The Discrete Brachistochrone Summer Reasearch

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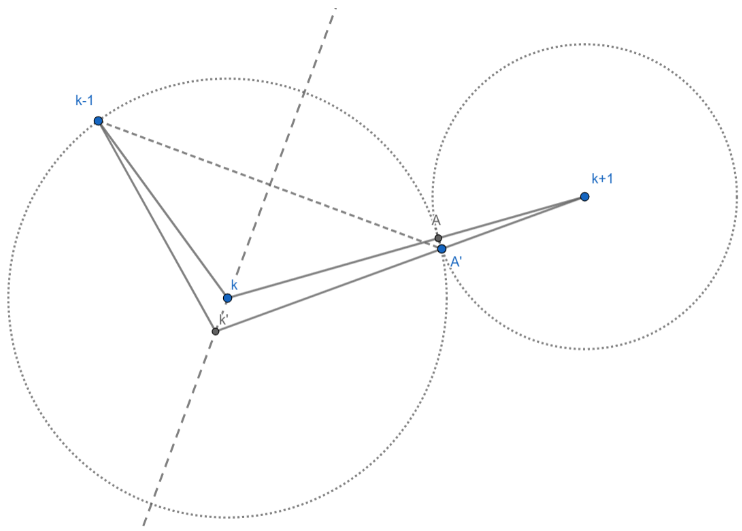
1 Introduction

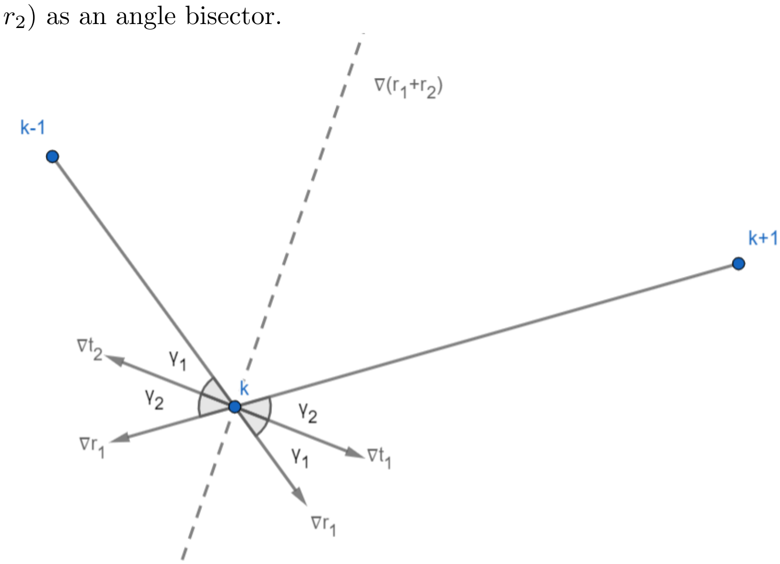
In recent years, a new discrete version of the classic Brachistochrone problem has been posed and found to have certain unique properties. Among these properties is the equal times property, which states that any segment of a discrete N-brachistochrone must take the same amount of time to slide down as other segments of that N-brachistochrone. To my knowledge, this was first proven in a 1977 paper by J.P. Ballentine and Taw-Pin Lim. This proof, while algebraicallytrue, lacked a sense of simplicity and elegance expected for such a straightforward property, and italso only applied to the case of $N = 2$. Further research is done on the discrete brachistochrone, in particular a paper by David Agmon and Hezi Yizhaq in 2019. This physics paper found new properties relating angles in a N-brachistochrone and claimed to offer another proof of the equal times property. This proof, while insightful regarding variations of the midpoint between two segments, lacked a sense of mathematical rigor and is thus incomplete. In the past several years, David Gaebler, Mark Panaggio, and Timothy Pennings studied the discrete brachistochrone and found unique properties regarding time gradients, pleochronic angles, and convergence of a discrete brachistochrone with its continuous variant. In their work, they developed more connections surrounding the equal times property, but they were not satisfied with a complete proof of it independent of Lim's paper. Under the guidance and help of Dr. Gaebler, I set out to research further the properties of this discrete brachistochrone, with particular attention to completing a more sound and elegant proof of the equal times property. After having studied the history of the classical Brachistochrone and the calculus of variations, I turned my attention to the specifics of the discrete brachistochrone. The following summary details what I have found during this summer research project—both progress I made, and some dead ends I hit—which can be used to aid further research.

