S^1$ defines a proper face $\mathrm{conv}(\{\mathrm{SM}_{2k}(t_1), \ldots, \mathrm{SM}_{2k}(t_n)\})$ of $\mathcal{B}_{2k}$ if and only if there exists a raked trigonometric polynomial $p \not\equiv 0$ of degree at most $2k-1$ such that p is non-negative on S^1 and $p(t_i) = 0$ for all $1 \leq i \leq n$.

The faces of $\mathcal{B}_{2k}$ are known for $k = 2$; a subset of these faces are visible in Figure 2 (which is in $\mathbb{R}^3$ instead of $\mathbb{R}^4$).

Theorem 13 ([10, 35])**.** *The proper faces of* $\mathcal{B}_4$ *are*

- *the 0-dimensional faces (vertices)* $\mathrm{SM}_4(t)$ *for* $t \in S^1$,
- *the 1-dimensional faces (edges)* $\mathrm{conv}(\mathrm{SM}_4(\{t_1, t_2\}))$ *where* $t_1 \neq t_2$ *are the edges of an arc of* S^1 *of length at most* $\frac{2\pi}{3}$, *and*
- *the 2-dimensional faces (triangles)* $\mathrm{conv}(\mathrm{SM}_4(\{t, t+\frac{2\pi}{3}, t+\frac{4\pi}{3}\}))$ *for* $t \in S^1$.

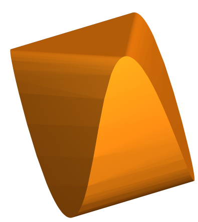

Figure 2. The convex hull of the map $f \colon S^1 \to \mathbb{R}^3$ defined by $f(t) = (\cos(t), \sin(t), \cos(3t))$.

Though the facial structure of the Barvinok–Novik orbitopes $\mathcal{B}_{2k}$ is not known for $k > 2$, certain neighborliness results have been established [9]. Sinn has shown that the orbitopes are simplicial [34]. Additionally, Vinzant proved that the edges of $\partial\mathcal{B}_{2k}$ consist of all line segments $\mathrm{conv}\,(\mathrm{SM}_{2k}(\{t_0, t_1\}))$ with $|t_0 - t_1| \leq \frac{2\pi(k-1)}{2k-1}$ [39]. In other words, the edges of $\mathcal{B}_{2k}$ are the same as the edges of $\mathrm{VR}(S^1; \frac{2\pi(k-1)}{2k-1})$. The following is an immediate corollary of the work of Sinn and Vinzant.

Corollary 14 ([34, 39])**.** *Every face of the Barvinok–Novik orbitope* $\mathcal{B}_{2k}$ *is a simplex whose diameter in* S^1 *(not in* $\mathbb{R}^{2k}$*) is at most* $\frac{2\pi(k-1)}{2k-1}$.

In this way, the facial structure of Barvinok–Novik orbitopes is related to Vietoris–Rips simplicial complexes defined on S^1 at scale $\frac{2\pi(k-1)}{2k-1} \leq r$.

2.10. **The Vandermonde matrices.** A recurring computational tool in our study of the Carathéodory and Barvinok–Novik orbitopes is the Vandermonde matrix. This matrix has a particularly simple determinant, and by converting trigonometric functions into complex exponential form, we are able to reduce certain matrices to Vandermonde (or near-Vandermonde) matrices in order to compute their determinants.

Definition 15. A *Vandermonde matrix* is an $n \times n$ matrix of the form

$$V = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}.$$

The determinant of the above matrix is $\det(V) = \prod_{1\le i<j\le n}(a_j - a_i)$; see for example [30, Section 2.8.1].

2.11. **The Borsuk–Ulam theorem and $\mathbb{Z}/2\mathbb{Z}$-equivariant maps.** The classical Borsuk–Ulam theorem is typically stated as follows.

Theorem 16 ([29, Theorem 2.1.1])**.** *Given a continuous map $f\colon S^n \to \mathbb{R}^n$, there exists $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$.*

We say a map $f\colon S^n \to \mathbb{R}^k$ is *odd* or *centrally-symmetric* if $f(-x) = -f(x)$ for all $x \in S^n$. An equivalent formulation of the Borsuk–Ulam states that given a continuous and odd map $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = \vec{0}$.

More generally, given topological spaces X and Y equipped with $\mathbb{Z}/2\mathbb{Z}$-actions μ and ν respectively, we say a map $f\colon X \to Y$ is *odd* or $\mathbb{Z}/2\mathbb{Z}$-*equivariant* if $f \circ \mu = \nu \circ f$. Throughout, we always equip $\mathbb{R}^n$ and S^n with the standard antipodal $\mathbb{Z}/2\mathbb{Z}$-action specified by $x \mapsto -x$.

The following theorem characterizes odd maps into S^n. We say a nonempty topological space is n-*connected* if it is nonempty, path connected, and its homotopy groups vanish up to and including dimension n.

Theorem 17 ([29, Proposition 5.3.2(iv)])**.** *Let X be a topological space equipped with a $\mathbb{Z}/2\mathbb{Z}$-action. If X is $(n-1)$-connected, then*

$$n \le \min\{m \in \{0,1,\dots\} \mid \textit{there exists an odd map } X \to S^m\}.$$

In particular, this implies a generalization of the Borsuk–Ulam theorem in which the domain of interest is not necessarily a geometric sphere.

Corollary 18. *Let X be a $(n-1)$-connected topological space equipped with a $\mathbb{Z}/2\mathbb{Z}$-action. Given an odd map $f\colon X \to \mathbb{R}^n$, there exists $x_0 \in X$ such that $f(x_0) = \vec{0}$.*

Proof. Theorem 17 implies that there is no $\mathbb{Z}/2\mathbb{Z}$-equivariant map from X into S^{n-1}. Hence, any odd map $f\colon X \to \mathbb{R}^n$ must hit the origin, because otherwise we would obtain an odd map $\frac{f}{|f|}\colon X \to S^{n-1}$. □

3. Problem statement

Broadly speaking, the primary motivating question for this research is the following.

