Handbook of Set Theory

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I. Partition Relations

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1. Introduction

The study of partition relations dates back to 1930, when F. P. Ramsey [50] proved his oft-cited theorem.

1.1 Theorem (Ramsey's Theorem). *Assume* $1 \leq r,\ k < \omega$ *and* $f : [\omega]^r \to k$ *is a partition of the* r *element subsets of* ω *to* k *pieces. Then there is an infinite subset* $X \subseteq \omega$ *homogeneous with respect to this partition. That is, for some* $i < k,\ f``[X]^r = \{i\}$ *.*

In 1941, B. Dushnik and E.W. Miller [9] looked at partitions of the set of all pairs of elements of an uncountable set, involving P. Erdős in solving one of their more difficult problems (see Theorem 7.4). In 1942, P. Erdős [10] proved some basic generalizations of Ramsey's Theorem, including among others the theorem generally called the Erdős-Rado Theorem for pairs. In the early fifties, P. Erdős and R. Rado [17], [19] initiated a systematic investigation of quantitative generalizations of this result. They called it the partition calculus. There are cases in mathematical history when a well-chosen notation can enormously enhance the development of a branch of mathematics and a case in point is the ordinary partition symbol (see Definition 1.3)

$$\alpha \to (\beta_\xi)^r_{\xi<\gamma}$$

invented by Richard Rado [18], reducing Ramsey's Theorem to $\omega \to (\omega)^r_\gamma$ for $1 \leq r, \gamma < \omega$. It became clear that a careful analysis of the problems according to the size and nature of the parameters leads to an inexhaustable array of problems, each seemingly simple and natural. These classical investigations were completed in the 1965 paper [15] of Erdős, Hajnal and Rado, and were extended in the book [14] written jointly with Attila Máté.

In 1967, after the first post Cohen set theory conference, held in Los Angeles, Erdős and Hajnal wrote a list of unsolved problems for the ordinary partition symbol and related topics. This paper [12] appeared in print four years later.

A great many new results were proved by the *then* young researchers. However, unlike many other classical problems, these problems yielded some but continued to resist. The introduction of new methods and the discovery of new ideas usually has given only incremental progress, and objectively, we are as far as ever from complete answers. However, small steps requiring new methods have been continuously made, quite a few of them during the writing of this paper, and we will concentrate on them.

For easy reference, in the ordinary partition relation $\alpha \to (\beta_\xi)^r_\gamma$, we call α the *resource*, β_ξ the *goals*, and γ the *set of colors*. We will be focusing on two main subjects:

1. New ZFC theorems obtained via the elementary submodel method both for ordinary partition relations and for polarized partition relations (see Definition 1.5).

2. The new results obtained in the late nineties for partition relations with a countable resource.

Section 2 describes the classical proofs of the (balanced) form of the Erdős-Rado Theorem and the Positive Stepping Up Lemma. These are the results where the resource is regular and the goals are equal and of the form τ, or $\tau + 1$ for some cardinal τ. In subsection 2.3 we state but do not prove the Negative Stepping Up Lemma complementing these results.

In Section 3, we describe the elementary submodel method and in particular, the use of nonreflecting ideals first introduced in [4]. We give an alternate proof of the balanced Erdős-Rado Theorem, and give a proof of the unbalanced form of it using the new method.

In Section 4, especially in subsection 4.2, we fully develop the method of elementary submodels. We give streamlined proofs of both the balanced and unbalanced forms of the Baumgartner-Hajnal-Todorcevic Theorems [4] in subsections 4.3 and 4.4. These results generalize the Erdős-Rado Theorem to allow goals which are ordinals more complex than cardinals τ and their ordinal successors, $\tau + 1$. We state a result of Foreman and Hajnal [20] for the successors of measurable cardinals. Using the methods of the Foreman-Hajnal proof, in subsection 4.5, we give a direct proof of a special case of the Baumgartner-Hajnal Theorem [2].

In Section 5, we discuss the Milner-Rado Paradox and the new ordinal $\Omega(\kappa) < \kappa^+$ introduced in the Foreman-Hajnal result [20], which is related to a form of the Milner-Rado Paradox.

In Section 6, we discuss a new development, the first in the twenty-first century. Solving a problem of Foreman and Hajnal, Shelah [56] proved that if there is a strongly compact cardinal, then there are cardinals κ such that $\kappa^+ \to (\kappa+2)^2_\omega$.

In Section 7, we briefly discuss the case of singular resources. We state but do not prove a compilation of theorems on this subject from the 1965 Erdős, Hajnal and Rado paper [15] and the 1975 Shelah paper [58].

In Section 8, we describe a new variant of the elementary submodel method called *double ramification*, which was invented by Baumgartner and Hajnal in 8.2.

In subsection 8.1, we use it for the proof of

$$(*) \quad \begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \kappa \\ \kappa \end{pmatrix}^{1,1}_\gamma$$

where κ is weakly compact and $\gamma < \kappa$. Result $(*)$ was previously known only if $\gamma < \omega$ (see the discussion before Theorem 8.2). In subsection 8.2, we use the method for the proof of Shelah's Theorem [60] stating that $(*)$ holds for κ a singular strong limit cardinal (of uncountable cofinality) which satisfies $2^\kappa > \kappa^+$ and for $\gamma < \mathrm{cf}(\kappa)$.

In Section 9, we discuss the spectacular progress by Carl Darby [7], [8] and Rene Schipperus [54], [52] on the cases where the resource α is a countable ordinal, listing their negative partition results in Theorem 9.9, and give a sample counterexample, $\omega^{\omega^2} \not\to (\omega^{\omega^2}, 6)^2$. This example is not optimal, but was chosen to illustrate the methods of Darby without all the complicating detail.

In Section 10, we outline a proof of a special case of the positive results by Schipperus that $\omega^{\omega^\beta} \to (\omega^{\omega^\beta}, 3)^2$ for $\beta \geq 2$ the sum of one or two indecomposable ordinals (Darby independently proved the result for $\beta = 2$).

We close this section with some background definitions.

1.1. Basic Definitions

1.2 Definition. Let X be a set, $r < \omega$ and β, γ be ordinals.

1. A map $f : [X]^r \to \gamma$ is called an *r-partition of X with γ colors.*

2. For $\xi < \gamma$, a subset $Y \subseteq X$ is called *homogeneous for f in color ξ* if $f``[Y]^r = \{\xi\}$.

3. The set $Y \subseteq X$ is *homogeneous for f* if it is homogeneous for f in some color $\xi < \gamma$.

4. A linearly ordered set X has *order type β*, in symbols, $\mathrm{ot}\, X = \beta$, if it is order isomorphic to β.

1.3 Definition. Let α, β_ξ for $\xi < \gamma$, and γ be ordinals and suppose $1 \leq r < \omega$. The *ordinary partition symbol*

$$\alpha \to (\beta_\xi)^r_\gamma$$

means that the following statement is true.

> For every r-partition of α with γ colors, $f : [\alpha]^r \to \gamma$, there exist $\xi < \gamma$ and $X \subseteq \alpha$ such that $\operatorname{ot} X = \beta_\xi$ and X is homogeneous for f in color ξ.

We write

$$\alpha \not\to (\beta_\xi)^r_\gamma$$

to indicate that the negation of this statement is true. If all β_ξ equal β, then we write

$$\alpha \to (\beta)^r_\gamma \quad (\text{or } \alpha \not\to (\beta)^r_\gamma).$$

A further more or less self explanatory abbreviation is $\alpha \to (\beta_0, (\beta)_\gamma)^2$ in case $\beta_\xi = \beta$ for $1 \leq \xi < \gamma$.

1.4 Remark. Note that the notation of Definition 1.3 is so devised that if we start with a positive partition relation $\alpha \to (\beta_\xi)^r_\gamma$, then the truth of the assertion is preserved under increasing the *resource* ordinal α on the lefthand side of the arrow ($\to$) and decreasing the ordinal *goals* β_ξ, or the colors γ on the righthand side of the arrow. And this latter statement holds, with some exceptions, for the exponent r as well (see [14]).

We stated Definition 1.3 in this generality, because it will suffice for most of what we will prove. It should be clear that further generalizations can be made. For example, a similar symbol $\Theta \to (\Theta_\xi)^\delta_\gamma$ can be defined where $\Theta, \Theta_\xi, \delta$ are order types, by starting with an arbitrary ordered set $\langle X, \prec \rangle$ for which $\operatorname{ot}(X, \prec) = \Theta$, partitioning its subsets of order type δ,

$$[X]^\delta = \{\, Y \subseteq X : \operatorname{ot}(Y, \prec) = \delta \,\},$$

into γ color classes, and as above, looking for homogeneous subsets of the prescribed color and order type. As general Ramsey theory developed in both finite and infinite combinatorics, problems were considered in which the set partitioned was a subset of $[X]^\delta$ rather than all of $[X]^\delta$, and the homogeneous sets consisted of possibly other kind of subsets of $[X]^\delta$. Partition relations proliferated. For a review of some of them we refer to [14], since we can not try to cover all of them in the limit space of this chapter.

In [15], among other generalizations, polarized partitions were introduced. In fact, this paper is the only place in the published literature where these relations are systematically discussed.

1.5 Definition. Let α, β be ordinals and suppose that $\alpha_0, \alpha_1 \leq \alpha$ and $\beta_0, \beta_1 \leq \beta$. The *polarized partition relation*

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \begin{pmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{pmatrix}$$

means that the following statement is true.

> For all ordered sets A and B of order type α, β respectively, and all partitions $f : A \times B \to 2$, there is an $i < 2$ and sets $A_i \subseteq A$, $B_i \subseteq B$ such that $\operatorname{ot} A_i = \alpha_i$, $\operatorname{ot} B_i = \beta_1$ and $f``A_i \times B_i = \{i\}$.

2. Basic Partition Relations

2.1. Ramsey's theorem

2.1 Definition. Assume $\langle X, \prec \rangle$ is an ordered set and $f : [X]^r \to \gamma$ is an r-partition of length γ of X, $1 \leq r < \omega$.

1. For $V \in [X]^{r-1}$, define $f_V : X \smallsetminus V \to \gamma$ by
$$f_V(u) = f(V \cup \{u\})$$
2. f is *endhomogeneous on* X if for every $V \in [X]^{r-1}$, the function f_V is homogeneous on $X| \succ V = \{\, u \in X : V \prec u \,\}$.
3. Let
$$X^- = \begin{cases} X - \{m\} & \text{if } X \text{ has a maximal element } m \\ X & \text{otherwise} \end{cases}$$
4. Assume f is endhomogeneous on X. Define $f^- : [X^-]^{r-1} \to \gamma$ by $f^-(V) = \eta$ iff $\forall u \in X| \prec V (f_V(u) = \eta)$ for $V \in [X^-]^{r-1}$.

The next lemma follows immediately from the definitions.

2.2 Lemma. *Using the above notation, if f is endhomogeneous on X, $Y \subseteq X^-$ and f^- is homogeneous on Y then f is homogeneous on Y and on $Y \cup \{m\}$ if m is the maximal element of X.*

We first give a direct proof of the well-known Ramsey's Theorem using non-principal ultrafilters and postponing the more natural *ramification* method to the next section for two reasons. First, Erdős and Rado considered this approach part of their "combinatorics", (Erdős called the ultrafilters "measures"). Second, having given a proof here, we do not have to adapt the forumulation of the ramification to cover the case when the resource is a regular limit cardinal.

2.3 Theorem (Ramsey's Theorem).

$$\omega \to (\omega)^r_k \quad \textit{for } 1 \le r, k < \omega$$

Proof. By induction on r. For $r = 1$ the claim is obvious. Assume $r > 1$ and $f : [\omega]^r \to k$. Let U be a non-principal ultrafilter on ω and $V \in [\omega]^{r-1}$. Define $\tilde{f}(V)$ and $A(V)$ as follows: let $\tilde{f}(V) = i$ for the unique $i < k$ for which the set $A(V, i) := \{ u \in \omega - V : f_V(u) = i \}$ is in U, and set $A(V) := A(V, \tilde{f}(V))$.

