The Magus of the Infinite:
Georg Cantor’s Mathematics and Actual Infinity

Jack Graham
Collegiate Scholars Program
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Not all infinities are created equal. In the nineteenth century, mathematician Georg Cantor published his controversial findings on his new transfinite numbers, and it followed as a logical consequence that some infinites are larger than others. Cantor's work on transfinite numbers sprung out of his development of set theory, which, thanks to the advocacy of David Hilbert and other mathematicians, is widely considered the foundation of all mathematics. In the heart of set theory lies the notion of *cardinality*, a quality corresponding to the size of a set. Roughly, two sets have the same cardinality if there exists a one-to-one correspondence between them. For finite sets, this works as expected. For infinite sets, however, common intuition breaks down. Sets of numbers become equal in size to proper subsets of themselves, and multiple levels of infinity arise. Flagged by peculiar implications, Cantor's theory garnered ample criticism from mathematicians, philosophers, and theologians alike. One central objection to Cantor's results emerged from Aristotelian philosophy. Ever since Aristotle, most philosophers have denied the notion of a completed infinity—an *actual infinity*—as self-contradictory and impossible. Moreover, many theologians reject that any actual infinity exists outside of God. Despite overwhelming opposition, Cantor's set theory, including his math on infinite sets, rose to such great prominence as to become the widely accepted foundation for all of mathematics. How could Cantor's theory survive against the onslaught of criticism, and how did it rebut the opposition? Cantor's controversial ideas of the infinite overcame the opposition because many contemporaries embraced an ontological separation of mathematical ideas from physical reality. To see this, one must begin with the philosophical and theological tradition from which the opposition arises.

Everything begins with the Greeks, and thoughts on the nature of the infinite are no exception. The first assertion about the infinite comes from Anaximander of Miletus. He argued that the infinite was the primal substance out of which all things were made, the boundless source of everything.[1] For Anaximander, infinity did not belong to a specific field of study—it was "at once scientific, philosophical, and ethical," and even had a divine character.[2] Primarily a matter of philosophical and religious consideration, infinity was not conceived of in a mathematical or scientific way.[3] In contrast, the Pythagoreans abhorred the infinite, seeing it opposing the order and harmony of the finite world. More importantly, though, they perceived infinity in a quantitative and spatial way.[4] This quantitative view forms one of the two early views. The other view, a metaphysical one, arises from Parmenides, who "believed that reality—The One—must be autonomous and explicable in its own terms, a perfect unified self-subsistent whole."[5] Reality was not infinite in some capacity of extant or quantity, it was boundless in definition. It was infinite in that it had no boundary—nothing else but itself could define it. Though Parmenides did not explicitly describe The One in this way, his student, Melissus of Samos, did. Melissus declared that reality was infinite, but metaphysically and not in any mathematical sense.[6] As a result of this view, Parmenides and his followers in the Eleatic school embraced a separation of appearance and reality. Reality had no part, so change and motion could not exist. To support these views, Zeno of Elea composed many paradoxes of change and motion, such as the famous argument of Achilles and the tortoise.[7] Zeno's paradoxes, though designed to support Parmenides' metaphysical view of infinity, cast doubt on the coherence of any mathematical view of infinity. This is exemplified in his argument that reality is a unity, not

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[1] A. W. Moore, *The Infinite*, Taylor & Francis Group, 1991. *ProQuest Ebook Central*, https://ebookcentral.proquest.com/lib/hillsdale-ebooks/detail.action?docID=179281, 17.
[2] Ibid., 18.
[3] Michael Heller and W. H. Woodin, *Infinity : New Research Frontiers*, (Cambridge: Cambridge University Press, 2011), 20.
[4] Moore, 19.
[5] Ibid., 23.
[6] Moore, 24-25.
[7] James Nickel, *Mathematics: Is God Silent?*, (Ross House Books, California 1990), 26.

a plurality: “If reality were a many (alternatively, if reality were how it appears to be—having parts between any two of which there is a third), then it would have to have infinitely many parts. *But there cannot be infinitely many of anything.* So reality must be a one.”[8] Plato, with his world of eternal and timeless Ideas, likewise was primarily interested in the metaphysical, not the mathematical. From its inception, the concept of infinity has been full of philosophical assertions, theological implications, and mathematical paradoxes.

Of all the Greek philosophers, it was Aristotle who established the traditional view of infinity. He rejected much of the primarily metaphysical views that came before him, instead offering a more rational and mathematical conception.[9] In Book III of the *Physics*, Aristotle lays out his approach to infinity. First, he argues that something unlimited seems to exist, because of concepts such as time, divisibility of magnitude, and there being no end of counting numbers.[10] Second, he admits it produces paradoxes, implying it cannot exist: “Theoretical knowledge of the unlimited, though, does give rise to a puzzle. For if we posit that it does not exist, many impossibilities result, as they do if we posit that it does exist.”[11]How does he resolve the dilemma? Aristotle makes a distinction between the *potentially infinite* and the *actually infinite*.[12] In general, the infinite is defined “in virtue of one thing always being taken after another, and each thing taken is always limited, but is always *one thing followed by another*.”[13] Put shortly, the infinite is that to which something can always be added or taken. It is the capacity to continue without an end. This is what Aristotle defines as potentially infinite—that which can always be

[8] Moore, 26.
[9] Ibid., 36-37.
[10] Aristotle, *Physics*, translated by C.D.C. Reeve, (Indianapolis: Hackett Publishing Company, Inc., 2018), III, 4, 203b1.
[11] Ibid., III, 4, 203b30.
[12] Heller, 22.
[13] Aristotle, III, 6, 205b25.

more. He finds no contradiction in this conception, given that the infinite is not present all at once.[14] An infinite process that is present all at once is fundamentally impossible—this Aristotle defines as actually infinite. Aristotle argues that something cannot be actually infinite, only potentially infinite, such as a line that can continually be divided or a number that can be increased by one as many times as necessary. To illustrate this, consider what it could mean to say that a body is infinitely divisible.[15] If the body's divisibility is potentially infinite, then, given any portion of it, one could always divide it into two. Aristotle argues this is consistent. If the body's divisibility is actually infinite, then the body itself must have infinitely many parts all at once, which many, such as Zeno, would claim is absurd. By defining potential infinity and rejecting actual infinity, Aristotle resolves most of the paradoxes of the unlimited, and establishes the course of thought for the western tradition.

Though Aristotle's view remained definitive, new theological views on the nature of infinity began to develop. The Hebrew Scriptures connected God metaphorically with infinity, and Christian theologians as early as Gregory of Nyssa were claiming God's infinity.[16] Plotinus was the first influential thinker to claim God's infinity from a nonbiblical, philosophical standpoint.[17] Plotinus departed from Aristotle for a more Platonic understanding of the infinite. Despite this, he also rejected an actually infinite extent of number, but allowed for a potentially infinite process of counting.[18] Sharing a similar belief, St. Augustine argued that "God was both actually infinite and transcendent." [19] Both the quantitative and metaphysical views are held of God in St. John of Damascus' *An Exact Exposition of the Orthodox Faith*: "God then is infinite

[14] Moore, 39.
[15] Ibid., 42.
[16] Heller, 28.
[17] Heller, 24.
[18] Heller, 26.
[19] Moore, 46.

and incomprehensible and all that is comprehensible about him is his infinity and incomprehensibility."[20] Not only is infinity an attribute of God, but it is also used to argue God's existence. In Islamic philosopher al-Ghazali's cosmological argument, he supports his premise that the world began to exist by saying there could not have been an infinite regress of events in time.[21] The idea of an infinite regress requires an actually infinite and completed series of causes. In short, Aristotle's distinction between actual and potential informs both the philosophical and theological development in the early western tradition.