Question 19. What can be said about the topology (e.g., the homotopy type, the homology, the connectivity, etc.) of metric thickenings of spheres at large scales?

Specifically, we intend to investigate the quantitative and qualitative aspects of the topology of metric thickenings of spheres that may be revealed through connections to convex geometry and Borsuk–Ulam type results. Along these lines, we ask the following related questions.

Question 20. Let $f\colon S^n \to \mathbb{R}^k$ be odd and continuous for some integers $n, k \geq 1$. By Proposition 28, proved in Subsection 4.2, there exists a number $s_{n,k} \in [0, \pi)$ such that $\vec{0} \in \mathrm{conv}(f(X))$ for some $X \subseteq S^n$ with $\mathrm{diam}(X) \leq s_{n,k}$, and this bound is sharp. Given integers $n, k \geq 1$, what is the value of $s_{n,k}$?

Question 21. Loosely speaking, we can use linear algebra and determinants to translate the problem of identifying the faces of the Barvinok–Novik orbitopes to a combinatorial problem related to Schur polynomials. Is it possible to make this translation precise enough so that we may obtain new results about the facial structure of these orbitopes?

In Section 4, we present some common threads between metric thickenings of spheres, convex geometry and orbitopes, and Borsuk–Ulam theorems. More precisely, we describe how full or partial answers to any one of the above questions may inform our understanding of (and provide partial answers to) the others.

4. Methods and preliminary results

We describe our methods and review some preliminary results and partial answers to Questions 19, 21, and 20.

4.1. **The homotopy type of the Vietoris–Rips thickening of S^1 at the first critical scale.** As described in Subsection 2.3, the homotopy type of $\mathrm{VR}^m(S^n; r)$ is known at scales up to and including the first critical scale r_0, which is the diameter of a regular $(n+1)$-simplex inscribed in S^n. The proof technique used to obtain this result does not immediately generalize to larger scales. In [6], we take a more geometric approach and exhibit a homotopy equivalence $\mathrm{VR}^m(S^1; \frac{2\pi}{3}) \to S^3$ factoring through Euclidean space. While not immediately applicable to higher-dimensional spheres, this technique has the additional benefit of revealing connections between $\mathrm{VR}^m(S^1; r)$, the Barvinok–Novik orbitopes, trigonometric polynomials, and Borsuk–Ulam type theorems. We now recall the main intermediate steps and results of this geometric approach.

First, we make the following conjecture based on the known homotopy type of $\mathrm{VR}(S^1; r)$ at all scales (cf. Theorem 3).

Conjecture 22. *As r increases, $\mathrm{VR}^m(S^1; r)$ obtains the homotopy type of odd dimensional spheres until becoming contractible at $r = \pi$. Specifically, for all integers $k \geq 1$,*

$$\mathrm{VR}^m(S^1; r) \simeq S^{2k-1} \quad \textit{if} \quad \frac{2\pi(k-1)}{2k-1} \leq r < \frac{2\pi k}{2k+1}$$

As partial evidence for Conjecture 22, we observe that the facial structure of the Barvinok–Novik orbitopes closely resembles the structure of Vietoris–Rips complexes defined on S^1 at the appropriate scales. Specifically, while the exact facial structure of $\mathcal{B}_{2k}$ remains unknown for $k > 2$, Corollary 14 implies that there exists a well-defined inclusion $\iota_k\colon \partial\mathcal{B}_{2k} \hookrightarrow \mathrm{VR}^m(S^1; r)$ for $\frac{2\pi(k-1)}{2k-1} \leq r$. Further, in [6], we determine sharp lower bounds on the diameter of the preimage of any Carathéodory subset of the symmetric moment curve in $\mathbb{R}^{2k}$.

Theorem 23 ([6, Theorem 5])**.** *Let $X \subseteq S^1$ be any set such that $\vec{0} \in \mathrm{SM}_{2k}(X)$. Then, $\mathrm{diam}(X) \geq \frac{2\pi(k-1)}{2k-1}$. Further, if Y denotes any set of $2k-1$ equally-spaced points in S^1, then $\mathrm{diam}(Y) = \frac{2\pi(k-1)}{2k-1}$ and $\vec{0} \in \mathrm{conv}(\mathrm{SM}_{2k}(Y))$, i.e., this diameter bound is sharp.*

In turn, this implies that SM_{2k} induces a continuous map $\mathrm{SM}_{2k}\colon \mathrm{VR}^m(S^1;r) \to \mathcal{B}_{2k} \setminus \{\vec{0}\}$ exactly when $r < \frac{2\pi k}{2k+1}$. Putting this all together, we define a continuous sequence of maps

$$\mathrm{VR}^m(S^1;r) \xrightarrow{\mathrm{SM}_{2k}} \mathbb{R}^{2k} \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_{2k} \xrightarrow{\iota} \mathrm{VR}^m(S^1;r),$$

for $\frac{2\pi(k-1)}{2k-1} \le r < \frac{2\pi k}{2k+1}$, where p denotes the radial projection map and ι denotes the well-defined inclusion.

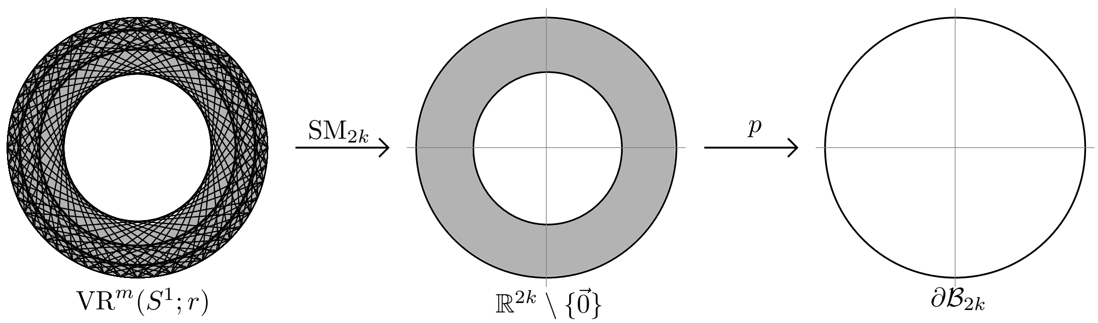

FIGURE 3. The composition of maps $\mathrm{VR}^m(S^1;r) \xrightarrow{\mathrm{SM}_{2k}} \mathbb{R}^{2k} \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_{2k}$, drawn in the case $k = 1$.