2 Fixing Agmon and Yizhaq

One topic of study was an attempt to make Agmon and Yizhaq's proof mathematically sound. Starting off, their proof can be translated into something more mathematical than their previous physics-based one, noting gaps in logic. Consider the 2-brachistochrone which has minimum time. As you vary the position of the point between the two segments, any small variation of total time is minimal, translating to $\nabla T = 0$, for T being total time. With t_1 and t_2 being the individual slide times for the two segments, and $T = t_1 + t_2, \nabla t_1 = -\nabla t_2$. One way this is true is if the variation takes place in a special direction, $\vec{w}$, for which $0 = D_{\vec{w}} t_1 = D_{\vec{w}} t_2$. In other words, vary the point in the direction that keeps individual slide times constant. Then, since $t_k = \frac{r_k}{v_k}$, where r_k is slide distance and v_k is average velocity for the kth segment, using the quotient rule and setting the derivative equal to 0 yields $\frac{D_{\vec{w}} r_k}{D_{\vec{w}} v_k} = \frac{r_k}{v_k} = t_k$. Thus, to show that $t_1 = t_2$, it must be shown that $D_{\vec{w}} r_1 = D_{\vec{w}} r_2$ and $D_{\vec{w}} v_1 = D_{\vec{w}} v_2$. First consider average velocity. Write instantaneous velocity as u_0, u_1, and u_2, for that at the beginning of the left line segment, that between the two, and that at the end, respectively. Then, $v_1 = \frac{1}{2}(u_0 + u_1)$ and $v_2 = \frac{1}{2}(u_1 + u_2)$. Due to conservation of energy and the endpoints being fixed, u_0 and u_2 are constant. As a result, $D_{\vec{w}} v_1 = \frac{1}{2} D_{\vec{w}} u_1 = D_{\vec{w}} v_2$. Second, consider slide distance, or more specifically, the quantity $D_{\vec{z}} r_1 - D_{\vec{z}} r_2$ for arbitrary direction $\vec{z}$. When $D_{\vec{z}} r_1 > D_{\vec{z}} r_2$, the quantity is positive, and when $D_{\vec{z}} r_1 < D_{\vec{z}} r_2$, the quantity is negative. Both conditions exist, and the quantity is a continuous function. So, by the intermediate value theorem, there must exist a direction of variation (a value of $\vec{z}$) for which $D_{\vec{z}} r_1 - D_{\vec{z}} r_2 = 0$ and $D_{\vec{z}} r_1 = D_{\vec{z}} r_2$. In Agmon and Yizhaq's proof, they assume this direction is the same direction as $\vec{w}$, but offer no justification. If it is, in fact, the same direction, then $t_1 = \frac{r_1}{v_1} = \frac{D_{\vec{w}} r_1}{D_{\vec{w}} v_1} = \frac{D_{\vec{w}} r_2}{D_{\vec{w}} v_2} = \frac{r_2}{v_2} = t_2$. In an attempt to complete this proof, I tried to connect the direction of $\vec{w}$ with that which made $D_{\vec{z}} r_1 = D_{\vec{z}} r_2$, and noted the following various connections. Since $\vec{w}$ is that direction in which time remains constant, it is along the level curve of time for both segments. As a result, since the gradient function is perpendicular to its level curve, $\vec{w} \perp \nabla t_1$ and $\vec{w} \perp \nabla t_2$, which is possible because the gradient vectors are antiparallel $\nabla t_1 = -\nabla t_2$. Furthermore, it is easy to show that the direction in which $D_{\vec{z}} r_1 = D_{\vec{z}} r_2$ is $\nabla(r_1 + r_2)$. Since ∇r_1 and ∇r_2 are unit vectors whose magnitudes are both 1, simply consider how $||\nabla r_1||^2 + \nabla r_1 \nabla r_2 = \nabla r_1 \nabla r_2 + ||\nabla r_2||^2$ implies $\nabla(r_1 + r_2)\nabla r_1 = \nabla(r_1 + r_2)\nabla r_2$, which in turn implies $D_{\nabla(r_1+r_2)} r_1 = D_{\nabla(r_1+r_2)} r_2$. With $\nabla(r_1 + r_2)$ being that direction midway between the two segments, it is easy to see how it is the angle bisector between them. This can be verified geometrically using the following diagram and considering a limiting case as k' approaches k. Namely, as the variation becomes sufficiently small, A' becomes sufficiently close to A, and the perpendicular bisector which keeps the change in the first segment's length equal to the change in the second segment's length becomes the angle bisector of the two.