The most important theologian to combine Aristotle's view of infinity with theology is St. Thomas Aquinas. In his *Summa Theologica*, Part 1, Question 7, *On the Infinity of God*, he discusses God's infinity and its connections to other infinities. Aquinas first affirms that God is infinite, citing St. John of Damascus.[22] Since God is existence itself, he is both limitless and perfect, not being limited by anything. Regarding other entities, Aquinas says, "things other than God can be unlimited in some, but not in all, respects."[23] This distinction is that of the relatively infinite from that of the absolute infinite. This distinction, however, does not mean other entities can be actually infinite. In his cosmological argument for God's existence, he rejects an actually infinite regress of causes, like the aforementioned argument by al-Ghazali. Similarly, he also rejects the existence of an actually unlimited number of things.[24] In fact, Aquinas denies that any material body can be infinite, and likewise neither any mathematical object:

> For the mathematical body things are no different. For if we imagine a mathematical body in actual existence we shall have to imagine it with a form, for actuality requires

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[20] John of Damascus, *Writings*, (Catholic University of America Press, 1958). *ProQuest Ebook Central*, https://ebookcentral.proquest.com/lib/hillsdale-ebooks/detail.action?docID=3134800.
[21] William Lane Craig, *Reasonable Faith: Christian Truth and Apologetics*, (Wheaton: Crossway Books, 1984) 96.
[22] Thomas Aquinas, *Summa Theologica*, translated by Fathers of the English Dominican Province, (Westminster: Christian Classics, 1981), 95.
[23] Ibid., 99.
[24] Ibid., 107.

> form. Now the form of anything extended as such is its shape, so the body will have to have a shape. And it will therefore be limited, because a shape must be contained within a boundary or boundaries.[25]

Adam Drozdek describes Aquinas' view well: "Since all knowledge, except that stemming from the light of grace, originates from the senses, so does mathematical knowledge. The object of mathematics is not independent of the real world. It is extracted from the world by abstraction... Moreover, since nothing in the world is infinite, no particular object of mathematics can be infinite."[26] So, Aquinas affirms God's unlimitedness, but denies that any actual infinity exists apart from Him, even in the realm of number. This view spreads, solidifying the Aristotelian denial of the actual infinite, but with an additional theological component.

This rejection of actual infinity, both philosophically and theologically, informed mathematicians up to and through the time of Cantor, even as they employed mathematical techniques using the infinite. For example, Rene Descartes asserted that only God is actually infinite,[27] clarifying that the size of his coordinate plane was only indefinite, or potentially infinite.[28] Additionally, the field of calculus raised many questions about the status of the infinitely small. Isaac Newton introduced his concept of the fluxion, by which "is to be understood the ratio of the quantities, not before they vanish, nor after, but that with which they vanish."[29] That is, the fluxion is a ratio between two numbers smaller than any natural number, but not quite $\frac{0}{0}$. Gottfried Leibniz, on the other hand, referred to "infinitely small quantities." Leibniz, however, did not consider these quantities very highly, saying of them, "it will be

[25] Ibid., 103.
[26] Adam Drozdek. "Number and Infinity: Thomas and Cantor." International philosophical quarterly 39, no. 1 (1999): 36.
[27] Anne A. Davenport, "The Catholics, the Cathars, and the Concept of Infinity in the Thirteenth Century. " *Isis* 88, no. 2 (1997): 263–95. http://www.jstor.org/stable/236574, 263.
[28] Ohad Nachtomy and Reed Winegar, Infinity in Early Modern Philosophy, (Cham: Springer International Publishing, 2018), https://doi.org/10.1007/978-3-319-94556-9_4, 46.

[29] William Dunham, *Journey through Genius: The Great Theorems of Mathematics*, (Penguin Books: 1990), 248.

sufficient simply to make use of them as a tool that has advantages for the purpose of calculation, just as the algebraists retain imaginary roots with great profit."[30] It is notable that Leibniz had such a view of infinitesimals, considering he had a more sympathetic view of the actual infinite. Though he denied actual infinite numbers and collections of numbers,[31] he vigorously championed a syncategorematic view of the actual infinite, insisting on an actual infinitude of monads.[32] These unclear concepts at the foundations of calculus unnerved many, captured in Bishop George Berkeley's famous comment, "May we not call [infinitesimals] the ghosts of departed quantities?"[33] This unease was mostly abated when Augustin Louis Cauchy and Karl Weierstrass formalized the notion of the limit,[34] so that the analytical techniques of calculus rested comfortably within the potentially infinite. Support of Aristotle's views echoed even until Friedrich Gauss, who declared, "I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking."[35] Though there were a few exceptions, the actual infinite was firmly rejected up until the time of Cantor.

Now that the historical opposition has been established, it is possible to analyze Cantor's ideas that departed from such a longstanding tradition. Georg Cantor was born in Russia in 1845, though his family quickly moved to Germany where he was raised. He completed his doctorate in 1867, studying under the great Weierstrass, himself. Having inherited an interest in analysis, Cantor began studying the existence and uniqueness of trigonometric functions. Noticing that

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[30] Ibid., 249.
[31] Mark van Atten, "A Note on Leibniz's Argument Against Infinite Wholes," in Essays on Gödel's Reception of Leibniz, Husserl, and Brouwer, 23–32, (Cham: Springer International Publishing, 2014), 23.
[32] Nachtomy, 156.
[33] Dunham, 250.
[34] Ibid., 252.
[35] Øystein Linnebo and Stewart Shapiro, "Actual and Potential Infinity." Noûs (Bloomington, Indiana) 53, no. 1 (2019): 160.

problems involving singularities required a more rigorous theory of the real numbers, he began examining the differences between separate sets of numbers, investigating in particular comparisons of size between those sets. In his *Contributions to the Founding of the Theory of Transfinite Numbers*, he lays out his refined, logical description of his theory of "aggregates" or sets. A set is "any collection into a whole of definite and separate objects m of our intuition or our thought. These objects are called the 'elements' of M."[36] Furthermore, he defines the *cardinality* of a set to be the number of elements in that set, or more abstractly: "Since every element m, if we abstract from its nature, becomes a 'unit,' the cardinal number $\overline{\overline{M}}$ is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M."[37] Finally, he writes that two sets are equal in cardinality "if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other."[38] In modern mathematical terminology, two sets have the same cardinality if they can be put into a one-to-one correspondence with each other. For example, the set $\{1,2,3,4,5\}$ has the same cardinality as the set $\{2,4,6,8,10\}$. This is because each element in the first set corresponds to the element in the second set that is twice the original number. Cantor's basic conception of cardinality underlies his departure from the tradition surrounding the actually infinite.