One may easily verify that the map $(p \circ \mathrm{SM}_{2k}) \circ \iota$ is the identity map on $\partial\mathcal{B}_{2k}$, i.e. that the space $\partial\mathcal{B}_{2k} \cong S^{2k-1}$ is a retract of $\mathrm{VR}^m(S^1;r)$. Hence, as a consequence of this geometric proof technique, we obtain the following corollary.

Corollary 24. *For $\frac{2\pi(k-1)}{2k-1} \le r < \frac{2\pi k}{2k+1}$, the $(2k-1)$-dimensional homology, cohomology, and homotopy groups of $\mathrm{VR}^m(S^1;r)$ are nontrivial.*

Finally, for $r = \frac{2\pi}{3}$, we prove $(\iota \circ p) \circ \mathrm{SM}_4 \simeq \mathrm{id}_{\mathrm{VR}^m(S^1;\frac{2\pi}{3})}$. Consequently, $\mathrm{VR}^m(S^1;\frac{2\pi}{3}) \simeq \partial\mathcal{B}_4 \cong S^3$. While we are unable to prove the analogous homotopy equivalence for scales beyond $r = \frac{2\pi}{3}$, we make the following conjecture.

Conjecture 25. *For $\frac{2\pi(k-1)}{2k-1} \le r < \frac{2\pi k}{2k+1}$, we have $(\iota \circ p) \circ \mathrm{SM}_{2k} \simeq \mathrm{id}_{\mathrm{VR}^m(S^1;r)}$.*

Note that Conjecture 25, together with the fact that $(p \circ \mathrm{SM}_{2k}) \circ \iota = \mathrm{id}_{\partial\mathcal{B}_{2k}}$ for this range of r values, would imply Conjecture 22.

We remark that an analogous technique for determining the homotopy type of the Čech metric thickenings of S^1, using the trigonometric moment curve and the Carathéodory orbitopes, has been considered. We are currently unable to prove the analogous homotopy $(\iota \circ p) \circ \mathrm{M}_{2k} \simeq \mathrm{id}_{\check{\mathrm{C}}^m(S^1;r)}$ in that case.

4.2. **A generalization of the Borsuk–Ulam theorem.** Given an odd map $f\colon S^n \to \mathbb{R}^k$ such that $k \le n$, the classical Borsuk–Ulam theorem guarantees the existence of a point $x_0 \in S^n$ such that $f(x_0) = \vec{0}$. On the other hand, if $k > n$, this is far from the truth: just the standard inclusion $\iota\colon S^n \hookrightarrow \mathbb{R}^{n+1}$ is an odd map that misses the origin. However, it is clear that the origin is contained in the convex hull of some number of points in the image of the inclusion. In the language of Subsection 2.4, $\iota(S^n)$ always contains a Carathéodory subset. In fact, this is trivially true for any $k > n$ and any odd map $f\colon S^n \to \mathbb{R}^k$, since $\vec{0} = \frac{1}{2}f(x_0) + \frac{1}{2}f(-x_0)$ for all points $x_0 \in S^n$.

In the case of the inclusion $\iota\colon S^1 \hookrightarrow \mathbb{R}^2$, however, we can do better: the origin is contained in the convex hull of the image of any set of three equally-spaced points $\{t_0, t_0 + \frac{2\pi}{3}, t_0 + \frac{4\pi}{3}\}$ in S^1. These points are "far from antipodal" in the sense that $\text{diam}(\{t_0, t_0 + \frac{2\pi}{3}, t_0 + \frac{4\pi}{3}\}) = \frac{2\pi}{3} < \pi$. Furthermore, the preimage of any other Carathéodory subset has diameter strictly greater than $\frac{2\pi}{3}$.

Somewhat surprisingly, this is *always* true. It follows from [6, Theorem 2] that, given any integers $k, n \geq 1$ and any odd map $f\colon S^n \to \mathbb{R}^k$, there exists a subset $X \subseteq S^n$ of diameter strictly less than π such that $f(X)$ is a Carathéodory subset, i.e., such that $\vec{0} \in \text{conv}(f(X))$. The proof of this theorem follows from facts about Vietoris–Rips metric thickenings of spheres. In this section, we explain the relationship between the homotopy connectivity of these metric thickenings and Borsuk–Ulam type theorems.

For notational convenience, we make the following definition.

Definition 26. Fix integers $n, k \geq 1$. We say $t \in [0, \pi]$ *satisfies Condition* $(*)$ if and only if, given any odd and continuous map $f\colon S^n \to \mathbb{R}^k$, there exists a subset $X \subseteq S^n$ with $\text{diam}(X) \leq t$ such that $\vec{0} \in \text{conv}(f(X))$.

We observe that $t \in [0, \pi]$ does not satisfy Condition $(*)$ if and only if there exists an odd continuous map $g\colon S^n \to \mathbb{R}^k$ such that, if $X \subseteq S^n$ with $\text{diam}(X) \leq t$, then $\vec{0} \notin \text{conv}(g(X))$.

Definition 27. We define

$$s_{n,k} = \min\{t \in [0, \pi) \,|\, t \text{ satisfies Condition } (*)\}.$$

We prove in the following proposition that $s_{n,k}$ is well-defined, i.e., that the minimum is attained. Because this number bounds the diameter of the preimage of any Carathéodory subset of $f(S^n)$, we call $s_{n,k}$ the *spherical Carathéodory diameter* for this choice of n and k.

