Cantor-Schröder-Bernstein Theorem, Part 1

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MHF 3202

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Proving Equinumerousity

Up to this point, the main method we have had for proving that two sets A and B are equinumerous is to show that there is a function

$$f : A \to B$$

that is one-to-one and onto.

In some cases, finding such a bijection can be rather difficult.

Today we will prove a theorem that will provide a new and simpler method for showing that two sets are equinumerous.

Theorem (Cantor-Schröder-Bernstein Theorem)

Suppose A and B are sets. If $A \precsim B$ and $B \precsim A$, then $A \sim B$.

A preliminary definition

Let A and B be sets. We say A is dominated by B, in symbols $A \precsim B$, if there is a one-to-one function $f : A \to B$.

A few examples:

- If $A \sim B$, then $A \precsim B$
- If $A \subseteq B$, then $A \precsim B$
- $\mathcal{P}(\mathbb{Z}^+) \precsim \mathbb{R}$

Question: Is $\precsim$ a partial order?

It is not too hard to show that $\precsim$ is reflexive and transitive.

Is $\precsim$ antisymmetric?

That is, if $A \precsim B$ and $B \precsim A$, then does it follow that $A = B$?

A counter-example

Consider $A = \mathbb{Z}^+$ and $B = \mathbb{Q}$.

- $\mathbb{Z}^+ \precsim \mathbb{Q}$ and
- $\mathbb{Q} \precsim \mathbb{Z}^+$, but
- $\mathbb{Z}^+ \neq \mathbb{Q}$.

Note however that $\mathbb{Z}^+ \sim \mathbb{Q}$.

Is this an instance of a more general fact? Yes!

The Cantor-Schröder-Bernstein Theorem

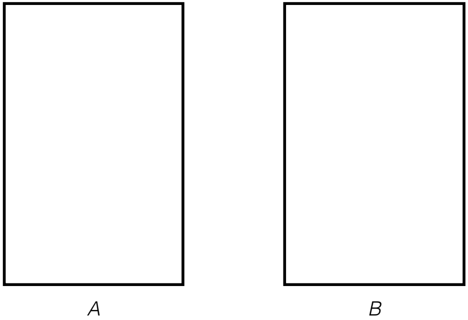
Theorem

Let A and B be sets. If $A \precsim B$ and $B \precsim A$, then $A \sim B$.

Our Approach

To help us understand the general strategy of the proof, we will make use of a series of diagrams.

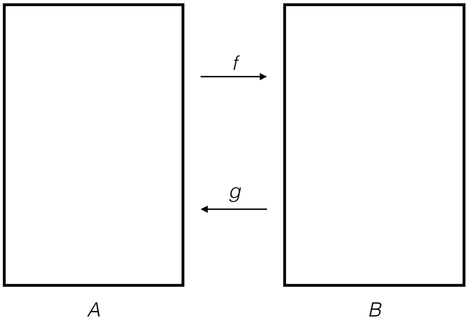
First, we will represent the sets A and B as follows.



Our Approach

Next, let

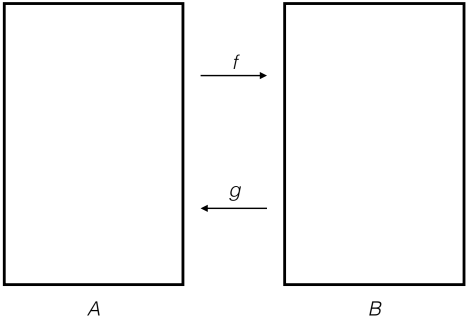
- $f : A \to B$ be a one-to-one function witnessing $A \precsim B$ and
- $g : B \to A$ be a one-to-one function witnessing $B \precsim A$.



Our Approach

Note that if either f or g is onto, it immediately follows that $A \sim B$.

So need to consider the possibility that neither f nor g are onto.



The plan

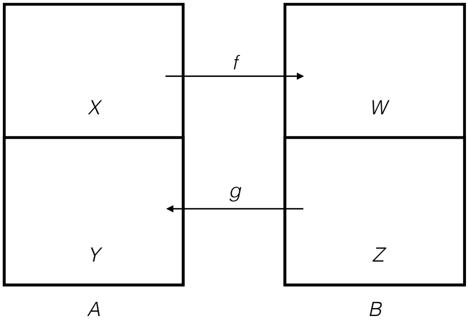
Our goal is to use f and g^{-1} to define a one-to-one and onto function $h : A \to B$:

To do so, we will

1. split A into two pieces X and Y;
2. split B into two pieces W and Z;
3. X will be matched up with W by f; and
4. Y will be matched up with Z by g.