We can choose by induction on n an increasing sequence $\langle x_n : n < \omega \rangle$ of integers satisfying $x_n \in \bigcap \{ A(V) : V \in [\{ x_j : j < n \}]^r \}$ for $n < \omega$. Let $X = \{ x_n : n < \omega \}$. Then $f^-|[X]^{r-1} = \tilde{f}|[X]^{r-1}$ and f is endhomogeneous on X. By the induction hypothesis, there is a $Y \subseteq X$ with $\mathrm{ot}(Y) = \omega$ so that Y is homogeneous for f^-. Finally, by Lemma 2.2, Y is the desired set homogeneous for f. ⊣

2.2. Ramification Arguments

2.4 Remark (A brief history). The first transfinite generalization of Ramsey's theorem appeared in the paper [9] of Dushnik and Miller. They proved $\kappa \to (\kappa, \omega)^2$ for regular κ and Erdős proved this for singular κ as well. His proof was included in [9]. This theorem, unique of its kind, logically belongs to Section 7 where we will discuss it briefly.

The basic theorems about partition relations with exponent $r = 2$ were first stated and proved in 1942 in an almost forgotten paper of Erdős [10]. There he proved $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ for $\kappa \ge \omega$; he indicated the counterexamples $2^\kappa \not\to (3)^2_\kappa$ and $2^\kappa \not\to (\kappa^+)^2_2$; and he proved $\omega_2 \to (\omega_2, \omega_1)^2$ assuming CH. The Erdős-Rado Theorem for exponent larger than 2 was proved later in [19]. (See Corollary 2.10.) Kurepa also worked on related questions quite early (see the discussion by Todorcevic in Section C of [38]).

Few theorems had so many simplified proofs as $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$, the Erdős-Rado Theorem. Erdős and Rado used the so called "ramification method". We will present this method in the proof of the next theorem. After some "streamlining," it still seems to be the simplest way for obtaining *balanced* partition relations for cardinals, ones in which all the goals are the same cardinal. For the *unbalanced* case, we will present a method worked out in [4]. This method will be used in the proofs of a number of more recent results which will be presented in later sections. Given limitations of time and energy, and a desire for coherence, we decided to focus on results amenable to this method.

2.5 Theorem. *Assume* $2 \le r < \omega,\ \kappa \ge \omega,\ \gamma < \kappa,\ \lambda = 2^{<\kappa}$ *and*

$$f : [\lambda^+]^r \to \gamma.$$

Then there exists an $X \subseteq \lambda^+$ *with* $\mathrm{ot}(X) = \kappa + 1$ *such that* f *is endhomogeneous on* X.

Proof. For $\alpha < \lambda^+$, define an increasing sequence $\overline{\beta}^\alpha = \langle \beta^\alpha_\eta : \eta < \varphi_\alpha \rangle$ of ordinals less then α and an ordinal φ_α by transfinite recursion on η. For $\alpha = 0$, set $\varphi_0 = 0$ and let $\overline{\beta}^0$ be the empty sequence. For positive α, to start the recursion, let $\beta^\alpha_q := q$ for $q < \max\{\alpha, r-1\}$, and for $\alpha < r-1$, let $\varphi_\alpha = \alpha$. To continue the recursion, assume $r - 2 < \eta$ and β^α_ζ is defined for $\zeta < \eta$. Let $\hat{\beta}^\alpha_\eta = \sup \left\{ \beta^\alpha_\zeta + 1 : \zeta < \eta \right\}$, and define sets

$$\begin{aligned} B^\alpha_\eta &:= \left\{ \beta^\alpha_\zeta : \zeta < \eta \right\} \\ A^\alpha_\eta &:= \left\{ \beta < \alpha : \hat{\beta}^\alpha_\eta \le \beta \wedge (\forall V \in [B^\alpha_\eta]^{r-1})(f_V(\beta) = f_V(\alpha)) \right\}. \end{aligned}$$

Let $\beta^\alpha_\eta := min A^\alpha_\eta$ if $A^\alpha_\eta \neq \emptyset$. If $A^\alpha_\eta = \emptyset$, put $\varphi_\alpha = \eta$. Clearly for each $\alpha < \lambda^+$, the set $B^\alpha_{\varphi_\alpha} \cup \{\alpha\}$ is an endhomogeneous set of order type $\varphi_\alpha + 1$, and we may define f^-_α on $[B^\alpha_{\varphi_\alpha}]^{r-1}$ as in Definition 2.1. If $\beta \in B^\alpha_{\varphi_\alpha}$, then it is easy to show by induction on $\eta < \varphi_\beta$ that $\beta^\beta_\eta = \beta^\alpha_\eta$. Thus if $\beta \in B^\alpha_{\varphi_\alpha}$, then f^-_α agrees with f^-_β on $[B^\beta_{\varphi_\beta}]^{r-1}$.

Define a relation $\prec$ on λ^+ by $\beta \prec \alpha$ iff $\beta \in B^\alpha_{\varphi_\alpha}$. It is easy to verify that $T := \langle \lambda^+, \prec \rangle$ is a tree on λ^+ and $\mathrm{rank}_T(\alpha) = \varphi_\alpha$ for $\alpha < \lambda^+$. T is called the *canonical partition tree* of f on λ^+, and T_φ, as usual, denotes the $\{ \alpha < \lambda^+ : \mathrm{rank}_T(\alpha) = \varphi \}$.

For $\alpha < \lambda^+$, let $C_\alpha : [\varphi_\alpha]^{r-1} \to \gamma$ be defined by $C_\alpha(U) = f^-_\alpha(V)$ where $V = \left\{ \beta^\alpha_\zeta : \zeta \in U \right\}$. It follows by transfinite induction on φ that for $\alpha, \beta \in T_\varphi$, if $C_\alpha = C_\beta$, then $\alpha = \beta$. Hence $|T_\varphi| \le |\gamma|^{|\varphi|} \le \lambda$ for $\varphi < \kappa$. Then $\left| \bigcup_{\varphi<\kappa} T_\varphi \right| \le \lambda$, $T_\kappa \neq \emptyset$ and for all $\alpha \in T_\kappa$, $B^\alpha_\kappa \cup \{\alpha\}$ is a set of order type $\kappa + 1$ which is endhomogeneous for f. ⊣

2.6 Remark. Note that $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$ can hold for singular κ. Indeed it is easy to see that either $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$ or $\mathrm{cf}\,(2^{<\kappa})^{<\kappa} = \mathrm{cf}(\kappa)$ and $2^{<\kappa} = \sup\{ (2^\tau)^+ : \tau < \kappa \}$. The proof described above gives Theorem 2.5 under the condition $\gamma \le \lambda$ provided $\lambda^{<\kappa} = \lambda$.

2.7 Theorem (The Stepping Up Lemma). *Assume* $\kappa \ge \omega$, $1 \le r < \omega$, $\gamma < \kappa$ *and* $\kappa \to (\alpha_\xi)^r_\gamma$. *Then*

$$\left(2^{<\kappa}\right)^+ \to (\alpha_\xi + 1)^{r+1}_\gamma.$$

This is an immediate consequence of Lemma 2.2 and Theorem 2.5.

2.8 Definition. Define $\exp_i(\kappa)$ by recursion on $i < \omega$:

$$\begin{aligned} \exp_0(\kappa) &= \kappa, \\ \exp_{i+1}(\kappa) &= 2^{\exp_i(\kappa)}. \end{aligned}$$

2.9 Theorem (The Erdős-Rado Theorem). *Assume $\kappa \geq \omega$, $\gamma < \mathrm{cf}(\kappa)$. Then for all $2 \leq r < \omega$,*

$$\exp_{r-2}\left(2^{<\kappa}\right)^{+} \rightarrow (\kappa + (r-1))^{r}_{\gamma}.$$

Proof. Starting from the trivial relation $\kappa \rightarrow (\kappa)^1_\gamma$ for $\gamma < \mathrm{cf}\,\kappa$, we get $(2^{<\kappa})^{+} \rightarrow (\kappa+1)^2_\gamma$, by Theorem 2.7. This is the case $r = 2$ of the theorem. The result follows by induction on r with repeated applications of Theorem 2.7. ⊣

A better known but weaker form of the theorem is the following.

2.10 Corollary. *Assume $\kappa \geq \omega$. Then for all $1 \leq r < \omega$,*

$$\exp_{r-1}(\kappa)^{+} \rightarrow \left(\kappa^{+} + (r-1)\right)^{r}_{\kappa}.$$

Note that while Theorem 2.9 guarantees for example that $\kappa^{+} \rightarrow (\kappa+1)^2_\gamma$ holds for $\gamma < \mathrm{cf}(\kappa)$ for a singular strong limit cardinal κ, Corollary 2.10 does not say anything about this case.

2.3. Negative Stepping Up Lemma

2.11 Theorem (The Negative Stepping Up Lemma). *Assume $\kappa > 0$ is a cardinal, $2 \leq r < \omega$, $1 \leq \gamma$ and $\kappa \nrightarrow (\lambda_\xi)^r_\gamma$, where each $\lambda_\xi > 0$ is a cardinal. Then $2^\kappa \nrightarrow (1 + \lambda_\xi)^{r+1}_\gamma$, provided at least one of the following conditions hold:*

1. *$\gamma \geq 2$, $\kappa, \lambda_0, \lambda_1 \geq \omega$ and λ_0 is a regular cardinal;*
2. *$\gamma \geq 2$, $\kappa, \lambda_0 \geq \omega$, λ_0 is a regular cardinal, and $r \geq 4$;*
3. *$\gamma \geq 2$, $\kappa, \lambda_0, \lambda_1 \geq \omega$, and $r \geq 4$;*
4. *$\kappa \geq \omega$ and $\lambda_\xi < \omega$ for all $\xi < \gamma$.*

For a proof, we refer the reader to the compendium by Erdős, Hajnal, Máté and Rado [14], which includes additional negative stepping up results. We do quote one open problem from that reference.

2.12 Question (Problem 25.8 in [14]). Assume GCH. Does

$$\aleph_{\omega_{\omega+1}+1} \nrightarrow \left(\aleph_{\omega_{\omega+1}+1}, (4)_\omega\right)^3?$$

The following theorem provides a context for this question.

2.13 Theorem. *Assume GCH. Then*

1. *$\aleph_{\omega+1} \nrightarrow (\aleph_{\omega+1}, (3)_\omega)^2$ and*
2. *$\aleph_{\omega_{\omega+1}} \nrightarrow \left(\aleph_{\omega_{\omega+1}}, (3)_\omega\right)^2$.*

3. Partition relations and submodels

For the rest of this paper we will adopt the following conventions. Whenever we write "$H(\tau)$", τ will be a regular cardinal, and "$H(\tau)$" will stand for a structure $\mathfrak{A}$ with domain the collection of sets $H(\tau)$ which are of hereditary cardinality $< \tau$. The structure $\mathfrak{A}$ will be an expansion of $\langle H(\tau), \in, \triangle \rangle$, where $\triangle$ is a fixed well ordering of $H(\tau)$. The expansion will depend on context, and will usually include all of the relevant "data" for the proof at hand. Note that the well ordering $\triangle$ yields well defined Skolem hulls for all sets $X \subseteq H(\tau)$.

3.1 Definition. Assume $\kappa \geq \omega$, $2^{<\kappa} = \lambda$. Let $H := H(\lambda^{++})$. A set N is said to be *suitable for* κ if it satisfies the following conditions: $\langle N, \in \rangle \prec H$, $|N| = \lambda$, $[N]^{<\mathrm{cf}(\kappa)} \subseteq N$, $[N]^{<\kappa} \subseteq N$ if $\lambda^{<\kappa} = \lambda$, $\lambda + 1 \subseteq N$, $\alpha := N \cap \lambda^+ \in \lambda^+$, $\mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)$. The ordinal $\alpha(N) = \alpha$ will be called the *critical ordinal of* N. Note that $\alpha \subseteq N$ by assumption.

We assume that the reader is familiar with the theory of stationary subsets of an ordinal. To make our terminology definite, for a limit ordinal α, a subset $B \subseteq \alpha$ is a *club* if B is cofinal (unbounded) and closed in the order topology of α. A set $S \subseteq \alpha$ is *stationary* if $B \cap S \neq \emptyset$ for every club subset of α. The notation $\mathrm{Stat}(\alpha)$ will denote the set of stationary subsets of α.

We will make use of the following facts about elementary submodels.