Proposition 28. *For all integers $n, k \geq 1$, the spherical Carathéodory diameter $s_{n,k}$ is a well-defined real number.*

Proof. As an intermediate step, define

$$\tilde{s}_{n,k} = \inf\{t \in [0, \pi] \,|\, t \text{ satisfies Condition } (*)\}.$$

It follows from [6, Theorem 2] that there exists a number $t \in [0, \pi)$ satisfying Condition $(*)$. Hence, $0 \leq \tilde{s}_{n,k} < \pi$.

It remains to prove that the infimum in the definition of $\tilde{s}_{n,k}$ is obtained. Toward that end, let an odd and continuous map $f\colon S^n \to \mathbb{R}^k$ and $\varepsilon > 0$ be given. Then, for each integer $m \geq 1$, there exists a subset $X_m \subseteq S^n$ of diameter at most $s_{n,k} + \frac{\varepsilon}{m}$ such that $\vec{0} \in f(X_m)$. Further, by Carathéodory's Theorem, we may assume $|X_m| \leq k + 1$. If $|X_m| < k + 1$, duplicate an arbitrary point in X_m to obtain a multi-set of size exactly $k + 1$. Arbitrarily order these points so that X_m can be thought of as a point in $(S^n)^{k+1}$. By compactness of this product of spheres, the sequence $\{X_m\}$ has a subsequence converging to a limit configuration $X \in (S^n)^{k+1}$ of diameter at most $s_{n,k}$ and with $\vec{0} \in \text{conv}(f(X))$. Removing duplicate points (and ignoring the ordering) gives us the desired subset $X \subseteq S^n$. ☐

Corollary 29. *Fix integers $n, k \geq 1$. Let $A \subseteq [0, \pi]$ denote the set of numbers satisfying Condition $(*)$, and let $B \subseteq [0, \pi]$ denote the set of numbers that do not satisfy Condition $(*)$. Then, $A = [s_{n,k}, \pi]$ and $B = [0, s_{n,k})$.*

The proof follows from the fact that $A \cap B = \varnothing$, and, if $t_0 \in A$ and $t_0 < t_1 < \pi$, then $t_1 \in A$.

Next, we relate spherical Carathéodory numbers to statements about the topology of metric thickenings of spheres. We will make use of Theorem 17, which requires an assumption about the homotopy connectivity of the domain of certain maps. For that reason, we make the following definition.

Definition 30. Fix integers $n, k \geq 1$. Define

$$r_{n,k} = \inf \{0 \leq t \mid \mathrm{VR}^m(S^n; t) \text{ is } (k-1)\text{-connected}\} .$$

Note that $r_{n,k} \in [0, \pi]$ because $\mathrm{VR}^m(S^n; \pi) \simeq \{\cdot\}$.

Remark 31. In all known cases, the homotopy type of $\mathrm{VR}^m(S^n; r)$ is right continuous. If this is true in general, it would be more convenient to work with the definition

$$r_{n,k} = \min \{0 \leq t \mid \mathrm{VR}^m(S^n; t) \text{ is } (k-1)\text{-connected}\} .$$

Theorem 32. *Fix integers $n, k \geq 1$. Given a continuous and odd map $f \colon S^n \to \mathbb{R}^k$, there exists a subset $X \subseteq S^n$ of diameter at most $r_{n,k}$ such that $\vec{0} \in \mathrm{conv}(f(X))$.*

Proof. If $r_{n,k} = \pi$, the result holds trivially, since $\vec{0} \in \mathrm{conv}(f(\{x_0, -x_0\}))$ for any $x_0 \in S^n$.

Now, suppose $r_{n,k} < \pi$. The space $\mathrm{VR}^m(S^n; r_{n,k})$ has a free $\mathbb{Z}/2\mathbb{Z}$-action that maps the convex combination $\sum_{i=1}^{j} \lambda_i \delta_{x_i}$ of Dirac measures for points $x_1, \ldots, x_j$ on S^n to $\sum_{i=1}^{j} \lambda_i \delta_{-x_i}$, that is, to the measure that is supported on the antipodal point sets with the same weights λ_i. This action is free since antipodal points on S^n are farther than $r_{n,k}$ apart.

Let $f \colon S^n \to \mathbb{R}^k$ be odd and continuous. By [3, Lem. 5.2], f induces a continuous map $F \colon \mathrm{VR}^m(S^n; r_{n,k}) \to \mathbb{R}^k$ defined by $F(\sum_{i=1}^{j} \lambda_i \delta_{x_i}) = \sum_{i=1}^{j} \lambda_i f(x_i)$. Notice that F commutes with the antipodal action on $\mathrm{VR}^m(S^n; r_{n,k})$ and S^n:

$$F\left(\sum_{i=1}^{j} \lambda_i \delta_{-x_i}\right) = \sum_{i=1}^{j} \lambda_i f(-x_i) = -\sum_{i=1}^{j} \lambda_i f(x_i) = -F\left(\sum_{i=1}^{j} \lambda_i \delta_{x_i}\right).$$

Thus, the map F, as a $\mathbb{Z}/2\mathbb{Z}$-equivariant map from an $(k-1)$-connected space to $\mathbb{R}^k$, has a zero by Theorem 17. That is, there are points $x_1, \ldots, x_j \in S^n$ that are pairwise at distance at most $r_{n,k}$ and such that $\sum_{i=1}^{j} \lambda_i f(x_i) = \vec{0}$ for some $\lambda_1, \ldots, \lambda_j \geq 0$ with $\sum_{i=1}^{j} \lambda_i = 1$. ∎

Note that Theorem 32 and Proposition 28 together imply that $s_{n,k} \leq r_{n,k}$ for all integers $n, k \geq 1$.

Conjecture 33. *For all integers $n, k \geq 1$, $s_{n,k} = r_{n,k}$.*

Conjecture 33 is true in all cases for which the values of $r_{n,k}$ and $s_{n,k}$ are known. Specifically, for each known value of $r_{n,k}$, there exists a continuous odd map $f \colon S^n \to \mathbb{R}^k$ such that all Carathéodory subsets of $f(S^n)$ have diameter greater than or equal to $r_{n,k}$.