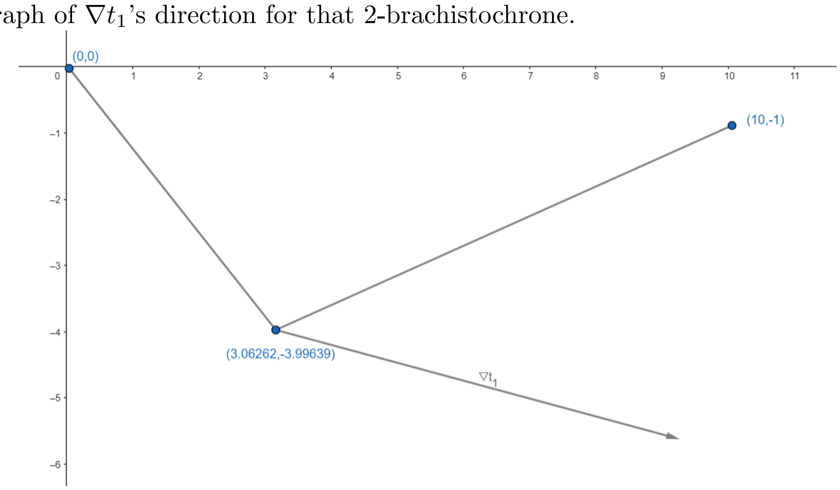
More succinctly, if one already assumes $t_1 = t_2$, then by Dr. Gaebler's work on pleochronic angles (the angle between a segment's slide time gradient and the extension of that segment, denoted γ_k),$\gamma_1 = \gamma_2$ implies that that line perpendicular to ∇t_1 and ∇t_2 is the angle bisector. The following diagram shows equal pleochronic angles and antiparallel time gradients in relation to $\nabla(r_1 + r_2)$ as an angle bisector.



Ultimately, if one could show this angle bisector, $\nabla(r_1 + r_2)$, is perpendicular to ∇t_1 and ∇t_2, one could prove that $\vec{w}$ satisfies both $D_{\vec{w}} r_1 = D_{\vec{w}} r_2$ and $D_{\vec{w}} v_1 = D_{\vec{w}} v_2$, thereby proving the equal time property. Algebraic attempts were made to show that $\nabla t_1 \nabla(r_1 + r_2) = 0$, but they were unsuccessful and lacked any straightforward elegance or symmetry. The connection with pleochronic angles, however, is easy to show using the formula for the angle between vectors. Since $\cos(\gamma_k) = \frac{\nabla t_k \nabla r_k}{||\nabla t_k|| ||\nabla r_k||}$ and $\nabla t_1 = -\nabla t_2$, we find $\cos(\gamma_1) - \cos(\gamma_2) = \frac{1}{||\nabla t_1||} \nabla t_1 \nabla (r_1 + r_2)$. This means that $\gamma_1 = \gamma_2$ is equivalent with perpendicularity, considering when either side of the equation is equal to zero. In the end, however, I could not bridge the gap in Agmon and Yizhaq's proof. Moreover, any attempt that showed progress was connected to pleochronic angles which, by Dr. Gaebler's work, had a much more direct connection to the equal time property. What had to be shown was that a Brachistochrone directly implied equal pleochronic angles.

3 A Numerical Calculation

Failing to complete Agmon and Yizhaq's proof, I moved to calculating a few things and finding numerical solutions for special cases. While certainly not difficult to calculate or new, I found the time gradient of the first segment of a 2-brachistochrone given initial point of $(0,0)$ and final point of (x,y), with an initial velocity of zero, can be expressed as $\nabla t_1 = (\frac{2x}{(\sqrt{x^2+y^2}\sqrt{2gy}}, \frac{y^2-x^2}{y\sqrt{x^2+y^2}\sqrt{2gy}})$. Geometrically, you can graph this time gradient's direction simply by constructing a perpendicular to the angle bisector of the two segments. I confirmed this numerically using and initial point of $(0,0)$, a final point of $(10,-1)$, and Mark Panaggio's numerical brachistochrone for a zero initial velocity. This was used merely to confirm or reject ideas by visual intuition. The following image is that graph of ∇t_1's direction for that 2-brachistochrone.



4 A Catalogue of Failed Attempts

After establishing some things for my own intuition, I moved to a few other approaches to finding an equal times proof. I will note a few ideas in which I found no success but might still be of merit.