3.2 Facts. Let $\lambda = 2^{<\kappa}$. For every set A with $|A| \leq \lambda$ and $A \in H(\lambda^{++})$, there is an elementary chain $\langle N_0, \in \rangle \prec \cdots \prec \langle N_\alpha, \in \rangle \prec \cdots \prec H$, with $A \subseteq N_0$, indexed by $\alpha < \lambda^+$ that is continuous, and internally approachable (i.e. $N_\beta \in N_{\alpha+1}$ for all $\beta \leq \alpha$), and the set

$$S_0 = \{\, \alpha < \lambda^+ : \alpha(N_\alpha) = \alpha \text{ and } N_\alpha \text{ is suitable for } \kappa \,\}$$

the intersection of a club in λ^+ with $S_{\mathrm{cf}(\kappa),\lambda^+} = \{\, \alpha < \lambda^+ : \mathrm{cf}(\alpha) = \mathrm{cf}(\kappa) \,\}$.

3.3 Definition. A subset $S \subseteq H(\lambda^{++})$ is *amenable* for this sequence if $S \cap \alpha \in N_{\alpha+1}$ for $\alpha \in S_0$. A function g is *amenable* if $g|\alpha \in N_{\alpha+1}$ for all $\alpha \in S_0$.

Note that S_0 itself may be assumed to be amenable.

In this section we will only use the existence of one N suitable for κ. The ideals defined below were introduced in [4] for regular κ. In most of the later applications we will only consider the regular case.

3.4 Definition. Let N be suitable for $\kappa \geq \omega$, $\lambda = 2^{<\kappa}$, $\alpha(N) = \alpha$. We define a set $I = I_\alpha = I(N) \subseteq \mathcal{P}(\alpha)$ as follows. For $X \subseteq \alpha$,

$$X \in I \Leftrightarrow (\exists Y)(Y \subseteq \lambda^+ \wedge Y \in N \wedge \alpha \notin Y \wedge |X - Y| < \kappa).$$

Note that for regular κ, the last clause can be replaced by $X \subseteq Y$.

3.5 Lemma. *Let N be suitable for $\kappa \geq \omega$, $\lambda = 2^{<\kappa}$, $\alpha(N) = \alpha$. We define a set $\mathcal{F} = \mathcal{F}_\alpha$ as follows:*

$$\mathcal{F}_\alpha := \{\, Z \in N : Z \subseteq \lambda^+ \wedge \alpha \in Z \,\}.$$

Then (i) $X \notin I = I_\alpha$ if and only if $|X \cap Z| \geq \kappa$ for all $Z \in \mathcal{F}_\alpha$; and (ii) the elements Z of $\mathcal{F}_\alpha$ are stationary subsets of λ^+.

Proof. Part (i) follows directly from Definition 3.4. To see that part (ii) also holds, we verify that $\alpha \in Z \subseteq \lambda^+$, $Z \in N$ imply that Z is stationary. Otherwise $Z \cap \beta = \emptyset$ for some club $B \in N$. Then $B \cap \alpha$ is cofinal in α, by elementarity and $\alpha \in B$ since B is closed. ⊣

3.6 Lemma. *If N is suitable for κ, then $I = I(N)$ is a* $\mathrm{cf}(\kappa)$*-complete proper ideal on $\alpha = \alpha(N)$. Moreover, if $\lambda^{<\kappa} = \lambda$, then I is κ-complete.*

Proof. The completeness clearly follows from $[N]^{<\mathrm{cf}(\kappa)} \subseteq N$ and $[N]^{<\kappa} \subseteq N$ respectively. To see that $\alpha \notin I$, let $Z \in N$ be a subset of λ^+ with $\alpha \in Z$. It is enough to show that $|Z \cap \alpha| = \lambda$. Since $Z \in N$, also $\sup(Z) \in N$. As $\alpha \in Z$ and $N \cap \lambda^+ = \alpha$, it follows that $\sup(Z) = \lambda^+$. Then *a fortiori* there is a one-to-one function $g : \lambda \to Z$. Hence there is a $g \in N$ like this. Using $\lambda + 1 \subseteq N$, we get that $\mathrm{ran}(g) \subseteq N \cap \lambda^+ = \alpha$. ⊣

In what follows we will often suppress details like those given above.

3.7 Definition. Assume N is suitable for κ, $\lambda = 2^{<\kappa}$ and $\alpha = \alpha(N)$. For $X \subseteq \alpha$, we say X *reflects the properties of* α if $X \cap Z \neq \emptyset$ for all $Z \in \mathcal{F}_\alpha$.

3.8 Lemma. *Assume N is suitable for κ, $\lambda = 2^{<\kappa}$ and $\alpha = \alpha(N)$. If $X \subseteq \alpha$ and $X \in I^+$, then X reflects the properties of α, so we call $I = I_\alpha$ the* non-reflecting ideal on α (induced by N).

Notation. Assume $f : [X]^2 \to \gamma$ is a function, $\eta < \gamma$ and $\alpha \in X$. For simplicity, we often write $f(\alpha, \beta)$ for $f(\{\,\alpha, \beta\,\})$, specifying which of the ordinals α, β is smaller, if necessary. Denote the set $\{\,\beta < \alpha : f(\alpha, \beta) = \eta\,\}$ by $f(\alpha; \eta)$.

3.9 Lemma (Connection Lemma). *Assume $\kappa \geq \omega$ and $\lambda = 2^{<\kappa}$. Further suppose that N is suitable for κ with $\alpha(N) = \alpha$, $f \in N$ is a 2-partition of λ^+ with $\gamma < \mathrm{cf}(\kappa)$ colors, and $X \subseteq f(\alpha; \eta) \cap \alpha$ for some $\eta < \gamma$ is such that $X \notin I = I(N)$. Then there is some $Y \subseteq X$ with $\mathrm{ot}(Y) = \mathrm{cf}(\kappa)$ so that $Y \cup \{\alpha\}$ is homogeneous for f in color η.*

Proof. Let Z be a subset of $X \cup \{\alpha\}$ maximal with respect to the following properties: $\alpha \in Z$ and Z is homogeneous for f in color η. If $|Z| \geq \mathrm{cf}(\kappa)$, then we are done. Assume by way of contradiction that $|Z| < \mathrm{cf}(\kappa)$. Then

$\sup(Z \cap \alpha) < \alpha$ and $Z \cap \alpha \in N$. Let $A = \bigcap \{ f(u;\eta) : u \in Z \cap \alpha \}$. Then $A \in N$ and $\alpha \in A$. Hence, by the reflection property, $A \cap (X - \sup(Z \cap \alpha)) \neq \emptyset$. If $y \in A \cap (X - \sup(Z \cap \alpha))$, then $\{y\} \cup Z$ is homogeneous for f in color η, contradicting the maximality of Z. ⊣

3.10 Theorem (Erdős-Rado Theorem (unbalanced form)). *Let κ be an infinite cardinal and $\gamma < \mathrm{cf}(\kappa)$. Then*

$$\left(2^{<\kappa}\right)^+ \to \left(\left(2^{<\kappa}\right)^+, (\mathrm{cf}(\kappa)+1)_\gamma\right)^2.$$

Proof. Let $\lambda = 2^{<\kappa}$, and suppose $f : [\lambda^+]^2 \to \gamma$ is a 2-partition of λ^+ into γ colors. Use Facts 3.2 to choose N suitable for κ with $f \in N$. For notational simplicity, let $\alpha = \alpha(N)$ and $I = I(N)$. If $f(\alpha;\eta) \cap \alpha \notin I$ for some $1 \leq \eta < \gamma$, then we are done by Lemma 3.9. By Lemma 3.6, we may assume that $\alpha - f(\alpha;0) \subseteq \bigcup \{ f(\alpha;\eta) \cap \alpha : 1 \leq \eta < \gamma \} \in I$. By Definition 3.4, there is a set $Z \in N$ with $Z \subseteq \lambda^+$ and $\alpha \in Z$ for which $|Z - f(\alpha;0)| < \kappa$. Define a set W in $H(\lambda^{++})$ as follows:

$$W := \{ \beta \in Z : |Z - f(\beta;0)| < \kappa \}.$$

Then $W \in N$ and $\alpha \in W$. Then by Lemma 3.5 we infer that $W \in \mathrm{Stat}(\lambda^+)$ and for $g(\delta) := \{ \beta < \delta : f(\beta,\delta) \neq 0 \}$, we have $|g(\delta)| < \kappa$ for all $\delta \in W$. By Fodor's Set Mapping Theorem [14], there is a stationary subset $S \subseteq W$ free for g (i.e. $\gamma \notin g(\delta)$ for all $\delta \neq \gamma \in S$), and S is homogeneous for f in color 0. ⊣

Note that with some abuse of notation we have proved the following stronger result.

3.11 Theorem. *Let $\kappa \geq \omega$, $\lambda = 2^{<\kappa}$ and suppose $\gamma < \mathrm{cf}(\kappa)$. Then*

$$\lambda^+ \to (\mathrm{Stat}(\lambda^+), (\mathrm{cf}(\kappa)+1)_\gamma)^2.$$

This theorem should be compared with the case $r = 2$ of Theorem 2.9 and it should be observed that while for regular κ, the above theorem is a strengthening of Corollary 2.10, for singular κ the results are incomparable. It should also be noted that using Theorem 2.7, the above result can be stepped up to the following.

3.12 Corollary. *Assume $\kappa \geq \omega$ and $\gamma < \mathrm{cf}(\kappa)$. Then for all $1 \leq r < \omega$,*

$$\exp_{r-2}\left(2^{<\kappa}\right)^+ \to \left(\left(2^{<\kappa}\right)^+, (\kappa + (r-1))_\gamma\right)^r.$$

Finally it should be remarked that we did not try to state the strongest possible forms of the Erdős-Rado theorems. Clearly the methods give similar results in cases where the resource cardinal κ is a regular limit cardinal. For a detailed discussion we refer to [14].

4. Generalizations of the Erdős-Rado Theorem

4.1. Overview

In this section we focus on the problem of what positive relations of the form

$$\left(2^{<\kappa}\right)^{+} \rightarrow (\alpha_\xi)^2_\gamma$$

can be proved for regular κ and $\gamma < \kappa$ in ZFC. The case for singular κ will be almost entirely omitted because of limitations of space. Many problems remain unsolved, and the simplest of these will be stated at the end of this subsection. We start by discussing limitations, the first of which comes from the next theorem.

4.1 Theorem (Hajnal[25], Todorcevic). *If $2^\kappa = \kappa^+$, then*

$$\kappa^+ \not\rightarrow (\kappa^+, \kappa + 2)^2.$$

Proof Outline. We only sketch the proof given in [25], omitting Todorcevic's proof for singular κ, which has been circulated in unpublished notes. Let $\{ A_\alpha : \alpha < \kappa^+ \}$ be a well-ordering of $[\alpha]^\kappa$. Define a sequence of sets $B_\alpha \in [\kappa^+]^\kappa$ for $\alpha < \kappa^+$ by transfinite recursion on α, in such a way that the following two conditions are satisfied:

1. $|B_\alpha \cap B_\beta| < \kappa$ for all $\beta < \alpha$;
2. $B_\alpha \cap A_\beta \neq \emptyset$ for all $\beta < \alpha$ for which $|A_\beta - \bigcup \{ B_\gamma : \gamma \in F \} | = \kappa$ for all $F \in [\alpha]^{<\kappa}$.

To complete the proof, for $\beta < \alpha < \kappa^+$, set $f(\beta, \alpha) = 1$ if and only if $\beta \in B_\alpha$.

The constraint that $|B_\alpha \cap B_\beta| < \kappa$ for all $\beta < \alpha < \kappa^+$ implies that f has no homogeneous subsets of order type $\kappa + 2$ for color 1. The assertion that it has no homogeneous subsets of order type κ^+ for color 0 follows from the claim below.

4.2 Claim. *Assume A is a subset of size κ^+. Then there is a subset B of A of size κ which is not almost contained in the union of fewer than κ many B_β's.*

On the one hand, if fewer than κ many B_β's meet A in a set of size κ, then any subset $B \subseteq A$ of size κ in the complement of the union of these B_α's proves the claim. Otherwise, choose a sequence $B_\beta(\eta)$ indexed by $\eta < \kappa$ of κ many sets whose intersection with A has cardinality κ, and let B be the union of the intersections $A \cap B_\beta(\eta)$. $\dashv$

Henceforth we will assume that the goals, α_ξ, are all ordinals, $\alpha_\xi < \kappa^+$ for $\xi < \gamma$.