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[36] Georg Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*, translated by Philip E. B. Jourdain, (Illinois: The Open Court Publishing Company, 1952), 85. Many mathematicians, including Cantor's contemporaries, criticized this definition as being too vague. If a set can be a collection of anything without qualification, paradoxes arise. The following is called Russel's Paradox, after Betrand Russel who discovered it. Consider the set of all sets which do not include themselves. Does it include itself? If it does, then it cannot include itself by definition. If it does not, then it must include itself by definition. This paradox is also well known as the barber's paradox.

[37] Ibid., 86.

[38] Ibid.

Though seemingly simple, this method for counting the size of a set has far-reaching consequences. Consider the infinite set N of all natural numbers, that is, $N = \{1,2,3,4,5 \dots\}$. Likewise, consider the infinite set E of all even natural numbers, that is, $E = \{2,4,6,8,10 \dots\}$. Consider the correspondence given in the previous example, so that each element of N is mapped to the element of E which is exactly twice as large. Then, 1 corresponds to 2, 2 corresponds to 4, and so on. As a result, there is a one-to-one correspondence between the two sets. Therefore, they have the same cardinality. A paradox immediately arises, however, because E is entirely contained in N.[39] Normally, the whole is greater than its part, but Cantor's definition of cardinality defies this. The consequences extend further. Consider the set of all integers, $Z = \{\dots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3 \dots\}$. Now, rearrange Z so that it begins with 0 and alternates between increasing positive and negative integers: $Z = \{0,1,-1,2,-2,3,-3 \dots\}$. Then, mapping N by this rearrangement, 1 corresponds to 0, 2 to 1, 3 to –1, and so on. By snaking between negative and positive numbers, the set of natural numbers has a one-to-one correspondence to the integers. As a result, Z, a set that extends to infinity in two directions, is the same size as N. For one last example, consider the set of all rational numbers, Q. Georg Cantor, using the process of diagonalization, was able to prove that Q, too, had the exact same cardinality as the natural numbers.[40] Cantor gave the name $\aleph_0$ to this cardinal number representing the size of N, Z, and Q. With his simple definition, Cantor showed that so many well-known infinite sets are equal.

However, not all infinities are created equal. Even before his work on set theory, he showed, in his 1874 paper *On a Property of the Collection of All Real Algebraic Numbers*, that

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[39] This confirmed similar geometric results discovered by Galileo Galilei a few centuries earlier. Galileo compared two line segments, one twice as long as the other. He drew lines though points on the two segments in such a way that there was exactly one line between any given point on the first segment and a corresponding point on the second. This geometric one-to-one correspondence suggested both lines had the same number of points, despite their differences in length.

[40] Dunham, 256.

there was a set of even greater cardinality than $\aleph_0$.[41] More technically, he proved the non-denumerability of the continuum. The continuum is the set of all real numbers, and a set is denumerable if it can be put in a one-to-one correspondence with the set of natural numbers. So, Cantor proved that the real number line was a separate cardinality from $\aleph_0$. To do this, he used a proof by contradiction, the informal idea of which is reproduced here. Suppose the continuum between 0 and 1 were denumerable, that is, suppose that there existed a one-to-one correspondence between the real number interval (0,1) and N. Then set up each real number in a list next to the natural number it corresponds to:

N	Real Numbers in (0,1)
1	0.5000000000…
2	0.3333333333…
3	0.1718281828…
4	0.7428571429…
5	0.1415326535…
…	…
n	$0.a_1a_2a_3a_4a_5 \ldots a_i \ldots$
…	…

Now, define the real number b defined as follows. Let $b = 0.b_1b_2b_3b_4b_5 \ldots b_i \ldots$ such that $b_i = 7$ if $a_n = 3$, otherwise $b_i = 3$. In other words, for each successive digit of b, make it 3 unless the corresponding digit of the corresponding natural number is 3. In that case, make the digit 7. In this example, the first decimal digit of the first real number is 5, so that $b_1 = 3$. The second decimal digit of the second real number is 3, so that $b_2 = 7$. In the end, $b = 0.37337 \ldots$. By construction, at least one digit of b is different than every real number in the correspondence. Thus, b does not have a corresponding natural number, which contradicts our assumption. Therefore, the continuum is not denumerable. Furthermore, since the set of all real numbers contains the set of natural numbers, it cannot have a smaller cardinality, and since it is neither

[41] Joseph Warren Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite*, (Cambridge, Mass: Harvard University Press, 1979), 50.

smaller nor equal, it must have a greater cardinality. Cantor eventually called this greater cardinality of the real numbers $\aleph_1$, which is the "next size"[42] of infinity from $\aleph_0$.

Moreover, Cantor showed that there were greater cardinalities than $\aleph_1$. One essential tool for doing so was the *power set*. The power set of a set M, denoted $P(M)$, is the set of all possible subsets of M. For example, let $M = \{1,2,3\}$. Then the power set is $P(M) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$, with $\emptyset$ being the empty set. Intuitively, $M < P(M)$ for any finite set M. In 1891, Cantor proved that the power set is strictly greater than the original set, even for infinite sets.[43] As a result, $\aleph_1 < P(\aleph_1)$, and $P(\aleph_1)$ is the "next size" of infinity. This process can be continued: $P(\aleph_1) < P(P(\aleph_1)) < P\left(P(P(\aleph_1))\right) < \cdots$. Furthermore, if the consecutive power sets are renamed and several ancillary theorems are proven, the following infinite, strictly increasing sequence is established: $\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \cdots < \aleph_n < \cdots$. Cantor called the numbers in this sequence *transfinite numbers*—numbers not equal to any finite number—since adding a new element to them yields the same number.[44] The transfinite numbers, described by some as "stepping stones to the throne of God,"[45] extend much further into infinities of larger and larger infinities.

Not only did Georg Cantor establish a whole new number system with his transfinite numbers, he affirmed that they were actual infinities. In his *Contributions to the Founding of the*

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[42] Cantor was not able to prove that $\aleph_1$ was the immediate next size of infinity up from $\aleph_0$. This infamous problem is known as the Continuum Hypothesis and was the first of Hilbert's famous 23 problems for mathematicians heading into the 20th century. Later, Paul Cohen proved that the Continuum Hypothesis was independent of Zermelo-Fraenkel axiomatizations of set theory, meaning it did not follow from set theoretic axioms as necessarily true or false.
[43] Dunham, 273. This tool led to additional criticism on Cantor's vague definition of a set, leading to a paradox of sets that Cantor himself discovered, often called Cantor's paradox. It goes as follows. Let A be the set of all sets. Since A is a set, $P(A)$ is also a set such that $A < P(A)$. However, since $P(A)$ is a set, it is a subset of A. So, $P(A) \leq A$. But this implies $A < A$, a contradiction. The solution to this paradox is that there is no set of all cardinalities.
[44] Cantor, 104.
[45] Joanna Van Der Veen and Leon Horston, *Cantorian Infinity and Philosophical Concepts of God*. https://philarchive.org/archive/VANCIA-15, 2.

Theory of Transfinite Numbers, he describes his theory as "the theory of the actually infinite or transfinite cardinal numbers."[46] Before his work, the commonly accepted notion of the infinite was that of the potentially infinite—a variable that could increase without bound. Joseph Dauben, an expert on Georg Cantor, writes, "in contrast to these, Cantor distinguished proper or *actual* infinities. The best examples of these, he suggested, were his new numbers, the transfinite numbers."[47] Cantor deliberately departed from the familiar notation of ∞ for infinity to reflect the completed, number-like nature of his transfinites.[48] In 1883, he published his *Foundation of a General Theory of Manifolds*, which prefigured *Contributions*, establishing not only his mathematics but also his philosophy. Dauben identifies that "one goal of [*Foundation*] was to demonstrate that there was no reason to accept the old objections to completed, actual infinities and that it was possible to answer mathematicians like Gauss, philosophers like Aristotle, and theologians like Thomas Aquinas in terms they would find impossible to reject."[49] Cantor's new math was unmistakably poised against the prevailing philosophical tradition.

