1.2 Definition of Vector Space

Definition: A vector space over a field F is a quadruple $(V, F, +, \cdot, O)$ where V is the set of vectors, $+$ is vector addition (a function from $V \times V$ to V), $\cdot$ is scalar multiplication (a function from $F \times V$ to V), and O is an element of V, such that the following properties hold:

VS1 (**Commutative** $+$): $(\forall x \in V)(\forall y \in V)(x + y = y + x)$.

VS2 (**Associative** $+$): $(\forall x \in V)(\forall y \in V)(\forall z \in V)[(x + y) + z = x + (y + z)]$.

VS3 (**Identity** $+$): There is some element in V, denoted O, such that $(\forall x)(x + O = x)$.

VS4 (**Inverses** $+$): $(\forall x \in V)(\exists y \in V)(x + y = O)$.

VS5 (**Scalar Identity**): $(\forall x)(1x = x)$.

VS6 (**Associative Scalar**): $(\forall a \in F)(\forall b \in F)(\forall x \in V)[(ab)x = a(bx)]$.

VS7 (**Distributive Scalar over vector sum**): $(\forall a \in F)(\forall x \in V)(\forall y \in V)(a(x + y) = ax + ay)$.

VS8 (**Distributive Vector over scalar sum**): $(\forall a \in F)(\forall b \in F)(\forall x \in V)[(a + b)x = ax + bx)$.

Sample Vector Spaces

1. (**Column vectors**): for every field F and positive integer n, $(F^n, F, +, \cdot, \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix})$ is a vector space with pointwise operations.

2. (**Matrices**): for every field F, and all positive integers m, n, $(M_{m\times n}(F), F, +, \cdot, O)$ is a vector space with pointwise operations where
$$O = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

3. (**Functions from S to F**): for every field F and every set S, $(\mathcal{F}(S, F), F, +, \cdot, O)$ is a vector space with usual function operations and O the constantly 0 function.

4. (**Polynomials in x**): for every field F, $(P(F), F, +, \cdot, O)$ is a vector space with the usual function operations and O the constant polynomial 0.

5. (**Sequences in F**): for every field F $(\mathcal{F}(Z^+, F), F, +, \cdot, O)$ is a vector space with usual function operations and O the constantly 0 sequence.