The following table lists some of the known values of $r_{n,k}$, where $D_n = \pi - \arccos\left(\frac{1}{n+1}\right)$ denotes the diameter of a regular inscribed $(n+1)$-simplex in S^n.

Known values of $r_{n,k}$ for $k \leq 9$ and $n \leq 6$									
	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$n=1$	0	$D_1 = \frac{2\pi}{3}$	$D_1 = \frac{2\pi}{3}$	$\frac{4\pi}{5}$	$\frac{4\pi}{5}$	$\frac{6\pi}{7}$	$\frac{6\pi}{7}$	$\frac{8\pi}{9}$	$\frac{8\pi}{9}$
$n=2$	0	0	D_2	D_2					
$n=3$	0	0	0	D_3	D_3				
$n=4$	0	0	0	0	D_4	D_4			
$n=5$	0	0	0	0	0	D_5	D_5		
$n=6$	0	0	0	0	0	0	D_6	D_6	

The values of $r_{n,k}$ appearing in this table follow from the known homotopy type of Vietoris–Rips metric thickenings of spheres at the first critical scale (and, in the case of $n = 1$, by considering the known homotopy type of all Vietoris–Rips complexes defined on S^1).

Here, we present an alternative proof of the values $s_{1,k}$ using Theorem 23. Notably, this proof does not depend upon prior knowledge of the connectivity of metric thickenings of the circle.

Theorem 34. *Let $f\colon S^1 \to \mathbb{R}^{2k+1}$ be odd and continuous for any positive integer k. Then, there exists a set $X \subseteq S^n$ such that $|X| \le 2k+1$, $\operatorname{diam}(X) \le \frac{2\pi k}{2k+1}$ and $\vec{0} \in \operatorname{conv}(X)$. Furthermore, this diameter bound is optimal.*

Proof. Consider the inclusion $\iota\colon \partial\mathcal{B}_{2k+2} \to \mathrm{VR}^m(S^1; \frac{2\pi k}{2k+1})$ defined by $\sum \lambda_i \mathrm{SM}_{2k+2}(t_i) \mapsto \sum \lambda_i t_i$. This map is well-defined by Corollary 14 and continuous by [6, Lemma 20]. Furthermore, as the composition of odd maps, observe that the induced map $\tilde{f} = F \circ \iota\colon \partial\mathcal{B}_{2k+2} \to \mathbb{R}^{2k+1}$ is odd. Hence, because $\partial\mathcal{B}_{2k+2} \simeq S^{2k+1}$, Corollary 18 implies that there exists $x_0 \in \partial\mathcal{B}_{2k+2}$ such that $\tilde{f}(x_0) = \vec{0}$. Further because the Barvinok–Novik orbitopes are simplicial, we may write $x_0 = \sum_{i=1}^{2k+1} \lambda_i \mathrm{SM}_{2k+2}(t_i)$ for some convex coefficients λ_i.

Last, to see that this diameter bound is optimal, observe that $\mathrm{SM}_{2k}\colon S^1 \to \mathbb{R}^{2k} \subset \mathbb{R}^{2k+1}$ is an odd map such that all Carathéodory subsets of $\mathrm{SM}_{2k}(S^1)$ have diameter at least $\frac{2\pi k}{2k+1}$ by Theorem 23. □

In a similar manner, we can use known facts about the facial structure of $\mathcal{B}_{2k}$ and Theorem 34 obtain additional nontrivial upper bounds for the numbers $s_{n,k}$. Proofs are omitted for brevity.

Theorem 35. *Let $X \subset S^1$ denote any set of $2k+1$ evenly-spaced points. If $f\colon S^{2n-1} \to \mathbb{R}^{2k+1}$ is odd and continuous, then there is a subset $X \subseteq S^{2n-1}$ of diameter at most $\operatorname{diam}(\mathrm{SM}_{2k}(X)) < \pi$ such that $\operatorname{conv}(f(X))$ contains the origin.*

Theorem 36 ([6, Theorem 2])**.** *If $f\colon S^{2n-1} \to \mathbb{R}^{2kn+2n-1}$ is odd and continuous, then there is a subset $X \subseteq S^{2n-1}$ of diameter at most $\frac{2\pi k}{2k+1}$ such that $\operatorname{conv}(f(X))$ contains the origin.*

Preliminary experimental evidence suggests that the diameter bound in Theorem 35 improves on the bound given in Theorem 36 for sufficiently large values of k.

We conclude this section by describing a new and exciting potential direction for this research. In a personal communication, Dr. Michael Crabb, from the University of Aberdeen, provided the authors of [6] with an alternative proof of certain values of $s_{n,k}$ using principal bundles and characteristic classes in cohomology. Whereas all previously known values of $s_{n,k}$ were determined using the topology of metric thickenings, Dr. Crabb's approach does not depend on this prior knowledge. In fact, he is able to precisely determine additional values of $s_{n,k}$ which were previously unknown. As an example, Dr. Crabb shows that $s_{3,4} = s_{3,5} = s_{3,6} = s_{3,7}$; previously, only the first equality was known.

We hope to gain a more thorough understanding of Dr. Crabb's proof techniques with an eye toward adapting them to prove new results about the topology of metric thickenings. Additionally, we note that a proof of Conjecture 33, together with Dr. Crabb's results, would imply previously unknown results about the homotopy connectivity of metric thickenings of spheres.

4.3. **A connection to trigonometric polynomials and convex geometry.** A key property of the symmetric moment curve SM_{2k} is that the preimage of any of its Carathéodry subsets is bounded below by the diameter of $2k-1$ equally-spaced points in S^1 (as stated in Theorem 23).

In particular, this theorem implies that the composition $\mathrm{VR}^m(S^1; r) \xrightarrow{\mathrm{SM}_{2k}} \mathbb{R}^{2k} \setminus \{\vec{0}\} \xrightarrow{p} \partial\mathcal{B}_{2k}$ is well-defined for $r < \frac{2\pi k}{2k+1}$. However, it also allows us to prove theorems about raked trigonometric polynomials, and the proof technique of Theorem 23 suggests applications to the ordinary trigonometric moment curve and faces of the related orbitopes.