Now, set theory is widely accepted among mathematicians. For many, it has secured its home as the foundation of all mathematical fields. The classic theorems of algebra, geometry, and calculus are all translatable into set theory. How could Cantor's ideas prevail against the overwhelming opposition to the actual infinite? An examination of the philosophical history reveals the answer. Cantor's controversial ideas overcame the opposition because many contemporaries embraced an ontological separation of mathematical ideas from physical reality. Individual ontologies varied, yet the positions most commonly adopted after Cantor have all incorporated one common agreement: mathematical objects need not reflect reality. This

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[46] Cantor, 97.
[47] Dauben, 97.
[48] Ibid., 99.
[49] Ibid., 122.

becomes evident when analyzing Cantor's own philosophy, the past opposition to his ideas, and the acceptance set theory eventually received.

To defend his transfinite numbers, Cantor had to find an explanation for the paradoxes of the infinite besides the Aristotelian distinction between potential and actual. His solution, as Dauben explains, was that "whatever mathematicians may have assumed in the past, finite properties could *not* be predicated in all cases of the infinite."[50] It is false that every property of finite numbers must apply to infinite numbers in the exact same way. Infinite numbers form a separate class than finite numbers. Consequently, it would be invalid to argue that infinite numbers cannot exist merely because they do not satisfy every property of finite numbers. To illustrate, consider the premise that the whole is greater than its part. Clearly, if a circle is cut in half, the resulting half circle must be smaller than the whole. Likewise, if one is subtracted from a number, the resulting number is, by definition, one less than the original number. This premise is a common notion of reasoning easily accepted by rational thinkers. In terms of infinite sets, however, this premise breaks down. As shown previously, the set E contains only every other element of the set N, yet they have the same cardinality. In other words, N, the whole, is not greater than E, its part. This is no contradiction. It only becomes one if it is assumed that the starting premise applies to infinite sets as it does to finite sets. Why should a premise derived from an intuition of finite numbers apply to infinite sets? Similarly, many argued that, during addition, "finite numbers would be swallowed up by any infinite number or magnitude."[51] Consider two positive numbers a and b. Clearly, their sum must be strictly greater than either of the two numbers: $a + b > a$ and $a + b > b$. With infinite sets, however, this also breaks down. In

[50] Ibid., 122.
[51] Ibid., 122.

general, adding one element to an infinite set does not change its cardinality: $\infty + b = \infty$ and $a + \infty = \infty$.[52] Dauben writes that "Cantor condemned this kind of argument... on the grounds that it was fallacious to assume that infinite numbers must exhibit the same arithmetical characteristics as did finite numbers."[53] With this explanation, Cantor resolved many paradoxes of the infinite by separating his new mathematical objects from material intuitions.

This separation, however, assumes that numbers can exist apart from material intuitions. Cantor's metaphysics support this. For Cantor, it was not material intuition that justified a number's existence, but the number's logical consistency:

> In particular, in introducing new numbers, mathematics is only obliged to give definitions of them, by which such a definiteness and, circumstances permitting, such a relation to the older numbers are conferred upon them that in given cases they can definitely be distinguished from one another. As soon as a number satisfies all these conditions, it can and must be regarded as existent and real in mathematics.[54]

If a mathematical concept makes sense abstractly, it exists. As Dauben explains, "for mathematicians, only one test was necessary: once the elements of any mathematical theory were seen to be consistent, then they were mathematically acceptable."[55] This view was not entirely original to Cantor. Earlier in the century, non-Euclidean geometry rose as a prominent mathematical theory. Non-Euclidean geometries were as logically consistent as Euclidean geometry, and yet no more than one could be true in physical reality. Consequently, math had to accept both and search for a new basis for truth—something more akin to logical consistency. Cantor reinforced this methodology with an ontological separation between mathematical objects

[52] Cantor did, in fact, design a way to "add one to infinity" and get a different number. This aspect of transfinite arithmetic can be found in *Contributions to the Founding of the Theory of Transfinite Numbers*, and explained by various authors, including Joseph Dauben. Here, however, I use $\infty + a = \infty$ to mean something more akin to the following. Let A be a set of finite cardinality $a \in N$, and B a set of cardinality $\aleph_0$. Then the cardinality of the set $A \cup B$ is also $\aleph_0$.
[53] Dauben, 122.
[54] Ibid., 128-129.
[55] Ibid., 128.

and their instantiations in reality. Numbers can be considered in two ways: as *immanent* and *transient*.[56] The immanent reality of a number is its existence insofar as it is well-defined in the mind. It is the number as a thought or ideal. In contrast, the transient reality of a number is that which "numbers could assume concretely, manifest in objects of the physical world."[57] Both of these realities exist for a number, so it is valid to study either the transient or immanent reality. In particular, one can study the immanent reality of a number without knowing anything about its transient reality. As a result, since math is the study of immanent number, mathematicians are free to define and invent new mathematical concepts without worrying about their physical manifestation. This argument culminates in Cantor's declaration that "the essence of mathematics lies entirely in its freedom."[58] He elaborates in *Foundations*: "because of this extraordinary position which distinguishes mathematics from all other sciences, and which produces an explanation for the relatively free and easy way of pursuing it, it especially deserves the name of *free mathematics*, a designation which I, if I had the choice, would prefer to the now customary 'pure' mathematics."[59] In his metaphysics, Cantor departs from the tradition that all numbers must reflect their material instantiations.

In fact, the traditional rejection of the actual infinite hinges on this specific ontology. This was true of Aristotle. In Book II of the *Physics*, Aristotle distinguishes the mathematician from the natural philosopher. Speaking of mathematical objects and physics, he writes:

> Now the mathematician too busies himself about these things, although not insofar as each of them is the limit of a natural body, nor does he get a theoretical grasp on the coincidents of natural bodies insofar as they are such. That is why he separates them. For

[56] Ibid., 132.
[57] Ibid.
[58] Maro Livio, *Is God a Mathematician?,* (Simon and Schuster Paperbacks: 2009), 169.
[59] Dauben, 132.

> they are separable in the understanding from movement, and so their being separated makes no difference, nor does any falsehood result from it.[60]

Aristotle noted how math, in some capacity, is separate from reality. This separation informs Aristotle's view of the actual infinite. Specifically, he believed that mathematical objects are abstractions.[61] Abstractions are inherently tied to and stem from reality. If, then, material things are "the primary substance and source of reality,"[62] any impossibility in reality must translate to those things that arise from it. Thus, a mathematical actual infinity is impossible for Aristotle, because an actual infinity is impossible in reality. For example, Aristotle believed that time was potentially infinite, since it always kept on ticking. It could not, however, be actually infinite, since that would require all of time to be completed, which is impossible in reality. As A. W. Moore puts it, "for Aristotle, the infinite was the untraversable. But traversal takes time. So there is no making sense of the claim that something is untraversable save with respect to the whole of time."[63] Moreover, Aristotle attributed many paradoxes of natural phenomena to the actual infinite. It is absurd to say that Achilles has to run through infinitely many points to pass the tortoise, and that is an actual infinity. With potential infinity, there is simply no end to how many times the path he runs can be divided.[64] These paradoxes motivate Aristotle to reject an actual infinite, paradoxes leading to physical absurdities. Fundamentally, math was tied to physical reality. It is this ontological understanding that underlies Aristotle's rejection of a mathematical actual infinite.