First, I tried to express total time in terms of the two pleochronic angles, hoping that an extremum in time would require equal pleochronic angles (which, by Dr. Gaebler's work, would result in equal times). This, physically, would mean there is some proportion between the pleochronic angle and how optimal a segment is, with the intention of proving that the angles must be equal for both segments to be optimal. Dr. Gaebler found that $\cot(\gamma_k) = \frac{2v_k}{gt_k \sin(\theta_k)}$, where θ_k is the angle between each line segment and the vertical. Thus, because $\nabla t_1 = -\nabla t_2$, it is true that $\nabla(\frac{2v_1 \tan(\gamma_1)}{g \sin(\theta_1)} = -\nabla(2v_2 \tan(\gamma_2) g \sin(\theta_2)$. At first, I attempted to simplify this equation algebraically, but that was unsuccessful and lacked any symmetry or elegance. Afterwards, I tried to utilize the well known optics principle of Snell's Law (which has been shown by Lim and Dr. Gaebler to apply to the discrete brachistochrone) that states that the ratio $\frac{v_k}{\sin(\theta_k)}$ is constant for all segments. Unfortunately, Snell's law only applies to the brachistochrone, not arbitrary variations thereof. As a result, I found no pattern behind its gradient. For clarity, I should mention that Dr. Gaebler's connection between pleochronic angles and time arrives through his expression for $\cot(\gamma)$ and Snell's law. Since velocity and angle θ are constant, times are equal if and only if pleochronic angles are equal: $t_1 \cot(\gamma_1) = t_2 \cot(\gamma_2)$. That is why it is enough to prove from the brachistochrone's basic properties that pleochronic angles are equal—because their equality implies the equal times property.

Second, I tried comparing how varying the point between segments in the directions of $\vec{r_1}$ and $\vec{r_2}$ impacted t_1 and t_2. Since total time is at a minimum, one can quickly establish the following relationships: $D_{\vec{r_1}} t_1 = -D_{\vec{r_1}} t_2$ and $D_{\vec{r_2}} t_1 = -D_{\vec{r_2}} t_2$. It is true that $\vec{r_k}$ extends in the same direction as ∇r_k, so I tried to find some useful connection between this system and $\nabla t_1 \nabla (r_1 + r_2)$. Though it seems like there should be an elegant proof that details how, on the optimal curve, there is an equal proportion between distance and time variation, this attempt was unsuccessful.

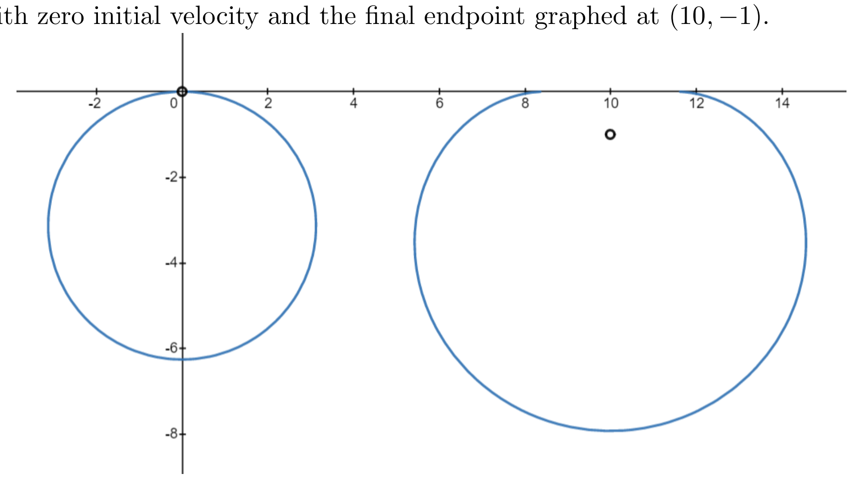
Third, I briefly considered an ellipse with foci of the two brachistochrone endpoints going through the point between line segments. This ellipse would actually be a level curve of an infinite number of ellipses, which perhaps would impact time differently based on the middle point that lies on them and defines them. Although ∇t_1 and ∇t_2 are tangent to the ellipse, I did not find any pattern that directly proved the equal time property. This approach, however, seems like it could still contain some merit.

Fourth, I tried to develop a straightforward proof by contradiction. The proof, ideally, would assume a brachistochrone did not have equal times and lead to some contradiction. While this may still exist as some rearrangement of the actual working proof I discuss at the end of this summary, no isolated proof by contradiction was discovered.

