Spotted Beebalm
Cantor-Schröder-Bernstein Theorem, Part 2

Jean A. Larson and Christopher C. Porter

MHF 3202

December 4, 2015
Theorem (Cantor-Schröder-Bernstein Theorem)

Suppose $A$ and $B$ are sets. If $A \preceq B$ and $B \preceq A$, then $A \sim B$. 
Opening of the Proof:

Recall that for any function $F : U \rightarrow V$ and any subset $D \subseteq U$, the image of $D$ under a $F$ is the set $F(D) := \{ F(d) \mid d \in D \}$.

Assume $A \preceq B$ and $B \preceq A$ ($\preceq$).

Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be one-to-one functions that witness the above relations ($\exists$).

**Case 1:** One of $f$ and $g$ is onto.
Then one of $f$ and $g$ is a witness that $A \sim B$.

**Case 2:** Neither $f$ nor $g$ is onto
Plan of the proof for Case 2:

- Define a set $X \subseteq A$ by recursion
- Set $W := f(X) = \{ f(x) \mid x \in X \}$ and show $f \cap (X \times W) : X \to W$ is one-to-one and onto.
- Set $Z = B \setminus W$, $Y = g(Z)$, and prove that $Y = g(Z) = A \setminus X$.
- Show $g \cap (Z \times Y)$ is one-to-one and onto, so $(g \cap (Z \times Y))^{-1} = (g \cap (Y \times Z))^{-1}$ is also one-to-one and onto.
- Show that $h = (f \cap (X \times W)) \cup (g \cap (Y \times Z))^{-1} : A \to B$ is the witness that $A \sim B$. 
Definitions of $R$, $A$ and $W$:

- Let $R = \text{Ran}(g) \subseteq A$.

- Define $X$ by recursion:
  - $A_1 := A \setminus R$;
  - for every $n \in \mathbb{Z}^+$, $A_{n+1} := g(f(A_n)) = \{g(f(a)) \mid a \in A_n\}$.

  Then $X := \bigcup\{A_n \mid n \in \mathbb{Z}^+\}$.

- Set $W := f(X) = \{f(x) \mid x \in X\} = \text{Ran}(f\mid_X)$. 
Claim 1: The function $f \cap (X \times W) : X \to W$ is one-to-one and onto.

Proof.
Since $f$ is one-to-one and $f \cap (X \times W)$ is a restriction of $f$, it follows from Exercise 5.2: 10a, that $f \cap (X \times W)$ is one-to-one.

Since $f(X) = W$ and $f$ and $f \cap (X \times W)$ agree with $f$ on $X$ by Exercise 5.1: 7a, it follows that $\text{Ran}(f \cap (X \times W)) = f(X) = W$.
Thus by Theorem 5.2.3, $f \cap (X \times W)$ is onto. $\square$
Claim 2: \( Y = g(Z) = \{g(z) | z \in Z\} \subseteq A \setminus X \) where \( Z := B \setminus W \).

Proof.
Assume toward a contradiction that \( Y = g(Z) \not\subseteq A \setminus X \) (o*).

Let \( g(z_0) \in g(Z) \subseteq A \) be a witness, i.e. assume \( z_0 \in Z \) and \( g(z_0) \not\in A \setminus X \) (a∃).

Since \( g(z_0) \not\in A \setminus X \), it follows that \( g(z_0) \in X \) (def set difference).

Since \( g(z_0) \) is in the range of \( g \), it is not in \( A_1 = Z \setminus \text{Ran}(g) \).

Since \( X = \bigcup \{A_n | n \in \mathbb{Z}^+\} \) and \( g(z_0) \not\in A_1 \), we can find a witness \( n_0 \in \mathbb{Z}^+ \) with \( g(z_0) \in A_{n_0+1} \) (a∃).
Claim 2 (proof continued)

Since $g(z_0) \in A_{n_0+1}$ and $A_{n_0+1} = g(f(A_{n_0}))$, we know $g(z_0) \in g(f(A_{n_0})) = \{g(f(x)) \mid x \in A_{n_0}\}$.
Let $x_0$ be a witness, i.e. assume $g(z_0) = g(f(x_0))$.

Since $g$ is one-to-one, it follows that $z_0 = f(x_0) \in W$, by the definition of $W$. Thus $z_0 \in Z = B \setminus W$ and $z_0 \in W$, which is a contradiction, since these two sets are disjoint ($c\ast$).

Thus our assumption was false and Claim 2 follows.
Claim 3: $A \setminus X \subseteq Y = g(Z) = \{g(z) \mid z \in Z\}$.