Similarly, Aquinas' rejection of the actual infinite stems from an Aristotelian ontology of math. In his rejection of material infinity, Aquinas says, "if we imagine a mathematical body in

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[60] Aristotle, II, 2, 141a1.
[61] Nickel, 34.
[62] Morris Kline, *Mathematics: The Loss of Certainty*, (New York: Oxford University Press, 1980) ,17.
[63] Moore, 40.
[64] Ibid., 42.

actual existence we shall have to imagine it with a form, for actuality requires form."[65] To consider mathematical bodies, Aquinas immediately imagines them in actual existence. While a geometer can invent his own shapes with which to reason, those shapes derive from reality. Furthermore, "since the form of quantity as such is figure, such a body must have some figure, and so would be finite; for figure is confined by a term or boundary."[66] Aquinas restricts mathematical objects to the figure thought up to express them. They are abstractions from reality, so they are tied to and limited by reality. Even the geometer's infinite line is only as long as is needed: "a geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite."[67] As abstractions from matter, mathematical objects cannot be actually infinite, because "the infinite of quantity... belongs to matter."[68] It is Aquinas' mathematical ontology that fuels his rejection of the actual infinite.

In contrast, Cantor's metaphysics allows for a belief in a mathematical actual infinite. Since a transient actual infinite is responsible for contradictions, it should instead be possible to establish an immanent actual infinity. More specifically, according to Cantor, transfinite numbers only depend on their own logical consistency for existence, not on their relationship to physical reality. Therefore, if a logically consistent system of transfinite numbers can be established, then they must exist. This is exactly what Cantor claimed he was developing. He supports his case by comparing transfinite numbers to irrational numbers. For the Pythagoreans, irrationals were problematic because they did not share all their properties with previously known numbers—

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[65] Aquinas, 103.
[66] Ibid.
[67] Ibid. Aquinas, here, is arguably incorrect. Euclid's Definition 23 requires that parallel lines be produced infinitely, so that they never meet in either direction they extend. Aquinas' view was, however, in line with Aristotle's view.
[68] Ibid.

they were not expressible as a ratio of two integers. However, they arose as a natural consequence of established theories, namely, the Pythagorean theorem. Though they had somewhat different properties, they were eventually accepted as a natural consequence. In later centuries, irrationals were formally defined using infinite sequences of rational numbers.[68] Cantor pointed out that this definition presupposed a completed infinity. In order for a sequence to keep producing rational numbers sufficiently close to the desired irrational, there had to be an infinite set of rationals from which to draw. Potentially infinite sequences necessitated actually infinite sets. Thus, transfinite numbers could be seen as an extension of irrational numbers:

> The transfinite numbers themselves are in a certain sense *new irrationals*, and in fact I think the best way to define the *finite* irrational numbers is entirely similar... One can absolutely assert: the transfinite numbers *stand or fall* with the finite irrational numbers; they are alike in their most intrinsic nature; for the former like these latter (numbers) are definite, delineated forms or modifications of the actual infinite.[69]

Though Cantor believed his transfinites had a material existence like irrationals,[70] only their logical consistency was necessary for existence. By his ontology and mathematics, Cantor argued for a belief in a mathematical actual infinite.

Cantor's metaphysics influenced each of the philosophical schools seeking to establish a foundation for mathematics. The first school was logicism. The school of logicism began with Bertrand Russel and Alfred North Whitehead in their effort to reduce all mathematics to logic. Russel thought very highly of Cantor, considering him "as one of the greatest intellects of the nineteenth century."[71] Whitehead, in *Science and the Modern World*, puts forward the logicist thesis on the ontological status of math:

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[69] Dauben, 128.
[70] Ibid., 126. Cantor argued that the set of all material monads had a power of $\aleph_0$, and the set of all aethereal monads had power $\aleph_1$. Thus, in the same way that $\sqrt{2}$ had a geometric instantiation as the diagonal of a triangle with length 1 legs, so certain transfinite numbers had instantiations that could be used to solve problems in physics.
[71] Ibid., 1.

> The originality of mathematics consists in the fact that in mathematical science connections between things are exhibited which, apart from the agency of human reason, are extremely unobvious. Thus the ideas, now in the minds of contemporary mathematicians, lie very remote from any notions which can be immediately derived by perception through the senses; unless indeed it be perception stimulated and guided by antecedent mathematical knowledge.[72]

Whitehead, here, maintains that math derives from reality. He admits, though, that mathematical objects are so separated from physical reality, that the connection between them has become "extremely unobvious." Math has been abstracted so far from reality that "the certainty of mathematics depends upon its complete abstract generality," and in its particular application, "we can have no *a priori* certainty that we are right in believing that the observed entities in the concrete universe form a particular instance of what falls under our general reasoning."[73] In short, math is only from reality in the sense that it is completely abstracted logic. This view developed Gottlob Frege's criticism of Cantor's methods. Cantor claimed that the cardinality of a set, no matter how big, could be simply "abstracted" from the set.[74] Frege, however, required a more definite and logical process to determine cardinality. For him, vague abstraction was in no way justifiable, because it lacked the logical rigor and complete generality required to justify mathematical concepts.[75] He believed, like Cantor, that logical consistency justified existence, but criticized Cantor for not being logical enough.[76] Similarly, other logicists such as Guiseppe Peano and Ernst Zermelo sought to fix and improve Cantor's theory, using logical axiomatic systems to do so.[77] In the end, the school of logicism adopted Cantor's math, despite

[72] Alfred North Whitehead. *Science and the Modern World*, (1925), 20.
[73] Ibid., 23.
[74] Cantor, 86.
[75] Dauben, 223.
[76] Ibid., 220.
[77] Ibid., 223.

philosophical disagreements, because it held to Cantor's metaphysic of how math derives justification.

The second school of math is formalism, championed by David Hilbert. Hilbert had a high view of Cantor, calling his theory of transfinite numbers "the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity."[78] When confronted with the infamous paradoxes found within set theory, Hilbert responds, "wherever there is any hope of salvage, we will carefully investigate fruitful definitions and deductive methods. We will nurse them, strengthen them, and make them useful. No one shall drive us out of the paradise which Cantor has created for us."[79] In short, he was committed to set theory. Despite this, Hilbert did not share all of Cantor's sentiments regarding the infinite. Whereas Cantor believed transfinites existed materially, Hilbert asserted that "the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought."[80] Instead, he concludes that "the role that remains for the infinite to play is solely that of an idea."[81] Modern science suggested everything was finite—non-Euclidean geometry provided grounds to doubt the infinite extension of space, and the discovery of atoms and quanta of energy threw doubt on infinite divisibility. Consequently, the infinite was not in reality, only in the mind. This corresponded to Hilbert's formalist view of mathematics. Instead of being reduced to logic, math was abstracted away from any inherent meaning at all. Rather, math was the manipulation of meaningless symbols that might coincidentally correspond to reality:

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[78] David Hilbert, "On the Infinite," in Philosophy of Mathematics, ed. with an introduction by Paul Benacerraf and Hillary Putnam (Englewood Cliffs, N.J.: Prentice-Hall, 1964), https://math.dartmouth.edu/~matc/Readers/HowManyAngels/Philosophy/Philosophy.html.
[79] Ibid.
[80] Ibid.
[81] Ibid.