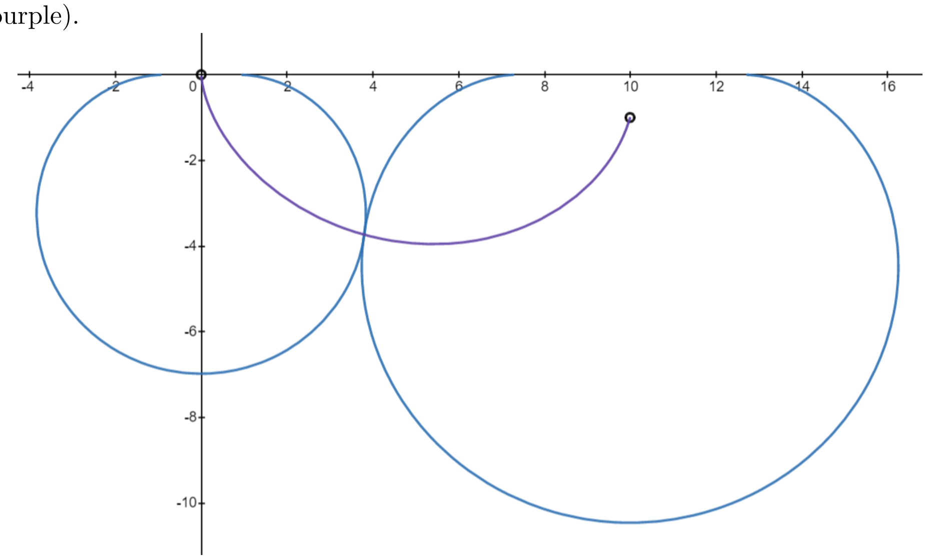
Finally, Dr. Gaebler and I briefly considered the Ham Sandwich theorem as a remedy to completing Agmon and Yizhaq's proof. The ham sandwich theorem essentially shows that, given a group of multiple objects (for example: a top slice of bread, some ham, and a bottom slice of bread), you can always find a plane that divides them all into bits of equal volume. The thought process was that the theorem lets you take multiple directions of variation and account for them all using one plane or line. If we could divide variation in both time (so time remains constant) and distance (so $D_{\vec{z}}r_1 = D_{\vec{z}}r_2$), we might be able to show that the direction which satisfies both conditions is the same. In the end, we did not easily find this to be the case.

5 Expanding Level Curves

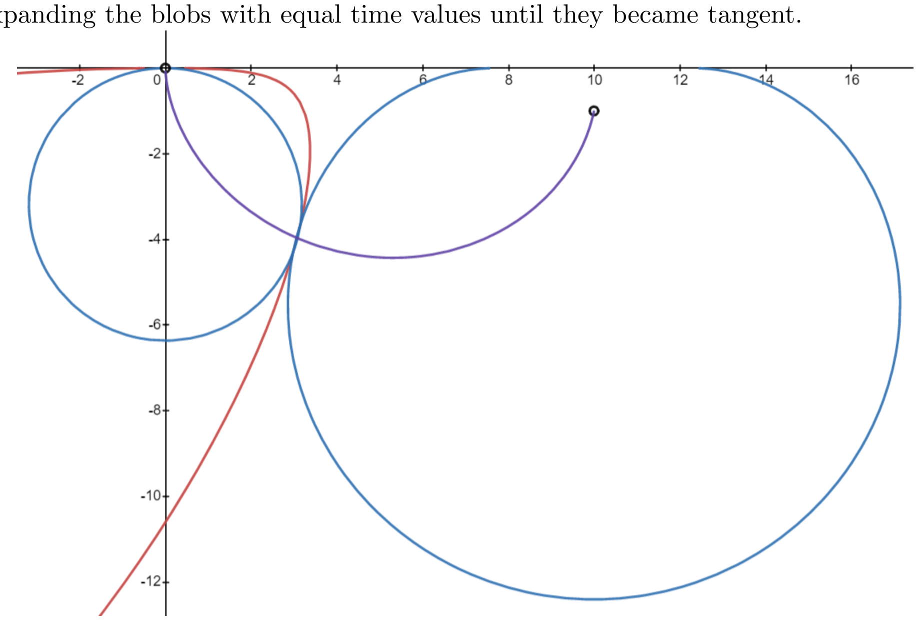
One approach, while not successful in producing an equal times proof, that did produce useful and interesting results was my attempt to expand level curves of time until they are tangent. Consider a 2-Brachistochrone with arbitrary initial velocity at the first point. Fix t_1 and t_2 at arbitrary values and graph their level curves. The graphs of the time levels curves (hereafter called "blobs") resembles squashed circles. The best physical interpretation of these blobs is that any point on them, a segment could be drawn to have an object slide down, and that slide time would be the same regardless of which point is chosen. Interestingly, if the initial velocity of the first segment is zero, then the blob for the first segment is a perfect circle (this goes back to discoveries made by Galileo while he studied properties of inclined planes). Given a starting point of $(0, 0)$, and a middle point of (x, y), the equation of this circle is $x^2 + (y - \frac{gt_1^2}{4})^2 = \frac{g^2t_1^2}{16}$. With a nonzero initial velocity, the equations for these blobs become much more complicated. I attempted to see if the vector for this circle's radius, $(x, y - \frac{gt_1^2}{4})$, could be easily and algebraically shown to be perpendicular to $\nabla(r_1 + r_2)$, but to no avail. I tried this because the radius of a circle is always perpendicular to its tangent line, and the tangent line of this circular blob, being a level curve of time, is perpendicular to ∇t_1. Thus, the time gradient is parallel to the radius, so I hoped it could also be used to complete Agmon and Yizhaq's proof. The following figure shows levels curves of t_1 and t_2, with the former being a circle with zero initial velocity and the final endpoint graphed at $(10, -1)$.



