Problem. Let $V$ be a vector space and $U$ be a finite subset of $V$. Prove that $\text{span}(U)$ is a subspace of $V$.

Proof. Let $F$ be the field over which $V$ is a vector space. Since $U$ is a finite subset of $V$, let $U = \{u_1, ..., u_n\} \subset V$. In order to prove $\text{span}(U)$ is a subspace of $V$, theorem 1.3 in the textbook tells us we must prove 1) $0 \in \text{span}(U)$, 2) $\text{span}(U)$ is closed under addition, and 3) $\text{span}(U)$ is closed under scalar multiplication.

1) Recall that $\text{span}(U) = \{a_1 u_1 + ...a_n u_n \mid a_1, ..., a_n \in F\}$ so that
$$\sum_{i=1}^{n} 0u_i = 0 \in \text{span}(U).$$
Note that this also works when $U$ is empty since the above sum would be the empty sum, to which we assign the value 0 by convention.

2) Let $x, y \in \text{span}(U)$. Then $x = a_1 u_1 + ...+ a_n u_n$ and $y = b_1 u_1 + ...+ b_n u_n$ for some $a_1, ..., a_n, b_1, ..., b_n \in F$ by definition of $\text{span}(U)$. Now it follows that
$$x + y = a_1 u_1 + ...+ a_n u_n + b_1 u_1 + ...+ b_n u_n = (a_1 + b_1)u_1 + ...+ (a_n + b_n)u_n.$$ Therefore, $x + y \in \text{span}(U)$ since each $a_i + b_i \in F$.

3) Let $c \in F$ and $x \in \text{span}(U)$. Then $x = a_1 u_1 + ...+ a_n u_n$ for some $a_1, ..., a_n \in F$. Now it follows that
$$cx = c(a_1 u_1 + ...+ a_n u_n) = ca_1 u_1 + ...+ ca_n u_n.$$ Therefore, $cx \in \text{span}(U)$ since each $ca_i \in F$. \qed