For $\kappa = \omega$, the best possible result, $\omega_1 \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and k finite was conjectured by Erdős and Rado [17] in 1952 and proved by Baumgartner and Hajnal [2] in 1971, already in a more general form. Using a self-explanatory extension of the ordinary partition relation for linear order types, it says

$$\Theta \to (\omega)^1_\omega \text{ implies } \Theta \to (\alpha)^2_k \text{ for all } \alpha < \omega_1,\ k < \omega.$$

Soon after it was generalized (also in a self-explanatory way) by Todorcevic to partial orders [64]. Schipperus [53] proved a topological version. The Baumgartner-Hajnal proof used "Martin's Axiom + absoluteness". An elementary proof not using this kind of argument was given by Fred Galvin [21] in 1975. We will treat this theorem later in Section 4.5, where we will also give a brief history of earlier work on this conjecture, because some of these approaches served as starting points for other investigations.

We will treat first the case $\kappa = \mathrm{cf}(\kappa) > \omega$. The reason for this strange order is really technical. The results to be presented for the case $\kappa > \omega$ were proved later and much of the method of using elementary substructures was worked out while proving them. We will give a new proof of the Baumgartner-Hajnal Theorem which can be extended to successors of measurable cardinals and uses the methods developed for the treatment of the cases $\kappa > \omega$.

For the cases $\kappa > \omega$, there are further limitations.

4.3 Theorem. *Assume that $\kappa = \tau^+ \geq \omega_1$ and GCH holds. Then there are κ-complete, κ^+-cc forcing conditions showing the consistency of the following negative partition relations:*

$$\kappa^+ \not\to (\kappa : \tau)^2_2 \text{ and } \kappa^+ \not\to (\kappa : 2)^2_\tau.$$

Here the relations mean that there are no homogeneous sets of the form $[A, B] := \{\, \{\alpha, \beta\} : \alpha \in A \wedge \beta \in B \,\}$ where $A < B$, $\mathrm{ot}(A) = \kappa$, and $\mathrm{ot}(B) = \tau$ or $\mathrm{ot}(B) = 2$ respectively. The forcing results are due to Hajnal and stated in [13]. The first result, $\kappa^+ \not\to (\kappa : \tau)^2_2$, was shown by Rebholz [51] to be true in L. It is interesting to remark that while the proofs of Theorem 4.1 really give $\kappa^+ \not\to (\kappa^+, (\kappa : 2))^2$ in the relevant cases, these two statements are really not equivalent. In [35], Komjáth proves it consistent with ZFC that $\omega_1 \not\to (\omega_1, \omega + 2)^2$ and $\omega_1 \to (\omega_1, (\omega : 2))^2$ hold.

In view of the limitations above, the following result of Baumgartner, Hajnal and Todorcevic [4], which we prove in Subsection 4.3 (see Theorem 4.12), is the best possible balanced generalization of the Erdős-Rado Theorem for finitely many colors to ordinal goals: for all regular uncountable cardinals κ and finite γ, if $\rho < \kappa$ is an ordinal with $2^{|\rho|} < \kappa$, then

$$\left(2^{<\kappa}\right)^+ \to (\kappa + \rho)^2_\gamma.$$

Note that for $\gamma = 2$, this result was proved much earlier by Shelah in section 6 of [57].

As a generalization of the unbalanced form, we prove in Subsection 4.4 (see Theorem 4.18) that for all regular uncountable cardinals κ and all finite m, γ,

$$\left(2^{<\kappa}\right)^+ \to (\kappa^{\omega+2}+1, (\kappa+m)_\gamma)^2.$$

In this discussion we have restricted ourselves to 2-partitions, since the situation is different for larger tuples. For example, Jones [28], [31] has shown that for all finite m, n, $\omega_1 \to (\omega+m, n)^3$, complementing the result of Erdős and Rado [19] who showed $\omega_1 \not\to (\omega+2, \omega)^3$. Milner and Prikry [44] proved that $\omega_1 \to (\omega+\omega+1, 4)^3$.

We conclude this subsection with some open questions.

4.4 Question. For which $\alpha < \omega_1$ and which $n < \omega$ does the partition relation $\omega_1 \to (\alpha, n)^3$ hold?

4.5 Question. Are the following statements provable in ZFC + GCH?

1. $\omega_3 \to (\omega_2+\omega, \omega_2+\omega_1)^2$?
2. $\omega_3 \to (\omega_2+2)^2_\omega$?

Though there are additional limitations for $\gamma \geq \omega$, which we will discuss in Section 5, both theorems may actually generalize for infinite γ with $2^{|\gamma|} < \kappa$, but nothing like this is known with the exception of the following very recent result a proof of which will be given in Section 6.

4.6 Theorem (Shelah [56]). *If $2^{<\kappa} = \kappa$, $\mu < \sigma \leq \kappa$, and σ is strongly compact, then*

$$\kappa^+ \to (\kappa+\mu)^2_\mu.$$

4.2. More elementary submodels

In this subsection we prove a generalization of Connection Lemma 3.9 for regular κ. Let $\lambda = 2^{<\kappa}$ and assume that $\langle \langle N_\alpha, \in \rangle : \alpha < \kappa^+ \rangle$ is a sequence of submodels of $H := H(\lambda^{++})$ satisfying the requirements outlined in 3.2, with $A = \{f\}$ where $f : [\lambda^+]^2 \to \gamma$ is a given 2-partition of λ^+ with γ colors. For notational convenience, we will let

$$S_0 := \left\{ \alpha < \lambda^+ : \alpha \cap N_\alpha = \alpha \text{ and } N_\alpha \text{ is suitable for } \kappa \right\}.$$

For $\alpha \in S_0$, we will write I_α for the ideal $I(N_\alpha)$ of Definition 3.4.

4.7 Lemma (Set Mapping Lemma). *Assume that $S \subseteq S_0$ is stationary and $g : S \to \mathcal{P}(\lambda^+)$ is a set mapping so that $g(\alpha) \subseteq \alpha$ and $g(\alpha) \cap S \in I_\alpha$ for all $\alpha \in S$. Then there is a stationary set $S' \subseteq S$ which is free for g. That is, $g(\alpha) \cap S' = \emptyset$ for all $\alpha \in S'$. Moreover, if S and g are amenable, then so is S'.*

Proof. Since S is a set of limit ordinals, for each $\alpha \in S$, we can choose $\beta_\alpha < \alpha$ and $Y_\alpha \subseteq \lambda^+$ so that $\alpha \notin Y_\alpha \in N_{\beta_\alpha}$ and $g(\alpha) \subseteq Y_\alpha$. By Fodor's Theorem, first β_α and then Y_α stabilize on a stationary set. That is, for some stationary $S' \subseteq S$ and some $Y \subseteq \lambda^+$, we have $\alpha \notin Y$ and $g(\alpha) \subseteq Y$ for all $\alpha \in S'$. ⊣

4.8 Corollary. *Suppose $S \subseteq S_0$. An element $\alpha \in S$ is a* reflection point of *S if $S \cap \alpha \notin I_\alpha$. Then the set $S - \tilde{S}$ is non-stationary, where $\tilde{S}$ denotes the set of reflection points of S. Moreover, if S is amenable, then so is S'.*

Proof. Assume by way of contradiction that $S' := S - \tilde{S}$ is stationary, and define $g(\alpha) := S' \cap \alpha$ for $\alpha \in S'$. By the Set Mapping Lemma 4.7, there is a stationary subset $S'' \subseteq S'$ so that S'' is free for g. On the other hand, if $\beta < \alpha$ are both in $S'' \subseteq S'$, then $\beta \in g(\alpha) := S' \cap \alpha$, contradicting the freeness of S'' for g. ⊣

4.9 Definition. For $\alpha < \lambda^+$ and $\sigma \in {}^{<\omega}\gamma$, we define ideals $I(\alpha, \sigma)$ by recursion on $|\sigma|$. To start the recursion, we set

$$I(\alpha, \emptyset) := \begin{cases} \mathcal{P}(\alpha) & \text{if } \alpha \notin S_0, \text{ and} \\ I_\alpha & \text{if } \alpha \in S_0. \end{cases}$$

If $\sigma = \tau^\frown \langle i \rangle$ and $I(\alpha, \tau)$ has been defined, then for all $X \subseteq \alpha$,

$$X \in I(\alpha, \sigma) \Leftrightarrow \{ \beta < \alpha : X \cap \beta \cap f(\alpha; i) \notin I(\beta, \tau) \} \in I(\alpha, \emptyset).$$

4.10 Lemma. *Suppose $\alpha < \lambda^+$ and $\sigma \in {}^{<\omega}\gamma$.*

1. *$I(\alpha, \sigma)$ is a κ-complete ideal;*
2. *if $\alpha \notin S_0$, then $I(\alpha, \sigma) = \mathcal{P}(\alpha)$;*
3. *$I(\alpha, \emptyset) \subseteq I(\alpha, \sigma)$.*

Proof. In the special case of $\sigma = \emptyset$, item (1) follows either from Lemma 3.6 or the triviality that $\mathcal{P}(\alpha)$ is κ-complete. Use recursion on $|\sigma|$ to complete the proof of (1), since at each successor stage, $I(\alpha, \tau^\frown \langle i \rangle)$ is gotten by averaging κ-complete ideals according to a κ-complete ideal.

Note that (2) follows immediately from the definition of $I(\alpha, \sigma)$.

Item (3) is also proved by induction on $|\sigma|$ simultaneously for all $\alpha < \lambda^+$. For $\alpha \notin S_0$, it follows from the second item, so assume $\alpha \in S_0$. It is trivial for $\sigma = \emptyset$, so assume it is true for $I(\alpha, \tau)$ where $\sigma = \tau^\frown \langle i \rangle$, and let $X \in I(\alpha, \emptyset) = I_\alpha = I(N_\alpha)$ be arbitrary. By definition of $I(N_\alpha)$, there is some $Y \subseteq \lambda^+$ so that $\alpha \notin Y \in N_\alpha$ and $X \subseteq Y$. Since α is limit, there is $\beta_0 < \alpha$ with $Y \in N_{\beta_0}$. Since the sequence of submodels is continuous, $Y \in N_\beta$ for all β with $\beta_0 < \beta < \alpha$, and for $\beta \notin Y$, we either have $X \cap \beta \in I_\beta$ if $\beta \in S_0$ or have $X \cap \beta \in I(\beta, 0)$ otherwise. Hence by the induction hypothesis, $X \cap \beta \in I(\beta, \tau)$ for $\beta \notin Y$ with $\beta_0 < \beta < \alpha$. That is, if $\beta < \alpha$ and $X \cap \beta \notin I(\beta, \tau)$, then $\beta \in Y \cup (\beta_0 + 1)$. So $X \in I(\alpha, \sigma)$, since $\alpha \notin Y - (\beta_0 + 1) \in N_\alpha$. ⊣

We postpone the proof that some of these ideals are proper.

4.11 Lemma (Second Connection Lemma). *Suppose $X \subseteq \alpha$, $X \notin I(\alpha, \sigma)$ and suppose $i \in \operatorname{ran}(\sigma)$. Then there is a subset $Y \subseteq X \cup \{\alpha\}$ with $\operatorname{ot}(Y) = \kappa + 1$ homogeneous for f in color i.*

Proof. The proof is by induction on $|\sigma|$. If $\sigma = \emptyset$, then there is nothing to prove. Next suppose $\sigma = \tau^\frown \langle j \rangle$ for some $j < \gamma$. By Lemma 4.10, we know that $X \cap \beta \notin I(\beta, \tau)$ for some $\beta < \alpha$ with $\beta \in X$. Thus the induction hypothesis gives the statement for $i \in \operatorname{ran}(\tau)$. Next assume $i = j$. Then by Lemma 4.10(3), we know that $X \notin I_\alpha$ and Connection Lemma 3.9 yields the desired result. ⊣

4.3. The Balanced Generalization

In this subsection we will prove, as announced earlier, the following balanced generalization of the Erdős-Rado Theorem.