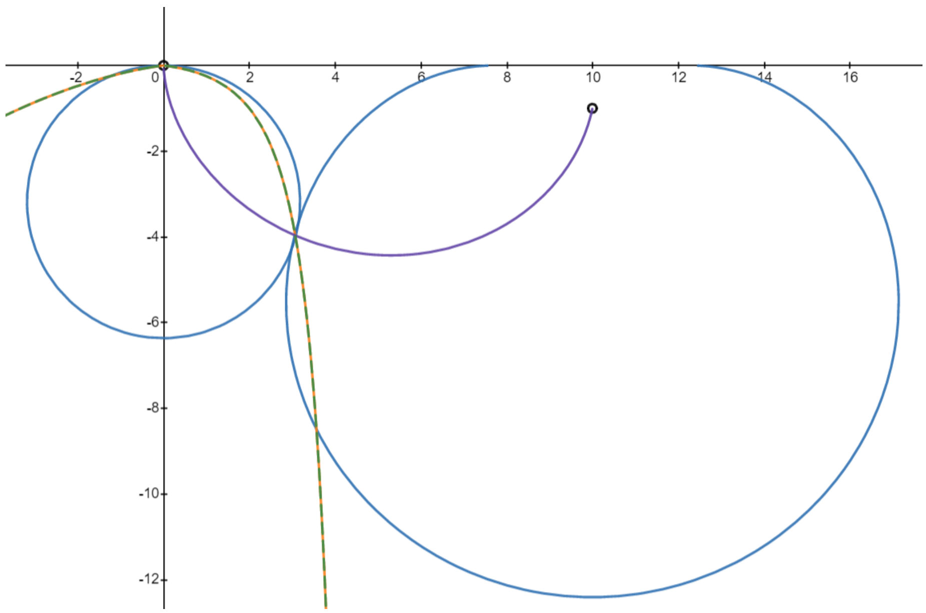
Though this special case did not work out easily, it did give me the idea to use tangent points and gradients of these blobs. I developed two arguments to show that the middle point of a 2-brachistochrone must occur at the intersection of two blobs which are tangent. First, a brachistochrone must have minimum time, meaning both $\nabla T = 0$, and $\nabla t_1 = -\nabla t_2$. Since ∇t_1 and ∇t_2 were always perpendicular to the tangent line of a blob, and they had to be antiparallel, then they had to be perpendicular to the same tangent line. In order for two blobs to have the same tangent line, they had to be tangent to each other. Second, consider the three possible cases of blob intersection. Case 1: They do not intersect. Case 2: They intersect at a single point (are tangent). Case 3: They intersect at more than one point. If the situation were case 1, then given the current amount of time, the segments drawn from both endpoints could not reach each other and complete a pathway. Thus, a brachistochrone, being a complete pathway, could not belong to case 1. Moreover, if the situation were case 3, and the blobs intersected at multiple points, you could reduce time by reducing the size of one blob until it was tangent with the other and only intersected at one point. Since a brachistochrone must have minimum time, it could not belong to case 3. Therefore, a brachistochrone belongs to case 2, and must exist at a point of tangency. Now, there are an infinite number of points at which two blobs could be tangent to each other, points which form a curve from the starting point to the end point. The following graph shows two blobs tangent to each other, intersecting on the curve of all points of mutual tangency (purple).



This curve lacked any noticeable algebraic significance. At first, I attempted to map it to an ellipse, with foci at the brachistochrone endpoints, but that failed. I also tried mapping it to a cycloid (as a little nod to the traditional brachistochrone) but that also failed, since the cycloid shape did not fit the curve. Another attempt was trying to derive a mathematical envelope that was tangent to each ∇t_1, but that also failed. To graph this, I merely expressed the components of $\frac{(\nabla t_1)}{||\nabla t_1||} = -\frac{(\nabla t_2)}{||\nabla t_2||}$ in terms of x and y (since that equation represented antiparallel time gradients without any restriction on their magnitude). This curve did, however, reduce the infinite number of locations for a middle point to a single line. If one more graphical condition could be discovered that intersects this line at one point, the middle point of the brachistochrone could be generated. My hope was that this second graphical condition would be a line of equal times. There were many attempts to express $T, ||\nabla t_1||, \gamma_1, \theta_1$, etc. as things that varied as a point moved along this line, but without success. One thought was to use the intermediate value theorem to find the place where $||\nabla t_1|| = ||\nabla t_2||$. The following figure shows a graph of the curve of equal time (red) intersecting with the curve of tangency. If the equal time property were proven, you could generate the middle point of a 2-brachistochrone simply by expanding the blobs with equal time values until they became tangent.