First, we describe results about the roots of trigonometric polynomials. Then, we summarize the proof technique of Theorem 23 and discuss potential applications to other convex bodies in Euclidean space.

4.3.1. **Roots of trigonometric polynomials**. Our primary result in [6] regarding the structure of roots of certain trigonometric polynomials is as follows. The proof of the first part of this theorem follows from Theorem 23, and the proof of the second part follows from Theorem 34.

Theorem 37. *Let $X \subseteq S^1$ be such that $\operatorname{diam}(X) < \frac{2\pi k}{2k+1}$. Then there is a raked homogeneous trigonometric polynomial of degree $2k-1$ that is positive on X. Moreover, there is a set $X \subseteq S^1$ of diameter $\frac{2\pi k}{2k+1}$ such that no raked homogeneous trigonometric polynomial of degree $2k-1$ is positive on X.*

As an intermediate step in proving Theorem 37, we also show the following.

Corollary 38. *Fix a list of odd degrees d_i for $1 \le i \le 2k+1$, and fix a list of trigonometric functions $f_i(t) = \sin(t)$ or $f_i(t) = \cos(t)$. Let P be the set of all polynomials of the form $p(t) = \sum_{j=1}^{2k+1} z_j f_j(d_j t)$ with $z_j \in \mathbb{R}$. Then there is a subset $X \subseteq S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that no polynomial in P is positive on X.*

For example, the above corollary applies if P is the set of all raked homogeneous trigonometric polynomials of degree at most $2k-1$, namely

$$p(t) = \sum_{j=1}^{k} a_j \cos\big((2j-1)t\big) + \sum_{j=1}^{k} b_j \sin\big((2j-1)t\big),$$

after noting that we are considering the special case in which one of the constants z_j defining $p(t) = \sum_{j=1}^{2k+1} z_j f_j(d_j t)$ is zero.

4.3.2. **Some interesting matrices and applications to convex bodies**. Next, we summarize the proof of Theorem 23 and consider applications of this proof technique to, e.g., the faces of Carathéodory orbitopes and Question 21. For notational convenience, we make the following definition.

Definition 39. Fix $\vec{t} = (t_0, \dots, t_{2k}) \in (S^1)^{2k+1}$. For $k \in \mathbb{N}$, define the $(2k+1) \times (2k+1)$ matrices

$$\mathbb{M}_{2k}\left(\vec{t}\right) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \mathrm{M}_{2k}(t_0) & \mathrm{M}_{2k}(t_1) & \dots & \mathrm{M}_{2k}(t_{2k}) \end{bmatrix}$$

and

$$\mathbb{SM}_{2k}\left(\vec{t}\right) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \mathrm{SM}_{2k}(t_0) & \mathrm{SM}_{2k}(t_1) & \dots & \mathrm{SM}_{2k}(t_{2k}) \end{bmatrix}.$$

Similarly, define the $2k \times (2k+1)$ matrices

$$\widetilde{\mathbb{M}}_{2k}\left(\vec{t}\right) = \begin{bmatrix} \mathrm{M}_{2k}(t_0) & \mathrm{M}_{2k}(t_1) & \dots & \mathrm{M}_{2k}(t_{2k}) \end{bmatrix}$$

and

$$\widetilde{\mathbb{SM}}_{2k}\left(\vec{t}\right) = \begin{bmatrix} \mathrm{SM}_{2k}(t_0) & \mathrm{SM}_{2k}(t_1) & \dots & \mathrm{SM}_{2k}(t_{2k}) \end{bmatrix}.$$

Throughout this section, we will make repeated use of the determinants of these matrices (or their submatrices). The expressions for these determinants, which are always a product of trigonometric functions, may be proved by rewriting each entry of the matrix as a complex exponential, using elementary row and column operations to reduce the matrix to a Vandermonde matrix, then rewriting the resulting product of complex exponentials in terms of trigonometric functions.

While all four of these matrices have applications to the Carathéodory and Barvinok–Novik orbitopes, exactly two of them have determinants with nice expressions as a product of trigonometric functions, while the other two are more difficult to factor. First, we describe applications of the "nice" determinants. Then, we consider the other two matrices and their desired applications.

The matrices $\widetilde{\mathbb{SM}}_{2k}\left(\vec{t}\right)$ and $\mathbb{M}_{2k}\left(\vec{t}\right)$. In [6], we characterize the nullspace of $\widetilde{\mathbb{SM}}_{2k}\left(\vec{t}\right)$ as follows. Compare this expression with $\det(V)$ in Definition 15.

Lemma 40. *If no two points $t_0, \ldots, t_{2k} \in S^1$ are equal or antipodal, then the nullspace of $\widetilde{\mathbb{SM}}_{2k}\left(\vec{t}\right)$ is one-dimensional and is spanned by $\vec{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{2k})^\intercal$, where*

$$\lambda_i = (-1)^i \prod_{\substack{0 \le j < l \le 2k \\ j,l \ne i}} \sin(t_l - t_j).$$

Now, toward proving Theorem 23, suppose there exists $X \subseteq S^1$ such that $\vec{0} \in \operatorname{conv}(\mathrm{SM}_{2k}(X))$. By Carathéodory's theorem, we may assume $|X| \le 2k+1$. Then, by definition, there exist convex coefficients $\lambda_0, \ldots, \lambda_{2k}$ and $t_i \in X$ such that $\vec{0} = \sum_{i=0}^{2k} \lambda_i \mathrm{SM}_{2k}(t_i)$. It follows that $\vec{0} \in \operatorname{conv}(\mathrm{SM}_{2k}(X))$ if and only if each λ_i in Lemma 40 has the same sign. Hence, because the sign of each λ_i depends only on the relative positions of the points $t_0, \ldots, t_{2k}$ by Lemma 40, we reduce the problem of proving $\vec{0} \notin \operatorname{conv}(\mathrm{SM}_{2k}(X))$ to a combinatorial problem about configurations of points in S^1.