> In exact analogy to the transition from material number theory to formal algebra, we now treat the signs and operation symbols of the logical calculus in abstraction from their meaning. Thus we finally obtain, instead of material mathematical knowledge which is communicated in ordinary language, just a set of formulas containing mathematical and logical symbols which are generated successively, according to determinate rules.[82]

Since math was inherently conceptual, mathematical infinity was inherently conceptual. This fits Cantor's idea of immanent number. Numbers, as formal symbols, need not be instantiated in reality. So, material paradoxes have no power to stop mathematicians from using concepts that find success in the abstract. Therefore, it was a separation of math from reality that allowed Hilbert to justify Cantor's transfinite numbers.

Cantor's work, however, was not without complete rejection—rejection found primarily in the third school of intuitionism. Early intuitionist Henri Poincaré saw Cantor's set theory as a disease in need of a cure: "the important thing is never to introduce entities not completely definable in a finite number of words. Whatever be the cure adopted, we may promise ourselves the joy of the doctor called in to follow a beautiful pathological case."[83] The seeds of intuitionism began with the bold finitist, Leopold Kronecker. Kronecker was Cantor's largest opponent, calling him "a scientific charlatan, a renegade, a 'corrupter of youth'."[84] Kronecker advocated that all numbers be constructable on the basis of the integers, having famously said, "God made the integers, but all else is the work of man."[85] E. J. Brouwer, a later mathematician, formalized this philosophy into the school of intuitionism. As Morris Kline describe it, "Brouwer conceived of mathematical thinking as a process of mental construction which builds its own universe, independent of experience and restricted only insofar as it must be based upon the

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[82] Ibid.
[83] Nickel, 186.
[84] Ibid.
[85] Ibid., 189.

fundamental mathematical intuition."[86] Brouwer allowed for certain potentially infinite sets, so long as things are not all present at once, as the mind abstracts from larger and larger finite sets to create an infinite set.[87] Intuitionism requires that all objects be constructible, not simply described.[88] It is for this reason that π is acceptable, since it can be calculated to any arbitrary degree of accuracy. It is notable that, though intuitionism rejects Cantor's transfinite numbers, it still maintains a portion of his metaphysics. As James Nickel describes it,

> Because of its Kantian underpinnings, this school presupposes that truth in mathematics can be known explicitly in the intuitive capabilities of man's mind. That mathematics reflects and expresses the laws inherent in the pre-established and ordered patterns of the universe is of no importance to the discussion. Hence, the question as to why mathematics works, i.e., why it describes the workings of the physical world so accurately, is left open and unanswered."[89]

Intuitionism, though in opposition to Cantor, does not reject a separation of mathematics from reality. Intuitionists may still reject transfinite numbers, but the basis of that rejection is no longer the philosophical notion of the actual infinite.

Mathematicians, however, were not the only ones to accept Cantor's work by considering his metaphysics—the Church itself had a significant collision with transfinite numbers. All throughout Cantor's life, religion was extremely important. William Dunham explains well why Cantor would maintain a strong connection between his theology and his mathematics: "Cantor had converted from Judaism to Protestantism, whereas his wife was born a Roman Catholic. With such an eclectic mix of religious perspectives, it is no surprise that young Georg developed a lifelong interest in theological matters."[90] This lifelong interest would heavily influence his later mathematical discoveries. As E. T. Bell remarks, Cantor had "acquired a singular taste for

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[86] Kline, 234.
[87] Ibid., 235.
[88] Ibid., 238.
[89] Nickel, 188-189.
[90] Dunham, 252.

the endless hairsplitting of medieval theology. Had he not become a mathematician it is quite possible that he would have left his mark on philosophy or theology."[91] Since religion was so important, Cantor wanted the Roman Catholic Church to examine the implications of his philosophy, to prevent him from falling into serious theological errors.[92] Cantor believed that his transfinite numbers existed as ideas in the mind of God,[93] and wanted to confirm he was not contradicting established dogma. On the contrary, for this belief about the mind of God, Cantor received approval from Catholic theologian Constantin Gutberlet. Gutberlet even used Cantor's math to defend his own use of actually infinite numbers.[94] He did so by asserting that an infinite sequence in God's intellect cannot be continually revealed and thus potentially infinite. Instead, it must exist all at once since God is unchanging: "in the absolute mind the entire sequence is always in actual consciousness, without any possibility of increase in the knowledge or contemplation of a new member of the sequence."[95] In Gutberlet's studies, Cantor found Church approval for the immanent reality of transfinite numbers.

In the transient reality of number, however, Cantor was at odds with the Church. Although Cantor and Gutberlet agreed that transfinite numbers actually existed in God's mind, they disagreed as to whether they existed in physical reality.[96] Cantor held that his transfinite numbers did have material existence, a position which Cardinal Johannes Franzelin called dangerous. Franzelin asserted that any actual, physical infinity "could not be defended and in a certain sense would involve the error of Pantheism."[97] It would be an attempt to equate God's

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[91] Bell, 559.
[92] Dauben, 231.
[93] Ibid., 228.
[94] Ibid., 144.
[95] Ibid., 143.
[96] Ibid., 145.
[97] Ibid. Comparisons can be drawn here to Spinoza's *natura naturans/natura naturata* distinction.

infinity with a temporal, physical infinity—if God's infinity were the same as nature's infinity, it would be the same as saying God was nature. To combat this danger, Cantor distinguished the actual infinite from what he called the *Absolute infinite.* The former is the created infinity of his transfinite numbers, applying to a quantity of objects in the universe, whereas the latter is reserved entirely for God and his attributes.[98] With this distinction between what is physically created and what belongs immaterially to God, Franzelin approved of Cantor:

> Thus the two concepts of the Absolute-Infinite and the Actual-infinite in the created world or in the *Transfinitum* are essentially different, so that in comparing the two one must only describe the former as *properly infinite*, the latter as improperly or equivocally infinite. When conceived in this way, so far as I can see at present, there is no danger to religious truths in your concept of the *Transfinitum*.[99]

With regard to the immanent existence of number, Cantor found acceptance only in separating his transfinites from existence in physical reality. Moreover, with regard to transient existence, he found acceptance only when he distinguished between what was physical and what was supernatural. In both instances, Cantor reconciled his theories with the Church by separating mathematical objects' physical existence from their ideal existence.