4.12 Theorem (Baumgartner, Hajnal, Todorcevic [4]). *Suppose κ is a regular uncountable cardinal, γ is finite and $\rho < \kappa$ is an ordinal with $2^{|\rho|} < \kappa$. Then*

$$\left(2^{<\kappa}\right)^+ \to (\kappa + \rho)^2_\gamma.$$

For notational simplicity, we are fixing κ, $\lambda = 2^{<\kappa}$, a 2-partition $f : [\lambda^+]^2 \to \gamma$, and ρ as in the statement of the theorem throughout this subsection, and we continue the notation introduced in subsections 4.1 and 4.2. In what follows, it will be convenient to look at the least indecomposable ordinal $\xi \geq \rho$, rather than ρ directly. In preparation for the proof, we give several preliminary facts about ideals.

4.13 Definition. For ordinals ξ, sets $x \subseteq \lambda^+$ and sequences $\sigma \in {}^{<\omega}\gamma$, define *$x$ is (ξ, σ)-canonical for f* by recursion on $|\sigma|$. To begin the recursion, we say *x is $(\xi, \emptyset)$-canonical for f* if $x = \{\alpha\}$ for some $\alpha < \lambda^+$. For $\sigma = \tau^\frown \langle i \rangle$, we say *$x$ is (ξ, σ)-canonical for f* if x is the union of a $<$-increasing sequence $\langle x_\eta : \eta < \xi \rangle$ so that each x_η is (ξ, τ)-canonical for $\eta < \xi$ and $f(u, v) = i$ for all $u \in x_\eta$ and $v \in x_\zeta$ with $\eta < \zeta < \xi$.

The following lemma is left to the reader as an exercise.

4.14 Lemma. *Assume that ξ is an indecomposable ordinal and $\sigma \in {}^n\gamma$ for some $n < \omega$. Then*

1. $\operatorname{ot}(x) = \xi^n$ *for all x which are (ξ, σ)-canonical for f;*

2. *if* x *is* (ξ,σ)*-canonical for* f*, then every* $y \subseteq x$ *with* $\mathrm{ot}(y) = \xi^n$*, is also* (ξ,σ)*-canonical for* f *and* $J := \{ z \subseteq y : \mathrm{ot}(z) < \xi^n \}$ *is a proper ideal;*

3. *if* x *is* (ξ,σ)*-canonical for* f*, then for every* $i \in \mathrm{ran}(\sigma)$*, there is some* $y \subseteq x$ *with* $\mathrm{ot}(y) = \xi$ *which is homogeneous for* f *in color* i*.*

4.15 Lemma (Reflection Lemma). *Assume* $X \notin I(\alpha,\sigma)$ *for some* $\alpha < \lambda^+$*,* $\sigma \in {}^{<\omega}\gamma$*, and further suppose that* $\xi < \kappa$ *is indecomposable. Then there is a set* $x \subseteq X$ *which is* (ξ,σ)*-canonical for* f*.*

Proof. The proof is by induction on $|\sigma|$. To start, notice the lemma is vacuously true for $\sigma = \emptyset$. Next suppose $\sigma = \tau^\frown\langle i \rangle$. Construct a sequence $\langle x_\eta : \eta < \xi \rangle$ by recursion on $\eta < \xi$. Assume that $\zeta < \xi$ and that the sets $x_\eta \subseteq X \cap f(\alpha; i)$ are (ξ,τ)-canonical for f for $\eta < \zeta$. Let $Z = \{ \beta < \lambda^+ : (\forall \eta < \zeta)(\forall \delta \in x_\eta)(f(\delta,\beta) = i) \}$. Since $\langle x_\eta : \eta < \xi \rangle \in N_\alpha$, we have $Z \in N_\alpha$ and $\alpha \in Z$. Since $\{ \beta < \alpha : X \cap \beta \cap f(\alpha; i) \notin I(\beta,\tau) \} \notin I_\alpha$, we can choose $\beta < \alpha$ so that $\beta \in Z \in N_\beta$, $X \cap \beta \cap f(\alpha; i) \notin I(\beta,\tau)$ and $\sup(\bigcup \{ x_\eta < \zeta \}) < \beta$. By the induction hypothesis, we can choose a set $x_\zeta \subseteq X \cap Z$ which is (ξ,τ)-canonical for f with $x_\eta < x_\zeta$ for all $\eta < \zeta$.

This recursion defines $\langle x_\zeta : \zeta < \xi \rangle$, and $x = \bigcup \{ x_\zeta : \zeta < \xi \}$ is the required set (ξ,σ)-canonical for f. ⊣

We need one more lemma which will be used in the proof of the unbalanced version (Theorem 4.18) as well.

4.16 Lemma. *Assume* $S \subseteq S_0$ *is stationary and* $\Gamma \subseteq \gamma$ *is non-empty. Then there are* $S' \subseteq S$ *stationary and* $\sigma \in {}^{<\omega}\Gamma$ *with* σ *one-to-one such that*

1. $S \cap \beta \cap f(\alpha; j) \in I(\beta,\sigma)$*, for every* $\beta, \alpha \in S'$ *with* $\beta < \alpha$ *and every* $j \in \Gamma - \mathrm{ran}(\sigma)$*, but*

2. $S \cap \alpha \notin I(\alpha,\sigma)$ *for* $\alpha \in S'$*.*

Moreover, if S *is amenable, then so is* S'*.*

Proof. Let σ be of maximal length so that $\mathrm{ran}(\sigma) \subseteq \Gamma$, σ is one-to-one, and

$$S'' := \{ \alpha \in S : S \cap \alpha \notin I(\alpha,\sigma) \} \text{ is stationary.}$$

For $j \in \Gamma - \mathrm{ran}(\sigma)$, let

$$g_j(\alpha) := \{ \beta < \alpha : S \cap \beta \cap f(\alpha; j) \notin I(\beta,\sigma) \}.$$

By the maximality of σ, it follows that $g_j(\alpha) \cap S'' \in I_\alpha$ for all but non-stationarily many $\alpha \in S$. By Lemma 4.7, there is a stationary subset $S' \subseteq S''$ which is free for g_j. ⊣

Let $\mathbb{S} := \{\, \sigma \in {}^{<\omega}\gamma : \sigma \text{ is one-to-one} \,\}$.

For $\alpha < \lambda^+$ and $\sigma \in \mathbb{S}$, say (X, Y) *fits* (α, σ) if $X \subseteq \alpha$, $X \notin I(\alpha, \sigma)$ and $f(\beta; j) \cap X \in I(\alpha, \sigma)$ for all $\beta \in Y$ and $j \notin \operatorname{ran}(\sigma)$.

From Lemma 4.16 we get the following corollary by applying the lemma with $\Gamma = \gamma$.

4.17 Corollary. *For every stationary set $S \subseteq S_0$, there are $\sigma \in \mathbb{S}$, $\alpha \in S$ and a stationary subset $S' \subseteq S$ so that $(S \cap \alpha, S')$ fits (α, σ).*

With these lemmas in hand, we turn to the proof of the main theorem of this subsection.

Proof of Theorem 4.12. Using Corollary 4.17, we define $\alpha_m \in S_0$, $\sigma_m \in \mathbb{S}$, and stationary $Z_m \subseteq S_0$ by recursion on m so that the following conditions are satisfied:

1. $\alpha_0 < \dots < \alpha_m < \dots$; $Z_0 \supseteq \dots \supseteq Z_m \supseteq \dots$; and

2. $(Z_m \cap \alpha_m, Z_{m+1})$ fits (α_m, σ_m).

Since $\mathbb{S}$ is finite, $\sigma_k = \sigma_n$ for some $k < n < \omega$. We conclude that there are a sequence $\sigma \in \mathbb{S}$, ordinals $\beta_0 < \beta_1$, and sets X_0, X_1 such that the following statement is true:

> (*) $X_0 < X_1$, $X_i \notin I(\beta_i, \sigma)$ for $i < 2$, and $f(\eta; j) \cap X_0 \in I(\beta_0, \sigma)$ for every $j \notin \operatorname{ran}(\sigma)$ and every $\eta \in X_1$.

Let ξ be the least indecomposable ordinal with $\rho \leq \xi$. By the Reflection Lemma 4.15, there is a $y \subseteq X_1$ such that y is (ξ, σ)-canonical for f.

We shrink X_0 to $X = X_0 - \bigcup \{\, f(\delta; j) : j \notin \operatorname{ran}(\sigma) \text{ and } \delta \in y \,\}$. Then $X \notin I(\beta_0, \sigma)$ since $I(\beta_0, \sigma)$ is κ-complete, $|y| < \kappa$ and $f(\delta; j) \in I(\beta_0, \sigma)$ for $j \notin \operatorname{ran}(\sigma)$, $\delta \in y \subseteq X_1$.

Let $J = \{\, Z \subseteq y : Z \text{ is not } (\xi, \sigma)\text{-canonical for } f \,\}$. By Lemma 4.14, J is a proper ideal on y.

For every $\delta \in X$, there is an $i(\delta) \in \operatorname{ran}(\sigma)$ so that $f(\delta; i) \cap y \notin J$. Thus for every $\delta \in X$, by Lemma 4.14(3), there is a $y(\delta) \subseteq y$ of order type ρ such that $\{\delta\} \cup y(\delta)$ is homogeneous for f in color $i(\delta)$.

Using the fact that $\omega^{|\rho|} = 2^{|\rho|} \cdot \omega < \kappa$, we now obtain $i_0 \in \operatorname{ran}(\sigma)$, $y' \subseteq y$ and $X' \subseteq X$ with $X' \notin I(\alpha, \sigma)$ so that $i(\delta) = i_0$ and $y(\delta) = y'$ for all $\delta \in X'$. Thus $f(\delta_0, \delta_1) = i_0$ for all $\delta_0 \in X'$ and $\delta_1 \in y'$.

By the Second Connection Lemma 4.11, we get an $X'' \subseteq X'$ of order type κ homogeneous for f in color i_0. Finally $X'' \cup y'$ is the required set of order type $\kappa + \rho$ homogeneous for f in color i_0. $\dashv$

4.4. The Unbalanced Generalization

4.18 Theorem (Baumgartner, Hajnal, Todorcevic [4]). *Suppose κ is a regular uncountable cardinal, and m, γ are finite. Then*

$$\left(2^{<\kappa}\right)^{+} \to (\kappa^{\omega+2}+1, (\kappa+m)_{\gamma})^{2}.$$

This subsection is devoted to the proof of this theorem, and for notational convenience we set $\lambda = 2^{<\kappa}$ throughout. Also, fix a partition $f : [\lambda^{+}]^{2} \to 1+\gamma$. We also continue to use the notation introduced in subsections 4.1, 4.2 and 4.3.

The strategy of the proof is to derive Theorem 4.18 from the following auxiliary assumption:

$$Q(\kappa) : 2^{<\kappa} = \kappa \text{ and } \forall \left\langle f_{\alpha} : \alpha < \kappa^{+} \right\rangle \subseteq {}^{\kappa}\kappa \; \exists g \in {}^{\kappa}\kappa \; (\; f_{\alpha} \prec g \;),$$

where $\prec$ is the relation of eventual domination on ${}^{\kappa}\kappa$.

Then as in the original proof of the Baumgartner-Hajnal Theorem [2], we observe that the assumption $Q(\kappa)$ is unnecessary, and therefore that Theorem 4.18 holds in ZFC.

Let us justify this observation before going on to prove the theorem from the assumption of $Q(\kappa)$.

Let P_0 be the natural κ-closed forcing for collapsing $2^{<\kappa}$ onto κ. Then in V^{P_0} we have $\lambda = \kappa$. Working in V^{P_0} and using a standard iterated forcing argument (as in [1]) we can force every sequence of functions in ${}^{\kappa}\kappa$ of length κ to be eventually dominated via a partial ordering P_1 that is κ-closed and has the λ^{+}-chain condition. Let $P = P_0 * P_1$. Then P is κ-closed and in V^{P}, both $\lambda = \kappa$ and $Q(\kappa)$ hold. Note that in V^{P}, we will have $2^{\kappa} > \kappa^{+}$, since this inequality is implied by $Q(\kappa)$.