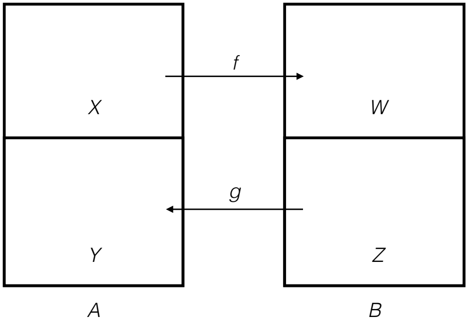
The plan

Here is a schematic diagram in which the splits have been made the functions map in their usual directions.



The plan

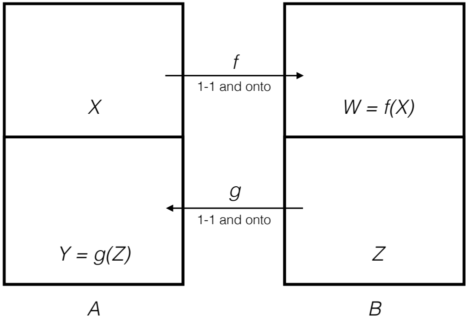
If we know what X is, we let $W = f(X) = \{f(x) \mid x \in X\}$.
Then we let $Z = B \setminus W$.
We know what Z is, so we let $Y = g(Z) = \{g(z) \mid z \in Z\}$.



The plan

It follows that

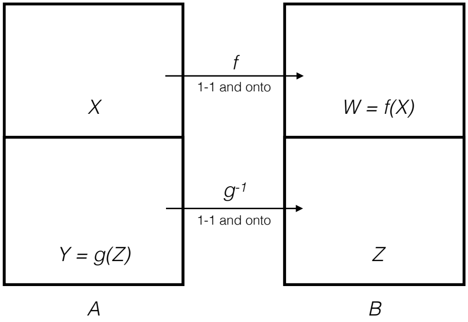
- $f \restriction_X : X \to W$ is one-to-one and onto and
- $g \restriction_Z : Z \to Y$ is one-to-one and onto.



The plan

Consequently,

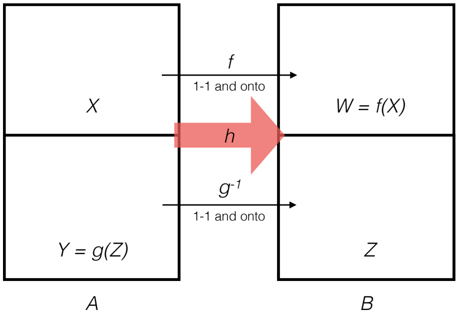
- $f\restriction_X : X \to W$ is one-to-one and onto and
- $(g\restriction_Z)^{-1} : Y \to Z$ is one-to-one and onto.



The desired function h

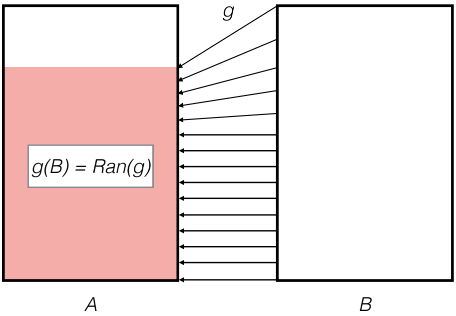
Therefore

- $h = f\restriction_X \cup (g\restriction_Z)^{-1} : X \cup Y \to W \cup Z$ is one-to-one and onto.
- We know $W \cup Z = B$, so
- if $X \cup Y = A$, then h is our witnessing function.



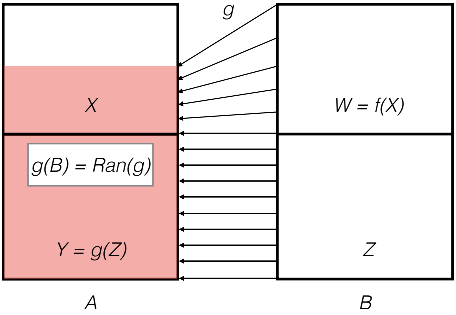
Choosing the sets X, Y, W, and Z

First we recall that we assumed g is not onto, since otherwise g ia a witness that $A \sim B$.



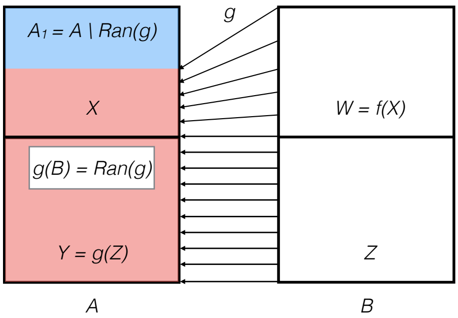
Choosing the sets X, Y, W, and Z

We want $Y \subseteq \mathrm{Ran}(g)$.