This ontological separation is found not only in Cantor's contemporaries, but in many mathematicians, philosophers, and theologians ever since the advent of set theory. For example, Christian apologist William Lane Craig makes such a distinction in *Reasonable Faith*. In his defense of the cosmological argument, Craig argues against an infinite regress of causes by arguing that an actually infinite number of things cannot exist. Craig acknowledges the work of Cantor on the topic, but asserts that math holds a different sphere of influence. He argues that many mathematicians "would simply insist that acceptance of the mathematical legitimacy of

[98] Ibid.
[99] Ibid., 146.

certain notions does not imply a commitment to the metaphysical reality of various objects."[100] He allows for Cantor's work in a formalist, anti-realist sense, claiming, "one may consistently hold that while the actual infinite is a fruitful and consistent concept within the postulated universe of discourse, it cannot be transposed into the real world, for this would involve counter-intuitive absurdities."[101] As another example, mathematician G. H. Hardy presents his beliefs in his essay, *A Mathematician's Apology*. He first establishes that there is a physical reality, "the material world, the world of day and night, earthquakes and eclipses, the world which physical science tries to describe."[102] In contrast, he acknowledges a mathematical reality, distinct from the physical reality. Hardy describes how mathematicians disagree about the nature of this mathematical reality, and that "a man who could give a convincing account of mathematical reality would have solved very many of the most difficult problems of metaphysics. If he could include physical reality in his account, he would have solved them all."[103] The separation of mathematical objects from reality that allows many to accept Cantor's set theory extends to many.

Though many mathematicians, philosophers, and theologians have rejected an actual infinity, and there are those even today who do not accept transfinite numbers, Cantor's work has staked a claim as one of the most influential theories in mathematics. Despite centuries of opposition from a wide variety of thinkers, set theory has settled down firmly at the foundation of the queen of the sciences. How did Cantor's theory overcome the opposition? Mathematics has come to accept set theory due to an understanding of an ontological separation between mathematical objects and reality. This understanding takes different forms for different thinkers,

[100] Craig, 117.
[101] Ibid.
[102] G. H. Hardy, *An Annotated Mathematician's Apology*, annotated by Alan J. Cain. Lisbon, (2019), 45.
[103] Ibid.

even those who have opposed Cantor. Regardless of its flavors, the result is the same—math need not reflect reality. This is not to say math cannot apply to reality. Indeed, math absolutely has an "unreasonable effectiveness" in its applications. However, there is something to be gained in seeing mathematics as possessing the nature of something else, something beyond the material. When sets are considered beyond their material instantiation, they become counter-intuitive. Infinite sets can be equal in size to proper subsets of themselves. The whole can be equal to the part, and a limitless number of points can lie inside a bounded interval. Instead of opposing these notions, mathematicians can use established logic to explore things beyond their prior understanding. Is this not what philosophy and theology do as well? The doctrine of the Trinity, for example, states that God is three persons, each equal in power and glory, unified in one Godhead. How can man understand this? Perhaps Cantor can offer an analogy. With infinite sets, it is perfectly logical that three sets be equal in cardinality yet unified in one set no greater than any of the three. When dealing with the infinite, conclusions are counter-intuitive, but thanks to Cantor, they need not be contradictory. Mathematicians, philosophers, and theologians alike must recognize what lies beyond them. They must marvel at how the finite intersects the infinite. They must wonder at the "stepping stones to the throne of God."[104]

[104] Veen, 2.

Bibliography

Aquinas, Thomas. *Summa Theologica*. Translated by Fathers of the English Dominican Province. Westminster, Md: Christian Classics, 1981.

Aristotle. *Aristotle Physics*. Translated by C. D. C. Reeve. Indianapolis: Hackett Publishing Company, 2018.

Bell, Eric Temple. *Men of Mathematics.* New York: Simon and Schuster, 1937.

Bigelow, John. "God and the New Math." Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition 84, no. 2/3 (1996): 127-54. http://www.jstor.org/stable/4320712.

Bruce Rosenstock. "Georg Cantor and the Mathematics of God." In Transfinite Life, 42–. Indiana University Press, 2017.

Cantor, Georg. *Briefe.* Translated by Herbert Meschkowski and Winfried Nilson. Berlin, Heidelberg: Springer Berlin Heidelberg, 1991.

———. *Contributions to the Founding of the Theory of Transfinite Numbers*. Translated by Philip E. B. Jourdain. La Salle, Ill: Open Court Publishing Company, 1952.

Cohen, Daniel. *Equations from God: Pure Mathematics and Victorian Faith.* Baltimore: The Johns Hopkins University Press, 2007. ProQuest Ebook Central.

Cohoe, Caleb. "There Must Be a First: Why Thomas Aquinas Rejects Infinite, Essentially Ordered, Causal Series." *British Journal for the History of Philosophy* 21, no. 5 (2013): 838–856.

Corry, Leo. *A Brief History of Numbers*. Oxford: Oxford University Press, 2015.

Craig, William Lane. *Reasonable Faith: Christian Truth and Apologetics*. Wheaton, Illinois: Crossway Books, 1984.

Dauben, Joseph Warren. *Georg Cantor: His Mathematics and Philosophy of the Infinite.* Cambridge, Mass: Harvard University Press, 1979.

———. "Georg Cantor and Pope Leo XIII: Mathematics, Theology, and the Infinite." *Journal of the History of Ideas* 38, no. 1 (1977): 85–108. https://doi.org/10.2307/2708842.

Davenport, Anne A. "The Catholics, the Cathars, and the Concept of Infinity in the Thirteenth Century." Isis 88, no. 2 (1997): 263–95. http://www.jstor.org/stable/236574.

Deiser, Oliver. "On the Development of the Notion of a Cardinal Number." *History and Philosophy of Logic* 31, no. 2 (2010): 123–143.

Drozdek, Adam. "Number and Infinity: Thomas and Cantor." *International Philosophical Quarterly* 39, no. 1 (1999): 35–46.

Dunham, William. *Journey through Genius: The Great Theorems of Mathematics*. New York: Penguin Books, 1990.

Ebert, Philip A, and Marcus Rossberg. “Cantor on Frege’s Foundations of Arithmetic: Cantor’s 1885 Review of Frege’s Die Grundlagen Der Arithmetik.” *History and Philosophy of Logic* 30, no. 4 (2009): 341–348.

Edwards, Harold. “Kronecker’s Place in History.” In *History and Philosophy of Modern Mathematics*, 139–. University of Minnesota Press, 1988.

Ewald, William Bragg. *From Kant to Hilbert: a Source Book in the Foundations of Mathematics.* Oxford: Clarendon Press, 2005.

Ferreirós, José. “On the Relations Between Georg Cantor and Richard Dedekind.” *Historia Mathematica* 20, no. 4 (1993): 343–363.

———. “The Motives Behind Cantor’s Set Theory – Physical, Biological, and Philosophical Questions.” *Science in Context* 17, no. 1-2 (2004): 49–83.

Frápolli, María José. “Non-Representational Mathematical Realism.” *Theoria: An International Journal for Theory, History and Foundations of Science* 30, no. 3 (2015): 331–48. http://www.jstor.org/stable/43686773.

Freydberg, Bernard. Brouwer’s Intuitionism Vis à Vis Kant’s Intuition and Imagination. The Mathematical Intelligencer (2009), 31(4), 28–36. https://doi.org/10.1007/s00283-009-9074-6.

Gomułka, Jakub. “Cantor’s paradise from the perspective of non-revisionist Wittgensteinianism." *Argument* (Kraków. 2011), 10(2), 351-372. https://doi.org/10.24917/20841043.10.2.5.

Grattan-Guinness, Ivor. *The Search for Mathematical Roots, 1870-1940: Logics, Set Theories, and the Foundations of Mathematics from Cantor Through Russell to Godel.* Princeton, New Jersey: Princeton University Press, 2000.