Assuming we have proved Theorem 4.18 under the assumption of $Q(\kappa)$, we may assume it holds in V^{P}. Suppose that $f : [\lambda^{+}]^{2} \to \gamma+1$ is a 2-partition in V. Then in V^{P}, there is some $A \subseteq \lambda^{+}$ such that either (a) A is homogeneous for f in color 0 and ot $A = \kappa^{\omega+2}+1$, or (b) A is homogeneous for f in color $i > 0$ and ot $A = \kappa+m$. Suppose (a) holds. Note that $\kappa^{\omega+2}+1$ is the same whether computed in V or in V^{P}. Let $h : \kappa \to \kappa^{\omega+2}+1$ be a bijection with $h \in V$. In V^{P}, fix an order-isomorphism $j : \kappa^{\omega+2}+1 \to A$. Now, working in V, find a decreasing sequence $\langle p_{\xi} : \xi < \kappa \rangle$ of elements of P and a sequence $\langle \alpha_{\xi} : \xi < \kappa \rangle$ of elements of λ^{+} such that for all ξ, $p_{\xi} \Vdash j(h(\xi)) = \alpha_{\xi}$. This is easy to do by recursion on ξ, using the fact that P is κ-closed. But now it is clear that $\{\, \alpha_{\xi} : \xi < \kappa \,\} \in V$ has order type $\kappa^{\omega+2}+1$ and is homogeneous for f in color 0. Case (b) may be handled the same way.

For the rest of this subsection, assume $Q(\kappa)$ holds. We may also assume that $\kappa > \omega$ since for $\kappa = \omega$ we have the much stronger result Theorem 4.30.

First we prove a consequence of $Q(\kappa)$.

4.19 Lemma. *Assume $Q(\kappa)$. For all positive $\ell < \omega$ and every sequence $\langle X_\alpha \subseteq \kappa^\ell : \alpha < \kappa^+ \rangle$ with $\operatorname{ot} X_\alpha < \kappa^\ell$ for $\alpha < \kappa^+$, there is a sequence $\langle Z_\nu \subseteq \kappa^\ell : \nu < \kappa \rangle$ with $\operatorname{ot} Z_\nu < \kappa^\ell$ for $\nu < \kappa$ such that every X_α is a subset of some Z_ν.*

Proof. Use induction on ℓ. For $\ell = 1$, the sets $X_\alpha \subseteq \kappa^1 = \kappa$ are bounded and we may define $Z_\nu := \nu$.

For the induction step, assume $\langle X_\alpha \subseteq \kappa^{k+1} : \alpha < \kappa^+ \rangle$ is a given sequence with $\operatorname{ot} X_\alpha < \kappa^{k+1}$. Write $\kappa^{k+1} = \bigcup_{\nu<\rho} U_\rho$ as the union of an increasing sequence $U_0 < \cdots < U_\rho < \dots$ in which $\operatorname{ot} U_\rho = \kappa^k$. For each $\alpha < \kappa^+$ and $\rho < \kappa$, define

$$Y_{\alpha,\rho} := \begin{cases} X_\alpha \cap U_\rho, & \text{if } \operatorname{ot} X_\alpha \cap U_\rho < \kappa^k \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since each U_ρ is isomorphic to κ^k, we may apply the induction hypothesis to each sequence $\langle Y_{\alpha,\rho} \subseteq U_\rho : \alpha < \kappa^+ \rangle$ to get $\langle W_{\mu,\rho} \subseteq U_\rho : \mu < \kappa \rangle$, so that every $Y_{\alpha,\rho}$ is a subset of some $W_{\mu,\rho}$.

For each $\alpha < \kappa^+$, define $g_\alpha : \kappa \to \kappa$ by $g_\alpha(\rho)$ is the least μ so that $Y_{\alpha,\rho} \subseteq W_{\mu,\rho}$. Choose an increasing $g : \kappa \to \kappa$ eventually dominating all the g_α for $\alpha < \kappa$. Define

$$Z_\nu := \bigcup_{\mu<\nu} \cup \bigcup \{ W_{\mu,\rho} : \rho \geq \nu \wedge \mu \leq g(\rho) \} .$$

Then $\langle Z_\nu \subseteq \kappa^{k+1} : \nu < \kappa \rangle$ satisfies the requirements of the lemma for $\ell = k+1$.

Therefore by induction, the lemma follows. ⊣

From this point forward in the subsection, we assume that there is no homogeneous set for color 0 of the order type required. We may also assume that the result is true for $\gamma' < \gamma$.

4.20 Lemma. *Assume $S \subseteq S_0$ is stationary. For all $\Sigma \subseteq [1,\gamma]$ with $\Sigma \neq \emptyset$, there are a stationary set $S' \subseteq S$ and a one-to-one function $\sigma \in {}^{<\omega}\Sigma$ such that the following two properties hold:*

1. *for every stationary $S'' \subseteq S'$ there is some $\alpha \in S''$ with $S'' \cap \alpha \notin I(\alpha, \sigma)$;*
2. *for all $j \in \Sigma - \operatorname{ran} \sigma$ and all $\beta, \alpha \in S'$, if $\beta < \alpha$, then $f(\alpha; j) \cap \beta \cap S' \in I(\beta, \sigma)$.*

Proof. By induction on $|\Sigma|$. For the basis case of $|\Sigma| = 1$, suppose $\Sigma = \{i\}$ for some postive $i \leq \gamma$. Then either $\operatorname{ran} \sigma = \{i\}$, the first property holds with $S' = S$ and the second holds vacuously, or by the Set Mapping Lemma 4.7, there is a stationary subset $S' \subseteq S$ free for color i.

For the induction step, assume the lemma is true for some non-empty proper subset $T \subseteq [1,\gamma]$ and let $i \in [1,\gamma] - T$. We must show the statement is also true for $\Sigma = T \cup \{i\}$. Let $S_T \subseteq S$ and τ witness that the lemma is true for T. Consider two cases depending on whether or not the following statement is true, where $\mathrm{Stat}(S_T) := \mathrm{Stat}(\lambda^+) \cap \mathcal{P}(S_T)$:

$$(*) \quad \forall S^* \in \mathrm{Stat}(S_T)\, \exists \alpha \in S^* \ (\ \{\beta < \alpha : S^* \cap \beta \cap f(\alpha; i) \notin I(\beta, \tau)\} \notin I_\alpha\ ).$$

For the first case, assume that $(*)$ holds. Then we can choose $S_\Sigma = S_T$ and $\sigma = \tau^\frown(i)$, since the first item holds by $(*)$ and the second remains true since no new j comes into play.

For the second case, assume that $(*)$ fails and choose a stationary $S^* \subseteq S_T$ showing the failure. Define

$$g(\alpha) := \{\beta < \alpha : S^* \cap \beta \cap f(\alpha; i) \notin I(\beta, \tau)\}.$$

Applying the Set Mapping Lemma 4.7 to g and S^*, we get a stationary $S_\Sigma \subseteq S^*$ free for g which together with $\sigma = \tau$ satisfy the required two conditions. ⊣

Our next lemma uses the fact that by $Q(\kappa)$, we have $2^{<\kappa} = \kappa$. For notational convenience, for each $\alpha \in S_0$, define

$$\mathcal{F}_\alpha := \{ Z \in N_\alpha : Z \subseteq \kappa^+ \wedge \alpha \in Z \}.$$

Also, for any $0 < \ell \leq \gamma$ and any one-to-one function $\sigma \in {}^{\ell-1}[1,\gamma]$, call a set Y *(α,σ)-slim* if $Y \subseteq S_0$, $\mathrm{ot}\, Y = \kappa^\ell$, $Y \notin I(\alpha,\sigma)$, and for all $W \subseteq Y$, the equivalence $W \notin I(\alpha,\sigma)$ if and only if $\mathrm{ot}\, W = \kappa^\ell$ holds.

4.21 Lemma. *For all one-to-one functions $\sigma \in {}^{<\omega}[1,\gamma]$, for all $X \subseteq S_0$ with $X \notin I(\alpha,\sigma)$, if $\ell - 1$ is the length of σ, then there exists $Y \subseteq X$ such that Y is (α,σ)-slim.*

Proof. To start the induction, note that if $X \notin I(\alpha,\emptyset) = I_\alpha$ for some $\alpha \in S_0$, then there is some $Y \subseteq X$ with $\mathrm{ot}\, Y = \kappa$ so that $Y \notin I_\alpha$. This implication is true because $\mathcal{F}_\alpha$ has cardinality at most κ and can be diagonalized in X. Then Y is $(\alpha,\emptyset)$-slim, by the κ-completeness of I_α. The rest follows by induction on the length of σ. ⊣

The following corollary is immediate from the previous two lemmas.

4.22 Corollary. *There are a stationary set $S_1 \subseteq S_0$, a nonempty subset $\Sigma \subseteq [1,\gamma]$ and a one-to-one function $\sigma \in {}^{\ell-1}\Sigma$ such that the following two conditions hold:*

1. *for all stationary $S \subseteq S_1$, there are $\alpha \in S$ and $X \subseteq \alpha$ of order type κ^ℓ so that $X \notin I(\alpha,\sigma)$;*

2. for all $\beta < \alpha \in S_1$ *and all* $j \in [1,\gamma] - \Sigma$*, one has* $f(\alpha; j) \cap \beta \in I(\beta, \sigma)$*.*

For notational convenience, write $X = \Sigma_{\nu<\kappa} X_\nu$ to indicate that $X_0 < \dots < X_\nu < \dots$ and $X = \bigcup_{\nu<\kappa} X_\nu$. For the remainder of this section, let $S_1 \subseteq S_0$, σ and ℓ as in the previous corollary be fixed.

4.23 Definition. For $\alpha \in S_0$, define $\mathcal{H}(\alpha, n)$ by recursion on $n < \omega$. To start the recursion, define

$$\mathcal{H}(\alpha, 0) := \{ X \subseteq S_1 : X \text{ is } (\alpha, \sigma)\text{-slim} \}.$$

If $\mathcal{H}(\alpha, n)$ has been defined, then $X \in \mathcal{H}(\alpha, n+1)$ if and only if the following conditions are satisfied:

1. $X \subseteq S_1$ and there exists $\langle X_\nu \in \mathcal{H}(\alpha, n) : \nu < \kappa \rangle$ with $X = \Sigma_{\nu<\kappa} X_\nu$;
2. for all $F \in \mathcal{F}_\alpha$, there exists ν_F so that $X_\nu \subseteq F$ for all $\nu > \nu_F$;
3. for all $\nu < \nu' < \kappa$ and $x \in X_\nu$, $y \in X_{\nu'}$, one has $f(x, y) = 0$.

Note that every $X \in \mathcal{H}(\alpha, n)$ has $\operatorname{ot} X = \kappa^{\ell+n}$ and X contains a subset of order κ^n homogeneous for f in color 0. Furthermore, every $Y \subseteq X$ of order type $\kappa^{\ell+n}$ has a subset in $\mathcal{H}(\alpha, n)$.

We now prove the lemma containing the main idea of the proof.

4.24 Lemma (Key Lemma). *Suppose* $\alpha \in S_1$, $n < \omega$ *and* $X \subseteq S_1$ *with* $X \in \mathcal{H}(\alpha, n)$*. Then there are* $\beta_0 \in S_1$ *with* $\beta_0 > \alpha$ *and* $\langle T_\nu \subseteq X : \nu < \kappa \rangle$ *with* $\operatorname{ot} T_\nu = \kappa^{\ell+n}$ *so that for all* $\beta \in S_1$ *with* $\beta > \beta_0$*, there is some* $\nu < \kappa$ *such that* $\operatorname{ot}(T_\nu - f(\beta; 0)) < \kappa^{\ell+n}$*.*

Proof. Let M be a maximal subset of S_1 with the property that for all $V \in [M]^{<\omega}$, $\operatorname{ot} \bigcap \{ X - f(\beta; 0) : \beta \in V \} = \kappa^{\ell+n}$. We claim that $|M| \leq \kappa$ and then we are done, by the maximality of M.