With that in mind, we are able to use straightforward combinatorial arguments about the relative configuration of points in S^1 to prove that, given $X = \{t_0, \ldots, t_{2k}\}$, we have $\vec{0} \notin \operatorname{conv}(\mathrm{SM}_{2k}(X))$ whenever $\operatorname{diam}(\{t_0, \ldots, t_{2k}) < \frac{2\pi k}{2k+1}$. This proves the first part of Theorem 23. To see that this bound is sharp, one checks that the image of $2k+1$ equally-spaced points in S^1 (which has diameter exactly $\frac{2\pi k}{2k+1}$) always contains $\vec{0}$.

In fact, the (signs of the) determinants of certain submatrices of $\widetilde{\mathbb{SM}}_{2k}(\vec{x})$ play a key role in the proof that $(\iota \circ p) \circ \mathrm{SM}_4 \simeq \mathrm{id}_{\mathrm{VR}^m(S^1; \frac{2\pi}{3})}$. Again, we have a nice description of these numbers in terms of a product of trigonometric functions.

Corollary 41. *For $0 \le i \le 2k$, let $\widetilde{\mathbb{SM}}^{i}_{2k}\left(\vec{t}\right)$ denote the $2k \times 2k$ matrix obtained by removing the i-th column of $\widetilde{\mathbb{SM}}_{2k}\left(\vec{t}\right)$. Then*

$$\det(\widetilde{\mathbb{SM}}^{i}_{2k}\left(\vec{t}\right)) = \kappa \prod_{\substack{0 \le j < l \le 2k \\ j,l \ne i}} \sin(t_l - t_j),$$

for some nonzero constant κ depending only on k.

As before, we are able to determine the sign of these numbers based on the configuration of the set of points $\{t_0, \ldots, t_{2k}\}$. Hence, because both the nullspace of $\widetilde{\mathbb{SM}}_{2k}\left(\vec{t}\right)$ and the determinants of some of its submatrices have a nice description in terms of products of trigonometric functions, we suggest the following general approach to the study of the Carathéodory and Barvinok–Novik orbitopes.

Remark 42. If a geometric statement about the Carathéodory or Barvinok–Novik orbitopes can be expressed in terms of the determinant of one of the above matrices, it may be possible to use the expressions for these determinants, which are often a product of trigonometric functions, to rephrase the original statement as an equivalent combinatorial statement about points in S^1.

As another example of the utility of this approach, we show how the determinant of the matrix $\mathbb{M}_{2k}\left(\vec{t}\right)$ may be used to easily determine a subset of the faces of the Carathéodory orbitope $\mathcal{C}_{2k}$ as follows.

Proposition 43. *For $k \geq 1$,*

$$\det(\mathbb{M}_{2k}\left(\vec{t}\right)) = \kappa \prod_{0\leq j<l\leq 2k} \sin\left(\frac{t_l - t_j}{2}\right)$$

for some nonzero constant κ depending on k.

Now, fix numbers $s_1, \dots, s_{2k} \in S^1$ and define $\vec{s} = (s_1, \dots, s_{2k})^\intercal$. By considering the cofactor expansion of the determinant of $\mathbb{M}_{2k}(t, \vec{s})$ along the first column, observe that $\det(\mathbb{M}_{2k}(t, \vec{s}))$ is a degree k trigonometric polynomial in t. Writing

$$f_{\vec{s}}(t) = \prod_{1\leq j\leq 2k} \sin\left(\frac{s_j - t}{2}\right),$$

it follows that

$$\det(\mathbb{M}_{2k}(t, \vec{s})) = \left(\kappa \prod_{1\leq j<l\leq 2k} \sin\left(\frac{s_l - s_j}{2}\right)\right) f_{\vec{s}}(t) = \tilde{\kappa} f_{\vec{s}}(t)$$

for some constant $\tilde{\kappa}$. This proves the following.

Proposition 44. *For any vector $\vec{s} = (s_1, \dots, s_{2k})^\intercal \in (S^1)^{2k}$,*

$$f_{\vec{s}}(t) = \prod_{1\leq j\leq 2k} \sin\left(\frac{s_j - t}{2}\right)$$

is a degree k trigonometric polynomial in t.

Note that the roots of $f_{\vec{s}}(t)$ are precisely the numbers $s_1, \dots, s_{2k}$. Now, choose k distinct points $s_1, s_3, \dots, s_{2k-1} \in S^1$ and define $s_{i+1} = s_i$ for $i = 1, 3, 5, \dots, 2k-1$. Then, for this choice of vector $\vec{s}$ observe that $f_{\vec{s}}(t)$ is a degree k trigonometric polynomial with k distinct roots $s_1, s_3, \dots, s_{2k-1}$ such that $f_{\vec{s}}(t) \geq 0$ on S^1. Hence, because the faces of the Carathéodory orbitope $\mathcal{C}_{2k}$ are in bijection with degree k trigonometric polynomials that are non-negative on S^1, these polynomials $f_{\vec{s}}(t)$ recover the maximal-dimensional faces of $\mathcal{C}_{2k}$. In a similar way, we may obtain all lower-dimensional faces, e.g., by choosing $s_1 = s_2 = s_3 = s_4, s_5 = s_6, s_7 = s_8$, etc.

In fact, because the faces of the Carathéodory orbitopes are already known (see Theorem 10), *all* faces of $\mathcal{C}_{2k}$ arise in this way (and, conversely, the polynomials $f_{\vec{s}}(t)$ are precisely those defining each face for these particular choices of $\vec{s}$).

Again, we are able to exploit the form of these determinants, a product of trigonometric functions, to reduce the problem of finding faces of $\mathcal{C}_{2k}$ to a combinatorial problem about points in S^1. In this case, the combinatorial problem is to find vectors $\vec{s} \in (S^1)^{2k}$ giving rise to trigonometric polynomials $f_{\vec{s}}(t)$ that are non-negative on S^1, and the solutions are easy to read off directly from the expression of the polynomial. While *a priori* this approach may reveal only a subset of faces of $\mathcal{C}_{2k}$, it is nonetheless sufficient to determine all faces in this case.