Through graphing many different level curves and equalities (and a mistaken idea that pleochronic angles were equal everywhere on this curve of tangency which lead us to try to use the implicit function theorem), we ended up finding that the graph representing equal pleochronic angles line up perfectly with the graph representing equal time gradient magnitudes. More succinctly, $\gamma_1 = \gamma_2 \iff ||\nabla t_1|| = ||\nabla t_2||$. The following figure shows the equal pleochronic angle curve (dashed green) overlapping perfectly with the equal time gradient magnitude curve (yellow).



The latter lacked any algebraic simplicity (although it is equivalent to the former) and was graphed using an unsimplified partial derivative. The former (pleochronic angle) curve was graphed using Dr. Gaebler's expression for $\cot(\gamma_k)$, which simplified to yield $\frac{(v_0+\sqrt{2gy})^2}{x} = \frac{(v_0+\sqrt{2gy}+\sqrt{2gY})^2}{X-x}$ in terms of x and y with an initial velocity of v_o and a final point of (X, Y). Their intersection, of course, meant that finding a point of tangency either with $\gamma_1 = \gamma_2$ or $||\nabla t_1|| = ||\nabla t_2||$ would be sufficient to find the middle point location of the 2-brachistochrone.

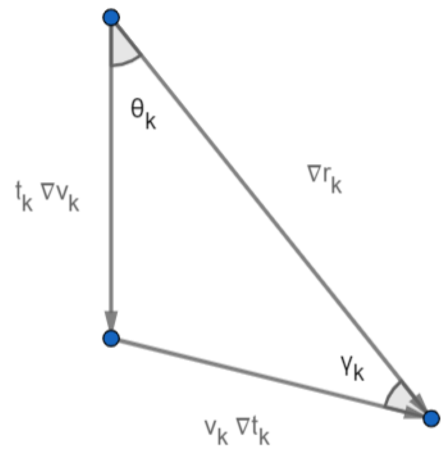
6 Graham's Lemma and the Proof of the Equal Time Property

What was more significant, however, was the implication of the relationship between pleochronic angles and time gradient magnitudes. We could already show that a discrete brachistochrone has equal time gradient magnitudes since $\nabla t_1 = -\nabla t_2 \implies ||\nabla t_1|| = ||-\nabla t_2|| = ||\nabla t_2||$. Furthermore, we could already show that $\gamma_1 = \gamma_2 \implies t_1 = t_2$, so a bridge between γ_k and $||\nabla t_k||$ would complete a proof of the equal times property. I then set out to prove the connection between pleochronic angles and time gradient magnitudes. I discovered the following proof combining some tricks Dr. Gaebler had used in his paper as well as some others.

Lemma 6.1 (Graham's Lemma)**.** *In a 2-brachistochrone,* $\gamma_1 = \gamma_2 \iff ||\nabla t_1|| = ||\nabla t_2||$.

Proof. Consider the equation $r_k = v_k t_k$, where each t_k, v_k, and r_k represents slide time, average velocity, and distance respectively in terms of the x and y

coordinates of the middle point between two segments. Take the gradient of r_k, and use the product rule to obtain the equation $\nabla r_k = v_k \nabla t_k + t_k \nabla v_k$. This equation can be thought of as a vector addition equation. The following figure displays this vector addition and marks relevant angles.



Notably, ∇r_k just extends in the direction of the line segment and is a unit vector. Furthermore, ∇v_k depends only on y, so it points directly downward. Mark down θ_k as the angle between ∇r_k and $t_k \nabla v_k$ since $t_k \nabla v_k$ is parallel to the vertical. Mark down γ_k as the angle between ∇r_k and $v_k \nabla t_k$. Then use law of sines to obtain the equation $\frac{v_k ||\nabla t_k||}{\sin(\theta_k)} = \frac{(t_k ||\nabla v_k||}{\sin(\gamma_k)}$. Then take Dr. Gaebler's expression $\cot(\gamma_k) = \frac{2v_k}{g t_k \sin(\theta_k)}$, solve it for $\sin(\theta_k)$ and substitute it into the previous equation. As a result, v_k and t_k will cancel out and $\cot(\gamma_k)$ will combine with $\sin(\gamma_k)$ to yield $\sec(\gamma_k) = \frac{g||\nabla t_k||}{2||\nabla v_k||}$. Now, due to conservation of energy, both $||\nabla v_k||$ and $||\nabla v_{k+1}||$ only depend on y, and thus $||\nabla v_k|| = ||\nabla v_{k+1}||$. Combining this with our expression for $\sec(\gamma_k)$ gives us $||\nabla t_k|| \cos(\gamma_k) = ||\nabla t_{k+1}|| \cos(\gamma_{k+1})$. With that equation, we have proven $\gamma_1 = \gamma_2 \iff ||\nabla t_1|| = ||\nabla t_2||$. ■