Assume for the sake of a contradiction that $|M| = \kappa^+$, and let

$$\Pi := \left\{ \bigcap_{\beta \in V} X - f(\beta; 0) : V \in [M]^{<\omega} \right\}.$$

Extend $\Pi \cup \{ X - Y : Y \subseteq X \wedge \operatorname{ot} X < \kappa^{\ell+n} \}$ to an ultrafilter U on X. Then for every $\beta \in M$, there is a $j(\beta) \in \Sigma$ so that $X \cap f(\beta; j(\beta)) \in U$. Hence there is some $j \in \Sigma$ so that the set $M_j := \{ \beta \in M : j(\beta) = j \}$ has cardinality κ^+. By $\kappa^+ \to (\kappa^+, n)^2$, there is a set $H \subseteq M_j$ of size n which is homogeneous for f in color j. Now $X \cap \bigcap \{ f(\beta; j) : \beta \in H \}$ is in U, so it must have order type $\kappa^{\ell+n}$. By Lemma 4.11 it contains a set W of type κ homogeneous for f in color j. This is the contradiction that proves the lemma. ⊣

4.25 Lemma. *Assume $S \subseteq S_1$ is stationary. Then for all $n < \omega$, there are $\alpha \in S$ and $X \subseteq S$ so that $X \in \mathcal{H}(\alpha, n)$.*

Proof. Work by induction on n. For the basis case, $n = 0$, the statement follows from Corollary 4.22 and Lemma 4.21.

For the induction step, a standard ramification argument gives the result. Assume the claim is true for some n. Let $\alpha \in S$ be arbitrary. We define a sequence

$$\{ X_\xi : \xi < \kappa \} \subseteq \mathcal{H}(\alpha_\xi, n)$$

by recursion on $\xi < \kappa$. Assume that $X_\eta \in \mathcal{H}(\alpha_\eta, n)$, $X_\eta \subseteq S \cap f(\alpha; 0)$ are defined for $\eta < \xi$. Let $S_\xi = \{ \beta \in S : \bigcup \{ X_\eta : \eta < \xi \} \subseteq f(\beta; 0) \}$. Then $\alpha \in S_\xi$ and $S_\xi \in N_\alpha$. Then S_ξ is stationary, and so by the induction hypothesis it contains a subset $X \in \mathcal{H}(\alpha_\xi, n)$ for some $\alpha_\xi \in \alpha \cap S_\xi$. By elementarity, we may assume $X \in N_\alpha$. By the Key Lemma, there are $T_\nu \subseteq X$ for $\nu < \kappa$ such that $\mathrm{ot}(T_\nu) = \kappa^{\ell+n}$ and $|S - \bigcup_{\nu<\kappa} Z_\nu| \leq \kappa$ where

$$Z_\nu = \left\{ \beta < \kappa : \mathrm{ot}(T_\nu - f(\beta; 0)) < \kappa^{\ell+n} \right\}.$$

Then, by elementarity $S - \bigcup_{\nu<\kappa} Z_\nu \subseteq \alpha$, hence $\alpha \in Z_\nu$ for some $\nu < \kappa$ and $X_\nu = T_\nu \cap f(\alpha; 0)$ satisfies the requirement. $\bigcup_{\nu<\kappa} X_\nu \in \mathcal{H}(\alpha, n+1)$ and as a bonus we have that $\bigcup_{\nu<\kappa} X_\nu \subseteq f(\alpha; 0)$. ⊣

The same ramification argument gives the next lemma as well.

4.26 Lemma. *Assume $S \subseteq S_1$ is stationary. Then there exist an increasing sequence $\langle \alpha_\xi \in S : \xi < \kappa \rangle$ and a family $\langle X_{\xi,n} \subseteq S : \xi < \kappa \wedge n < \omega \rangle$ with each $X_{\xi,n} \in \mathcal{H}(\alpha_\xi, n)$ so that if either $\xi < \eta$ or $\xi = \eta$ and $k < \ell$, then $X_{\xi,k} < X_{\eta,\ell}$ and $f(x, y) = 0$ for all $x \in X_{\xi,k}$, $y \in X_{\eta,\ell}$.*

The above lemma gives the result for $\kappa^{\omega+1}$, since the set

$$X := \bigcup \{ X_{\xi,n} : \xi < \kappa \wedge n < \omega \}$$

is homogeneous for f in color 0.

To finish the proof, we use yet another ramification argument.

4.27 Lemma. *Let X be a set of order type $\kappa^{\omega+1}$ as described above, and let $X_n := \bigcup \{ X_{\xi,n} : \xi < \kappa \}$. Note that $\mathrm{ot}\, X_n = \kappa^{\ell+n+1}$. Let*

$$J := \left\{ Y \subseteq X : \exists n_0 < \omega \ \forall n > n_0 \ (\mathrm{ot}\, Y \cap X_n < \kappa^{\ell+n}) \right\}.$$

Then J is an ideal and there are $\{ T_\nu \in J^+ : \nu < \kappa \}$ and $\beta_0 \in S_1$ such that for all $\beta \in S_1$ with $\beta > \beta_0$, the set $T_\nu - f(\beta; 0)$ is in J.

Let M be a maximal subset of S_1 so that $\bigcap_{\beta \in V} X - f(\beta; 0) \notin J$ for finite $V \subseteq M$.

To see that $|M| = \kappa$, we proceed just like in the proof of Lemma 4.24. We only need the fact that if $Z \subseteq X$ and $Z \notin J$, then for all $j \in \Sigma$, the set Z contains a subset of type κ homogeneous for f in color j.

Since $|M| = \kappa$, the set $\Pi := \left\{ \bigcap_{\beta \in V} X - f(\beta; 0) : V \in [M]^{<\omega} \right\}$ is a family of size κ such that for all $\beta \notin M$, there is some $Z \in \Pi$ so that $Z - f(\beta; 0) \subseteq Y$ for some $Y \in \Pi$.

The next lemma is the final tool we need.

4.28 Lemma. *Assume $T \in J^+$. Then there is a $\overline{J} \subseteq J$ with $|\overline{J}| \leq \kappa$ such that for all $\beta \in S_1$ with $T - f(\beta; 0) \in J$, there is a $Y \in \overline{J}$ so that $T - f(\beta; 0) \subseteq Y$.*

Proof. Choose $J_n \subseteq [X_n]^{\ell+n+1}$ with $|J_n| \leq \kappa$ so that for all $\beta \in S_1$ with $\operatorname{ot}(X_n - f(\beta; 0)) < \kappa^{\ell+n+1}$ there is a $Y_n \in J_n$ with $T - f(\beta; 0) \subseteq Y_n$. Let

$$J_0 := \left\{ \bigcup_{n<\omega} Y_n : \forall n < \omega \; Y_n \in J_n \right\}.$$

Note that $|J_0| = \kappa^\omega = \kappa$. Finally, set

$$\overline{J} := \left\{ A \cup B : A \in J_0 \text{ and } B = \bigcup \{ X_i : i \leq n \} \text{ for some } n < \omega \right\}.$$

Then $\overline{J}$ will do the job. ⊣

4.5. The Baumgartner-Hajnal Theorem

Here is a brief overview of the history of the Baumgartner-Hajnal Theorem and some of its generalizations. Erdős and Rado conjectured that $\omega_1 \to (\alpha)^2_k$ and $\lambda_0 \to (\alpha)^2_k$, for λ_0 the order type of the reals, and for all $k < \omega$, $\alpha < \omega_1$.

Fred Galvin figured out, for order types Θ, that $\Theta \to (\omega)^1_\omega$ would be the right necessary and sufficient condition for $\Theta \to (\alpha)^2_k$ to hold for all $\alpha < \omega_1$.

Hajnal [25] proved in 1960 that $\lambda_0 \to (\eta_0, \alpha \vee \alpha^*)^2$ where η_0 is the order type of the rationals. More significantly, Galvin proved $\lambda_0 \to (\alpha)^2_2$, for $\alpha < \omega_1$, but contrary to the first expectations, this proof provided no clues for the general case. For the resource ω_1, Galvin could only prove $\omega_1 \to (\omega^2, \alpha)^2$ for $\alpha < \omega_1$.

Another result of Prikry [49] said $\omega_1 \to (\alpha, (\omega : \omega_1))^2$. This result was later generalized by Todorcevic [65] to

$$\omega_1 \to ((\alpha)_k, (\alpha : \omega_1))^2 \text{ for all } \alpha < \omega_1.$$

Finally we mention a very significant consistency result of Todorcevic [64] that PFA (Proper Forcing Axiom) implies

$$\omega_1 \to (\omega_1, \alpha)^2 \text{ for all } \alpha < \omega_1.$$

(For context, recall that PFA implies that $\mathfrak{c} = \omega_2$.)

Before going back to the main line of discussion, we make another detour. It was already asked in the Erdős-Hajnal problem lists [12], [13] if the partition relations $\omega_2 \to (\alpha)^2_2$ were consistent for $\alpha < \omega_2$. Though there is nothing to refute such consistency, the results going in this direction are weak and rare.

The first consistency result was obtained by R. Laver [41] in 1982, and independently discovered by A. Kanamori [33], using what is now called a *Laver ideal* I on κ (a non-trivial, κ-complete ideal with the strong saturation property that given κ^+ sets not in the ideal, there are κ^+ of them so that the intersection of any $< \kappa$ of these is also not in the ideal). He proved that if there is a Laver ideal on κ, then

$$\kappa^+ \to (\kappa \cdot 2 + 1, \alpha)^2 \text{ holds for all } \alpha < \kappa^+.$$

Laver also proved the consistency of the hypothesis that there is a Laver ideal on ω_1 and derived as a corollary the consistency (relative to a large cardinal, of course) of

$$\omega_2 \to (\omega_1 \cdot 2 + 1, \alpha)^2 \text{ holds for all } \alpha < \omega_2.$$

Foreman and Hajnal [20] tried to get a stronger consistency result for ω_2 from the stronger assumption that ω_1 carries a dense ideal, and indeed, they proved that in this case

$$\omega_2 \to ({\omega_1}^2 + 1, \alpha)^2 \text{ holds for all } \alpha < \omega_2.$$

They however discovered that their proof gives a much stronger result for successors.

4.29 Theorem (Foreman and Hajnal [20]). *Suppose $\kappa > \omega$ is measurable and $m < \omega$. Then $\kappa^+ \to (\alpha)^2_m$ for all $\alpha < \Omega(\kappa)$.*

Here $\kappa < \Omega(\kappa) < \kappa^+$ is a rather large ordinal. We will comment about these results in detail in Section 5, but for lack of space and energy we will not include proofs.

4.30 Theorem (Baumgartner and Hajnal [2]). *If an order type Θ satisfies $\Theta \to (\omega)^1_\omega$, then it also satisfies $\Theta \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and finite k.*

4.31 Corollary. *For all $\alpha < \omega_1$ and $m < \omega$,*

$$\omega_1 \to (\alpha)^2_m.$$

So we decided to give a proof of Corollary 4.31 using the ideas of the Foreman-Hajnal proof. This will serve two purposes. It will make the text almost complete as far as the old results are concerned, and it will communicate most of the ideas of the new Foreman-Hajnal proof.

Notation. Let $\langle\, \langle N_\alpha, \in \rangle : \alpha < \omega_1 \,\rangle$ be a sequence of elementary submodels of $H(\omega_2)$ satisfying 3.2 with $\lambda = \kappa = \omega$, $A = \{f\}$ where $f : [\omega_1]^2 \to m$, and

$$S_0 := \{\, \alpha < \omega_1 : \omega_1 \cap N_\alpha = \alpha \text{ and } N_\alpha \text{ is suitable for } \omega \,\}\,.$$

Here S_0 is a club set in ω_1. We may assume S_0 is amenable.

4.32 Definition. We define S_ρ by transfinite recursion on $\rho < \omega_1$: S_0 has already been defined; $S_{\rho+1} := \tilde{S}_\rho$, the set of reflection points of S_ρ (see 4.8); and $S_\rho := \bigcap_{\sigma<\rho} S_\sigma$ if ρ limit.

4.33 Lemma. *For all $\rho < \omega_1$, the set S_ρ is amenable.*

Proof. Use induction on ρ and 4.8 to prove that $\langle\, S_\sigma : \sigma < \rho \,\rangle \subseteq N_{\alpha+1}$ for $\alpha \in S_\rho$. The details and the remainder of the proof are left to the reader. $\dashv$

Next we are going to define *diagonal sets*, *cross sets*, and *cross systems*.

4.34 Definition. For $\alpha \in S_0$, for the sake of brevity, we put

$$\mathcal{F}_\alpha := \{\, Z \in N_\alpha : Z \subseteq \omega_1 \ \wedge\ \alpha \in Z \,\}\,.$$

(Note that for $X \subseteq \alpha$, we have $X \notin I_\alpha$ if and only if $X \cap Z \neq \emptyset$ for all $Z \in \mathcal{F}_\alpha$; see the discussion of notation after Lemma 3.6.)