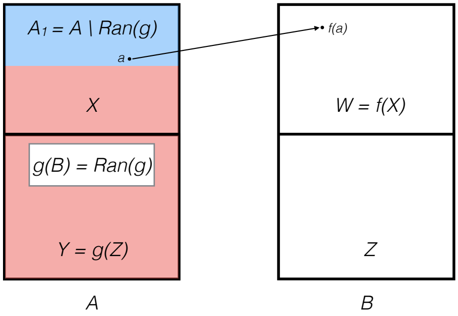
Choosing the sets X, Y, W, and Z

If we let $A_1 = A \setminus \mathrm{Ran}(g)$, then we must have $A_1 \subseteq X$.



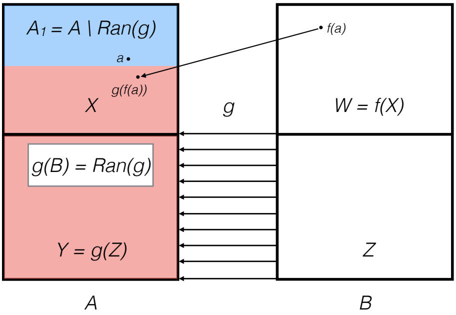
Choosing the sets X, Y, W, and Z

Given an arbitrary $a \in A_1$, since $a \in X$, it follows that $f(a) \in W$.



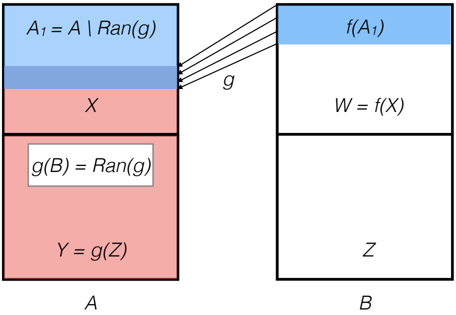
Choosing the sets X, Y, W, and Z

- For every $z \in Z = B \setminus W$, $z \neq f(a) \in W$.
- So, since g is one-to-one, for all $z \in Z$, $g(f(a)) \neq g(z)$.
- Thus $g(f(a)) \in X$.



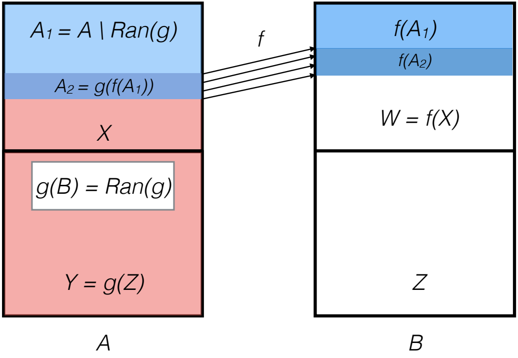
Choosing the sets X, Y, W, and Z

- Since a was arbitrary, we have $f(a) \in W$ and $g(f(a)) \in X$ for all $a \in A_1$.
- That is, $f(A_1) \subseteq W$ and $g(f(A_1)) \subseteq X$.



Using Recursion

$$A_1 = A \setminus \mathrm{Ran}(g);\ A_{n+1} = g(f(A_n)).$$



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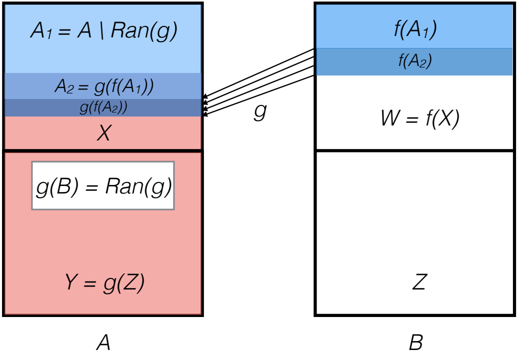

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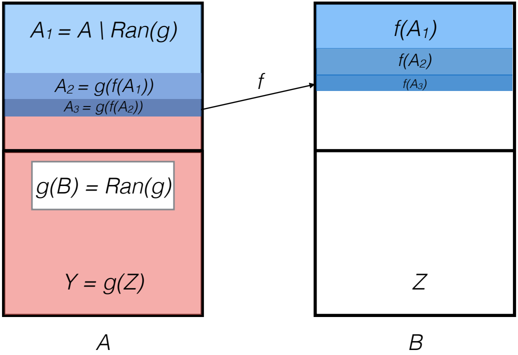