Gray, Robert. “Georg Cantor and Transcendental Numbers.” *The American Mathematical Monthly* 101, no. 9 (1994): 819–832.

Hardy, G. H.. *An Annotated Mathematician's Apology*. Annotated by Alan J. Cain. Lisbon, 2019.

Hauser, Kai. “Cantor’s Concept of Set in the Light of Plato’s Philebus.” The Review of metaphysics 63, no. 4 (2010): 783–805.

———, and W. Hugh Woodin. “Strong Axioms of Infinity and the Debate about Realism.” *The Journal of Philosophy* 111, no. 8 (2014): 397–419.

Hawking, Stephen. *God Created the Integers: the Mathematical Breakthroughs That Changed History*. Philadelphia, Pennsylvania: Running Press, 2005.

Heller, Michał, and W. H. Woodin. *Infinity: New Research Frontiers*. Cambridge: Cambridge University Press, 2011.

Hilbert, David. “On the Infinite,” in Philosophy of Mathematics, ed. with an introduction by Paul Benacerraf and Hillary Putnam, New Jersey: Prentice-Hall, 1964, https://math.dartmouth.edu/~matc/Readers/HowManyAngels/Philosophy/Philosophy.html.

Hitoshi, Ochiai. “Theology of Georg Cantor.” *Apeiron Centre*, 24 Nov. 2018, https://apeironcentre.org/theology-of-georg-cantor/.

Jané, Ignacio. “The Role of the Absolute Infinite in Cantor’s Conception of Set.” Erkenntnis (1975-) 42, no. 3 (1995): 375–402. http://www.jstor.org/stable/20012628.

John of Damascus. *Writings*. Translated by Frederic H. Chase, Jr. Washington, D.C: Catholic University of America Press, 1958.

Kline, Morris. *Mathematics: The Loss of Certainty*. New York: Oxford University Press, 1980.

Lauri, Josef. "Florenskij and Georg Cantor: Naming Infinity." Melita Theologica; Journal of the faculty of theology, University of Malta, 69/1 (2019): 11-15.

Le Blanc, Jill. “Infinity in Theology and Mathematics.” *Dartmouth Department of Mathematics*, https://math.dartmouth.edu/~matc/Readers/HowManyAngels/Blanc.html.

Lee, M. “Georg Cantor.” Georg Cantor, August 2017, 1–3. https://search.ebscohost.com/login.aspx?direct=true&AuthType=sso&db=f6h&AN=20211641&site=ehost-live.

Linnebo, Øystein, and Stewart Shapiro. “Actual and Potential Infinity.” *Noûs* 53, no. 1 (2019): 160–191.

Livio, Mario. *Is God a Mathematician?*. New York: Simon & Schuster Paperbacks, 2009.

Luna, Laureano. “Grim’s Arguments against Omniscience and Indefinite Extensibility.” International Journal for Philosophy of Religion 72, no. 2 (2012): 89–101. http://www.jstor.org/stable/23324791.

Mar, Gary. “Why ‘Cantorian’ Arguments Against the Existence of God Do Not Work.” *International philosophical Quarterly* 33, no. 4 (1993): 429–442.

Meyers, Dale. “Infinity: A Loss for Skeptics.” *University of Hawaii at Manoa Department of Mathematics*, https://math.hawaii.edu/~dale/godel/infinity.html.

Moore, A. W.. *The Infinite.* New York: Taylor & Francis Group, 1991. ProQuest Ebook Central.

Nachtomy, Ohad and Reed Winegar. *Infinity in Early Modern Philosophy.* Cham: Springer International Publishing, 2018.

Naylor, Marshall. “God, Evil, and Infinite Value.” *Religions* 9, no. 1 (2018): 20–.

Newstead, Anne. “Cantor on Infinity in Nature, Number, and the Divine Mind.” *American Catholic Philosophical Quarterly* (2009) 83 (4):533-553. https://philarchive.org/archive/NEWQOIv1.

Nickel, James. *Mathematics: Is God Silent?*. Ross House Books: Vallecito, California 1990.

Pantsar, Markus. “In Search of $\aleph_0$: How Infinity Can Be Created.” *Synthese* 192, no. 8 (2015): 2489–2511.

Poythress, Vern. *Redeeming Mathematics: a God-Centered Approach*. Crossway: Wheaton, Illinois 2015.

Proietti, Carlo. “Natural Numbers and Infinitesimals: A Discussion Between Benno Kerry and Georg Cantor.” *History and Philosophy of Logic* 29, no. 4 (2008): 343–359.

Reid, Constance. *Hilbert*. New York: Springer-Verlag, 1970.

Rosenstock, Bruce. “Georg Cantor and the Mathematics of God.” In *Transfinite Life*, 42–. Indiana University Press, 2017.

Russell, Bertrand. “The Axiom of Infinity.” *Hibbert Journal* 2, no. 4 (Jul 1904): 809-12. https://users.drew.edu/jlenz/br-axiom-of-infinity.html

Séguin, Philippe. “The Idea of Number from Gauss to Cantor. The Leibnizian Heritage and Its Surpassing.” In *New Essays on Leibniz Reception*, 1–11. Basel: Springer Basel, 2012.

Shapiro, Stewart. “Theology and the Actual Infinite: Burley and Cantor.” *Theology and Science* 9, no. 1 (2011): 101–108.

Steinhart, Eric C. “A Mathematical Model of Divine Infinity.” *Theology and Science* 7, no. 3 (2009): 261-274.

Srivastava, S. M. “Georg Cantor (1845–1918).” *Resonance* 19, no. 11 (2015): 972–973.

Stillwell, John. *Roads to Infinity: the Mathematics of Truth and Proof* Natick, Massachusetts: A.K. Peters, 2010.

Strauss, Danie F.M. “Background perspectives on infinity and God.” *In Die Skriflig* 53, no. 1 (2019): 1–8.

Tall, David. “Natural and Formal Infinities.” *Educational Studies in Mathematics* 48, no. 2/3 (2001): 199–238.

Tapp, Christian. “Infinity in Mathematics and Theology.” *Theology and Science* 9, no. 1 (2011): 91–100.

Trlifajová, Katerina. Bolzano’s Infinite Quantities. *Found Sci* 23, 681–704 (2018). https://doi.org/10.1007/s10699-018-9549-z.

van Atten, Mark. “A Note on Leibniz’s Argument Against Infinite Wholes.” In *Essays on Gödel’s Reception of Leibniz*, 23–32. Cham: Springer International Publishing, 2014.

Veen, Joanna Van Der and Leon Horston. “Cantorian Infinity and Philosophical Concepts of God.” https://philarchive.org/archive/VANCIA-15, 2.

Voss, Sarah. *What Number Is God?: Metaphors, Metaphysics, Metamathematics, and the Nature of Things.* Albany: State University of New York Press, 1995.

Whitehead, Alfred North. “Mathematics as an Element in the History of Thought,” in *Science in the Modern World*. New York: The Free Press, 1925.

Zhu, Wujia, Yi Lin, Ningsheng Gong, and Guoping Du. “Wide-Range Co-Existence of Potential and Actual Infinities in Modern Mathematics.” *Kybernetes* 37, no. 3/4 (2008): 433–437.