One particular note that is necessary for a rigorous proof: there is a slight geometric difference between the left pleochronic angle of a segment and the right pleochronic angle. Angle θ_k appears in a different location outside the vector triangle when considering the left pleochronic angle, making the angle used in the law of sines $\pi - \theta_k$ instead of merely θ_k. Luckily, $\sin(\pi - \theta_k) = \sin(\theta_k)$, so this argument still holds for all γ_k. One other equation that arises from this law of sines approach, which applies to the brachistochrone curve specifically, is $\frac{t_k}{||\nabla t_k|| \sin(\gamma_k)} = \frac{t_{k+1}}{||\nabla t_{k+1}|| \sin(\gamma_{k+1})}$. This equation can be seen as showing the connection between the three quantities of time, time gradient magnitude, and pleochronic angle. With this proof in place, the final proof for the equal times property is as follows.

Theorem 6.2 (Equal Time Property)**.** *In a discrete brachistochrone,* $t_k = t_{k+1}$*.*

Proof. A discrete brachistochrone must have minimum time. Thus, $\nabla T = 0 \Longrightarrow \nabla t_1 = -\nabla t_2 \Longrightarrow ||\nabla t_1|| = ||-\nabla t_2|| = ||\nabla t_2||$. Since $\gamma_1 = \gamma_2 \Longleftrightarrow ||\nabla t_1|| = ||\nabla t_2||$ by the previous demonstration, $\gamma_1 = \gamma_2$. Furthermore, since $t_k \cot(\gamma_k) = t_{k+1} \cot(\gamma_{k+1})$ by Dr. Gaebler's expression for $\cot(\gamma_k)$ and Snell's law, $t_k = t_{k+1}$. ■

7 Further Work and Concluding Remarks

Dr. Gaebler and I believe there is much room for continued and further research. As of the time of this summary, the preceding work is all that has been finished. First, the discovery about the relationship between pleochronic angles and time gradient magnitudes is very new. It is possible there is a direct way to compact Dr. Gaebler's algebra and the previous vector addition/law of sines work into one process, instead of leaving it as multiple separate steps. This, if done, would likely take the form: consider the vector addition triangles of left and right pleochronic angles, you can derive these two expressions for pleochronic angles, due to minimum time the expressions are the same, and when combined they produce the equal time property. Similarly, we expect there might be some constant usable quantity such as a proportion between $\frac{\sin(\theta_k+\gamma_k)}{\sin(\theta_k-\gamma_k)}$ and $\frac{||\nabla_R t_k||}{||\nabla_L t_(k)||}$ or something (where the subscripts of R and L denote right and left, respectively). Moreover, we believe there is definitely something to be developed from time level blobs. With this equal times proof in place, you can generate a unique 2-brachistochrone by expanding level blobs maintaining equal times until they become tangent. Is there a good way to quantify this that would make calculating specific points easier? Also, this process might be difficult to expand to 3 and above brachistochrones. That is because you do not know from what point to expand middle time blobs. Take the 3-brachistochrone, for example. While you could expand equal time blobs from both known, fixed endpoints, you cannot do so for either of the two middle points, since you do not know their location prior to expansion. This difficulty has lead to the idea of studying discrete brachistochrones to and from curves (instead of merely points). If, after all, you could find the optimal discrete curve between one level curve and another, you could generate brachistochrones one segment at a time. It would also be interesting to see if there is some kind of symmetry argument to be made for the time blob approach using an analogy to physics. For example, what if there level blobs were really expanding forces or surface areas or volumes that must be equal to rest in equilibrium? Furthermore, a completion of Agmon and Yizhaq's proof would be very nice, since their explanation is very insightful. Perhaps some geometric translation of the $\gamma_1 = \gamma_2 \Longleftrightarrow ||\nabla t_1|| = ||\nabla t_2||$ principle could be uncovered. Finally, it is unclear if the current version of this particular equal times proof is the most elegant. Is it possible to translate it into dot-products or some other operation and simplify the process? Overall, this summer research project has been edifying, informative, and (by the grace of God) successful.

8 References

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