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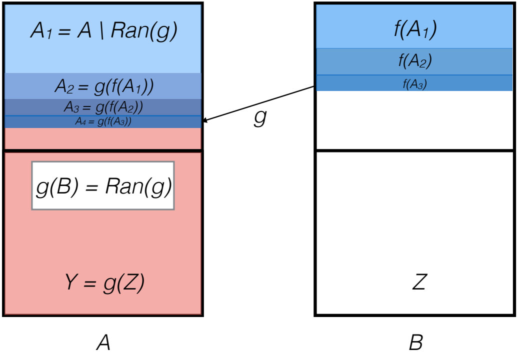

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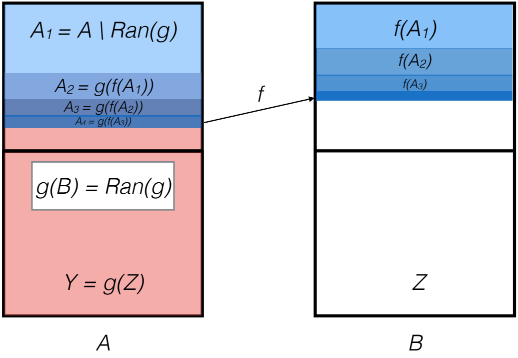

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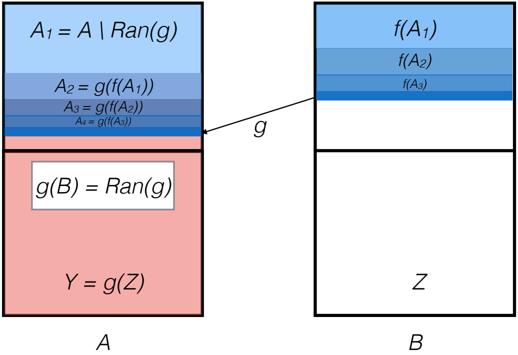

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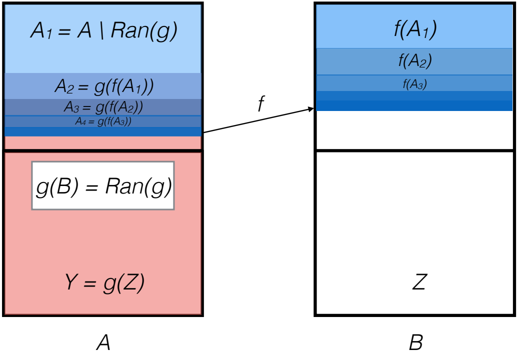

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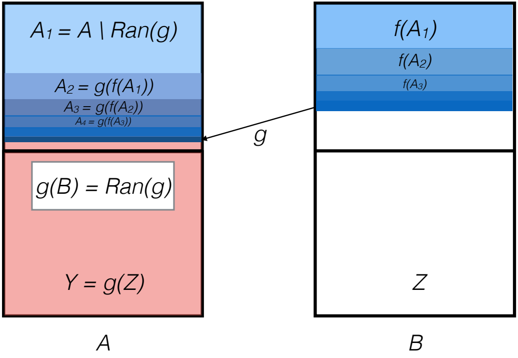

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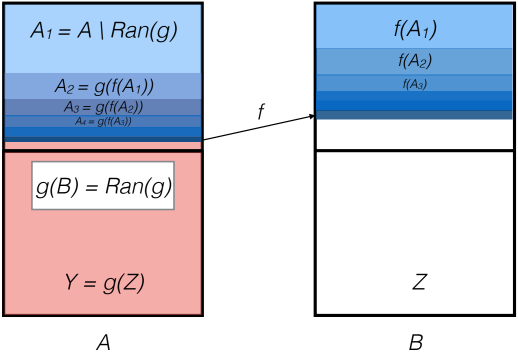
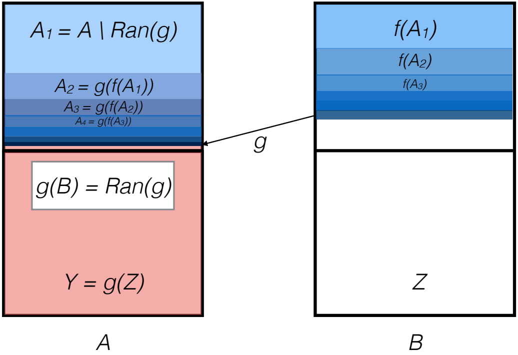


Using Recursion

$A_1 = A \setminus \mathrm{Ran}(g)$; $A_{n+1} = g(f(A_n))$; and $X = \bigcup\{A_n \mid n \in \mathbb{Z}^+\}$



Taking the union of the family

$X = \bigcup\{A_n \mid n \in \mathbb{Z}^+\}$