Assume toward a contradiction that $A \setminus X \not\subseteq g(Z)$ (o*).

Let $a_0 \in A \setminus X$ be a witness, i.e. assume $a_0 \notin g(Z)$ (a∃).

Since $y_0 \in A \setminus X$, it follows that $y_0 \notin X$ and in particular, $y_0 \notin A_1 = A \setminus \text{Ran}(g)$, so $y_0 \in \text{Ran}(g)$.

Let $b_0 \in B$ be a witness, i.e. $g(b_0) = y_0$ (a∃).

Since $y_0 \notin g(Z)$, it follows that $b_0 \notin Z = B \setminus W$, so $b_0 \in W = f(X)$. Let $x_0 \in X$ be a witness, i.e. assume $f(x_0) = b_0$ and let $m_0$ be such that $x_0 \in A_{m_0}$ (a∃).

Thus $a_0 = g(b_0) = g(f(x_0)) \in g(f(A_{m_0})) = A_{m_0+1} \subseteq X$, so $a_0$ is in both $X$ and $A \setminus X$ which is a contradiction since these sets are disjoint (c*). So our assumption was false and Claim 3 follows.
Claim 4: The function \( g \cap (Z \times Y) : Z \to Y \) is one-to-one and onto and so is its inverse, \( g^{-1} \cap (Y \times Z) : Y \to Z \).

Proof.
By Claims 2 and 3, \( Y = g(Z) = A \setminus X \).
Since \( g \) is one-to-one and \( g \cap (Z \times Y) \) is a restriction of \( g \), it follows from Exercise 5.2: 10a, that \( g \cap (Z \times Y) \) is one-to-one.

Since \( g(Z) = Y \) and, by Exercise 5.1: 7a, \( g \) and \( g \cap (Z \times Y) \) agree with \( g \) on \( Z \), it follows that \( \text{Ran}(g \cap (Z \times Y)) = g(Z) = Y \), so \( g \cap (Z \times Y) \) is onto by Theorem 5.2.3.

Since \( g \cap (Z \times Y) \) is one-to-one and onto, its inverse is a function, \( (g \cap (Z \times Y))^{-1} : Y \to Z \), and it is one-to-one and onto, by Theorem 5.3.4. \( \square \)
Claim 5: \((f \cap (X \times W)) \cup (g \cap (Y \times Z))^{-1}: A \to B\) is one-to-one and onto.

**Proof.**

By Claims 2 and 4, the functions \(f \cap (X \times W)\) and \((g \cap (Y \times Z))^{-1}\) are one-to-one and onto.

Note that \(X\) and \(Y = A \setminus X\) are disjoint, as are \(\text{Ran}(f \cap (X \times W)) = W\) and \(\text{Ran}(g^{-1} \cap (Y \times Z)) = Z\).

By Exercises 5.1: 9a and 5.2:12, \((f \cap (X \times W)) \cup (g \cap (Y \times Z))^{-1}\) is a one-to-one function from \(A = X \cup Y\) to \(B = W \cup Z\).

Since the range of \((f \cap (X \times W)) \cup (g^{-1} \cap (Y \times Z))\) is \(W \cup Z = B\), by Theorem 5.2.3, it is onto and Claim 5 follows. 

\(\square\)
Claim 6: The function $h : A \rightarrow B$ defined for all $a \in A$ by $h(a) = f(a)$ and if $a \in X$ and $h(a) = g^{-1}(a)$ if $a \in A \setminus X$ is one-to-one and onto.

Proof.

By Claim 5, $(f \cap (X \times W)) \cup (g^{-1} \cap (Y \times Z))$ has the same domain and codomain as $h$.

Note that $h$, $f$, and $f \cap (X \times W)$ agree on $X$. Also $h$, $g^{-1}$ and $g^{-1} \cap (Y \times Z)$ agree on $Y$.

Thus $h$ and $(f \cap (X \times W)) \cup (g^{-1} \cap (Y \times Z))$ agree on $A = X \cup Y$, so $h = (f \cap (X \times W)) \cup (g^{-1} \cap (Y \times Z))$, by Theorem 5.1.4. \qed
Closing of the Proof:

By Claim 6, \( h : A \rightarrow B \) is one-to-one and onto, so it witnesses that \( A \sim B \) \( (p\exists) \). This assertion completes Case 2.

By exhaustive case analysis, \( A \sim B \).

We assumed \( A \preceq B \) and \( B \preceq A \) and proved \( A \sim B \), so the implication and the theorem follow \( (c\rightarrow) \).