The matrices $\mathbb{SM}_{2k}\left(\vec{t}\right)$ and $\widetilde{\mathbb{M}}_{2k}\left(\vec{t}\right)$. Because the precise facial structure of the Barvinok–Novik orbitope $\mathcal{B}_{2k}$ is currently unknown for $k \geq 3$, we attempt to use this technique to obtain raked trigonometric polynomials defining faces of $\mathcal{B}_{2k}$. In this case, for a fixed vector $\vec{s} \in (S^1)^{2k}$, it follows from Corollary 41 that

$$\det(\mathbb{SM}_{2k}(t, \vec{s})) = \kappa \prod_{\substack{1\leq j<l\leq 2k \\ j,l\neq i}} \sin(s_l - s_j) + \kappa \sum_{i=1}^{2k} (-1)^i \prod_{\substack{1\leq l\leq 2k \\ l\neq i}} \sin(s_l - t) \prod_{\substack{1\leq j<l\leq 2k \\ j,l\neq i}} \sin(s_l - s_j)$$

for some nonzero constant κ depending only on k. By considering the cofactor expansion of $\mathbb{SM}_{2k}(t,\vec{s})$ along the first column, we see that $\det(\mathbb{SM}_{2k}(t,\vec{s}))$ is a raked trigonometric polynomial of degree $2k-1$ in t. Unfortunately, the problem of finding vectors $\vec{s}$ such that $\det(\mathbb{SM}_4(t,\vec{s}))$ is non-negative on S^1 is much more difficult in this case, and the roots of this expression for a given vector $\vec{s}$ are not obvious.

In a similar way, the determinants of submatrices of $\widetilde{\mathbb{M}}_{2k}(t,\vec{s})$ are relevant for proving the homotopy type of Čech metric thickenings of S^1 at large scales (as described at the end of Subsection 2.1), but, again, it is difficult to control the sign and roots of the resulting trigonometric polynomials.

However, we are still optimistic that this approach may yield results. In the case $k=2$, Mathematica simplifies the above determinant as

$$\det(\mathbb{SM}_4(t,\vec{s})) = \kappa \left(\prod_{1\le l\le 4} \sin\left(\frac{s_l - t}{2}\right) \prod_{1\le j<l\le 4} \sin\left(\frac{s_l - s_j}{2}\right)\right)\left(2 + \sum_{1\le j\le 4} \cos(s_l - t) + \sum_{1\le j<l\le 4} \cos(s_l - s_j)\right).$$

As before, factoring out the constant $\kappa \prod_{1\le j<l\le 4} \sin\left(\frac{s_l-s_j}{2}\right)$ from this expression proves that

$$g_{\vec{v}}(t) = \left(\prod_{1\le l\le 4} \sin\left(\frac{s_l - t}{2}\right)\right)\left(2 + \sum_{1\le j\le 4} \cos(s_l - t) + \sum_{1\le j<l\le 4} \cos(s_l - s_j)\right)$$

is a raked trigonometric polynomial of degree $2k-1=3$ in t. Because of the product of sine functions, we note that $g_{\vec{s}}(t)$ has a root at each s_i. However, the sum of cosines makes it difficult to ensure $g_{\vec{s}}(t) \ge 0$ on S^1.

We are hopeful that the higher-dimensional analogues of these polynomials will have similar factorizations. Ultimately, we hope to develop techniques for identifying vectors $\vec{s}$ such that $g_{\vec{s}}(t) \ge 0$ on S^1, which would, in turn, allow us to identify faces of the Barvinok–Novik orbitopes.

To conclude this section, we describe how the problem of identifying these vectors $\vec{s}$ may be related to existing objects of study in representation theory. Specifically, we encounter Schur functions, which are symmetric polynomials forming a basis for the space of all symmetric polynomials [28, Theorem 3.3] and which arise in the context of representation theory as linear combinations of symmetric group representations [31, Theorem 4.6.4]. A straightforward computation shows

$$\det(\mathbb{SM}_{2k}(t,\vec{s_0}) = e_k \frac{i^k}{2^k} \omega \det \begin{pmatrix} 1 & e^{2it} & e^{4it} & \cdots & e^{(2k-2)it} & e^{(2k-1)it} & e^{(2k)it} & \cdots & e^{(2(2k-1))it} \\ 1 & e^{2is_1} & e^{4is_1} & \cdots & e^{(2k-2)is_1} & e^{(2k-1)is_1} & e^{(2k)is_1} & \cdots & e^{(2(2k-1))is_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{2is_{2k}} & e^{4is_{2k}} & \cdots & e^{(2k-2)is_{2k}} & e^{(2k-1)is_{2k}} & e^{(2k)is_{2k}} & \cdots & e^{(2(2k-1))is_{2k}} \end{pmatrix},$$

where $e_k \in \{-1,+1\}$ and $\omega = e^{-(2k-1)i(t+s_1+s_2+\cdots+s_{2k})}$. The matrix in this expression is almost a Vandermonde matrix: it has an additional column, and is referred to as a *generalized Vandermonde matrix*. It is known (see [22, 12]) that the determinant of a generalized Vandermonde matrix factors as a product of an ordinary Vandermonde determinant and a Schur polynomial. In fact, the trio of papers [12, 13, 14] present algorithms for computing the coefficients in these polynomials and formulas for approximating their roots. Along these lines, we hope to gain better control over the roots and signs of the particular family of Schur polynomials arising as factors in the determinants $\det(\mathbb{SM}_{2k}(t,\vec{s_0})$ (interpreted as raked trigonometric polynomials), consequently improving our understanding of the facial structure of the Barvinok–Novik orbitopes.

5. Conclusion

We demonstrate how the topology of metric thickenings of spheres implies Borsuk–Ulam type theorems and results about convex bodies and trigonometric polynomials, and how these geometric results, in turn, inform our understanding of metric thickenings at large scales. However, many conjectures remain open, and certain connections have not yet been fully explored. Preliminary results suggest that the intersection of these areas form a fertile landscape for both discovering new conjectures and proving new results.

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