Call $D \subseteq \alpha$ a *diagonal set for $\alpha \in S_0$* if $\sup D = \alpha$ and $|D - Z| < \omega$ for all $Z \in \mathcal{F}_\alpha$.

Clearly every diagonal set D for α has order type ω, and every cofinal subset of it is also diagonal. Moreover, a diagonal set D for α is *reflecting for α* in the sense described after Lemma 3.6.

4.35 Lemma. *For all $\alpha \in S_0$ and $X \subseteq \alpha$ with $X \notin I_\alpha$, there is a diagonal set $D \subseteq X$ for α. If $X \in N_{\alpha+1}$, then D can be chosen in $N_{\alpha+1}$.*

Proof. Since $|\mathcal{F}_\alpha| = \omega$, we can diagonalize it. $\dashv$

Notation. Assume that $\langle D_n : n < \omega \rangle$ is a sequence of sets of ordinals and $\alpha \in S_0$. Then the sequence *converges to α in N_α*, in symbols, $D_n \implies \alpha$, if and only if for every $Z \in \mathcal{F}_\alpha$ there is some n_0 so that for all $n > n_0$, $D_n \subseteq Z$.

For a set $D \subseteq \mathrm{ON}$, we denote by $\overline{D}$ its closure in the ordinal topology.

4.36 Definition. By transfinite recursion on $\rho < \omega_1$, we define, for $\alpha \in S_\rho$, the concept *D is a cross set of rank ρ for α* as follows:

1. For $\alpha \in S_0$, the set $\{\alpha\}$ is cross set of rank 0 for α.
2. For $\rho > 0$, the set D is cross set of rank ρ for α if $\alpha \in S_\rho$ and there is a witnessing sequence $\langle D_n : n < \omega \rangle$ satisfying the following conditions:

(a) each D_n is a cross set of rank ρ_n for α_n for some $\rho_n < \rho$ and for $\alpha_n := \sup D_n$;
(b) $D_0 \cup \{\alpha_0\} < \cdots < D_n \cup \{\alpha_n\} < \dots$;
(c) $\overline{D}_n \implies \alpha$;
(d) if $\rho = \sigma + 1$, then $\rho_n = \sigma$ for all $n < \omega$; if ρ is a limit, then $\rho = \sup \rho_n$;
(e) $D = \bigcup_{n<\omega} D_n$.

4.37 Remark. Note that a cross set D of rank 1 for α is a diagonal set for α, and if $\{ \alpha_n : n < \omega \}$ is the set of $\alpha_n := \sup D_n$ for a witnessing sequence for D, then $\{ \alpha_n : n < \omega \}$ is also a diagonal set for α.

The next lemma is proved by induction on ρ.

4.38 Lemma. *If D is a cross set for α of rank ρ, then* $\operatorname{ot} D = \omega^\rho$.

We now define the concept of a cross system of rank ρ for α. Informally, this is just the closure of a cross set of rank ρ for α, equipped with functions that remember the sets appearing in the definition of the cross set of rank α.

4.39 Definition. By transfinite recursion on $\rho < \omega_1$, we define, for $\alpha \in S_\rho$, the concept $\mathcal{D} = \langle \overline{D}, <_D, \mathrm{rank}_D, \mathrm{succ}_D \rangle$ *is a cross system of rank ρ for α* as follows:

1. For $\alpha \in S_0$, a quadruple $\mathcal{D} = \langle \overline{D}, <_D \mathrm{rank}_D, \mathrm{succ}_d \rangle$ is a *cross system of rank* 0 *for α* if and only if $D = \{\alpha\}$, $<_D = \emptyset$, $\mathrm{rank}(\alpha) = 0$, and $\mathrm{succ}(\alpha) = \emptyset$.

2. For $\rho > 0$, a quadruple $\mathcal{D} = \langle \overline{D}, <_D, \mathrm{rank}_D, \mathrm{succ}_D \rangle$ is a *cross system of rank ρ for α* with underlying cross set D if there is a witnessing sequence $\langle \mathcal{D}_n : n < \omega \rangle$ of cross systems so that

   (a) $\mathcal{D}_n$ is a cross system of rank ρ_n for α_n for all $n < \omega$;
   (b) $D = \bigcup \{ D_n : n < \omega \}$ is a cross set with witnessing sequence $\langle D_n : n < \omega \rangle$, where D_n underlies $\mathcal{D}_n$;
   (c) $\overline{D} = \bigcup \{ \overline{D}_n : n < \omega \} \cup \{\alpha\}$;
   (d) $<_D$ is defined by $\alpha <_D \beta$ for all $\beta \in \overline{D} - \{\alpha\}$, and $<_D |\overline{D}_n = <_{D_n}$ for $n < \omega$.
   (e) under $<_D$, D is a (rooted) tree with root α;
   (f) $\mathrm{rank}_D : \overline{D} \to \rho + 1$ is defined by $\mathrm{rank}_D(\alpha) = \rho$, and $\mathrm{rank}_D |\overline{D}_n = \mathrm{rank}_{D_n}$ for $n < \omega$.

   Finally, $\mathrm{succ}_D(\beta)$ is just a redundant notation for the set of immediate successors of β in the tree under $<_D$.

Note that for $\rho > 0$ and $n < \omega$, under the notation of Definition 4.36, $\mathrm{succ}_D(\alpha) = \{\, \alpha_n : n < \omega \,\}$ and $\mathrm{rank}_D(\alpha_n) = \rho_n$.

Note that the underlying set of a cross system is definable as the set of elements in $\overline{D}$ of rank 0.

4.40 Lemma. *Assume $\mathcal{D} = \langle \overline{D}, <_D, \mathrm{rank}_D, \mathrm{succ}_D \rangle$ is a cross system of rank ρ for α. Then for all $\beta \in \overline{D}$, $\mathrm{rank}_D(\beta) = \emptyset$ if and only if $\beta \in D$.*

The next two lemmas are proved by induction on ρ.

4.41 Lemma (Reflection Lemma). *Assume $\mathcal{D}$ is a cross system of rank ρ for α. Then for $\gamma \in \overline{D} - D$, $\mathrm{succ}_D(\gamma)$ is a diagonal set for γ.*

4.42 Definition. Assume $\mathcal{D}$ is a cross system of rank ρ for α with underlying set D. We say that C is a *full subset of $\overline{D}$* if $\alpha \in C$ and $C \cap \mathrm{succ}_D(\beta)$ is infinite for $\beta \in C$ with $\mathrm{rank}_D(\beta) > 0$.

4.43 Lemma (Induction lemma for cross systems). *Assume $\mathcal{D}$ is a cross system of rank ρ for α with underlying set D. For every full subset C of $\overline{D}$, there is a set $B \subseteq C \cap D$ so that $\overline{B} \subseteq C$ and B is the underlying set for a cross system of rank ρ for α.*

4.44 Definition. By recursion on $\rho < \omega_1$ define, for $\alpha \in S_\rho$, the concept *$\mathcal{D}$ is an f-canonical cross system of rank ρ for α* as follows.

1. For $\alpha \in S_0$, the unique cross system of rank 0 for α is an f-canonical cross set of rank 0.
2. For $\rho > 0$, $\mathcal{D}$ is an f-canonical cross system of rank ρ for α if it is a cross system of rank ρ for α with a witnessing sequence $\langle\, \mathcal{D}_n : n < \omega \,\rangle$ for which the following additional conditions hold:
   - (g) for $n < \omega$, $\mathcal{D}_n$ is an f-canonical cross system of rank ρ_n for α_n ;
   - (h) there is some i so that $f(\beta, \gamma) = i$ for all $\beta \in D_n$ and $\gamma \in D_p$ with $n < p < \omega$.

This usage is slightly different from the use of the word "canonical" in 4.13. In this section we do not use the term (ξ, σ)-canonical.

The following is one of the oldest ideas in the subject.

4.45 Lemma (Homogeneity Lemma). *For all $\sigma < \omega_1$ there is some $\rho < \omega_1$, so that if $\mathcal{D}$ is an f-canonical cross system of rank ρ, then there is a set $H \subseteq D$ of order type ω^σ which is homogeneous for f.*

The proof is left to the reader. Detailed proofs can be found in both [2] and in [21] of F. Galvin, where the first elementary proof of Theorem 4.30 was given.

We need one more technical lemma, a strengthening of Lemma 4.43, before launching into the main proof.

4.46 Lemma (Induction lemma for canonical cross systems).
Assume $\mathcal{D}$ is an f-canonical cross system of rank ρ for α. Suppose C is a full subset of $\overline{D}$. Then there is a set $B \subseteq C \cap D$ so that $\overline{B} \subseteq C$ and B is the underlying set of an f-canonical system of rank ρ for α.

Proof. Use induction on ρ and the fact that every cofinal subset of a diagonal set for β is diagonal for β. $\dashv$

By the Homogeneity Lemma 4.45, the following lemma will be sufficient to prove Corollary 4.31.

4.47 Lemma (Main Lemma). *For all $\rho < \omega_1$, $\alpha \in S_\rho$ and $F \in \mathcal{F}_\alpha$, there is an f-canonical system $\mathcal{D}$ of rank ρ for α with $\overline{D} \subseteq S_0 \cap F$ and $\mathcal{D} \in N_{\alpha+1}$.*

Note that it would be sufficient to prove 4.47 without the last clause, which is needed to support induction.

The rest of this section is devoted to the proof of 4.47. We need further preliminaries. In what follows, U is a fixed non-principal ultrafilter on ω with $U \in N_0$.

4.48 Definition. Define, by recursion on $\rho < \omega_1$, *deference functions* $i_{\mathcal{D}}$ where $\mathcal{D}$ is a cross system of rank ρ for α. For $\alpha \in S_0$ and a cross system $\mathcal{D}$ of rank 0 for α, define $i_{\mathcal{D}}(\xi)$ for ξ with $\alpha < \xi < \omega_1$ by $i_{\mathcal{D}}(\xi) = i$ if and only if $f(\{\alpha, \xi\}) = i$. Assume $\rho > 0$ and deference functions have been defined for cross systems of rank $\sigma < \rho$. For a cross system $\mathcal{D}$ of rank ρ for α, define $i_{\mathcal{D}}(\xi)$ for ξ with $\alpha < \xi < \omega_1$ by $i_{\mathcal{D}}(\xi) = i$ if and only if $\{n < \omega : i_{\mathcal{D}_n}(\xi) = i\} \in U$ where $\langle \mathcal{D}_n : n < \omega \rangle$ is the witnessing sequence of cross systems for $\mathcal{D}$.

Notice that if $\mathcal{D} \in N_{\alpha+1}$, then the deference function $i_{\mathcal{D}} : \omega_1 - (\alpha+1) \to m$ is also in $N_{\alpha+1}$. Note also that $i_{\mathcal{D}}(\xi)$ can be defined "inside $\mathcal{D}$" for a fixed ξ, as follows.

4.49 Definition. Assume $\mathcal{D}$ is a cross system of rank ρ for α and $\alpha < \xi < \omega_1$. Define $j_{\mathcal{D}}(\beta, \xi)$ for $\beta \in \overline{D}$ by transfinite recursion on $\mathrm{rank}_D(\beta)$ as follows. If $\mathrm{rank}_D(\beta) = 0$, then $j_{\mathcal{D}}(\beta, \xi) = f(\{\beta, \xi\})$. For $\sigma > 0$ and β with $\mathrm{rank}_D(\beta) = \sigma$, set $j_{\mathcal{D}}(\beta, \xi) = j$ for that $j < m$ so that $\{n < \omega : j_{\mathcal{D}}(\beta_n, \xi) = j\} \in U$, where β_n is the nth element of $\mathrm{succ}_D(\beta)$.

The proof that these two definitions coincide is left to the reader.

4.50 Lemma. *Assume $\mathcal{D}$ is a cross system of rank ρ for α. Then for all ξ with $\alpha < \xi < \omega_1$, $j_{\mathcal{D}}(\alpha, \xi) = i_{\mathcal{D}}(\xi)$.*

Note that $j_{\mathcal{D}}$ is an element of $N_{\alpha+1}$ if $D \in N_{\alpha+1}$.

Next we use a fixed enumeration of pairs of natural numbers to define a *standard well-ordering* for $\overline{D}$ where $\mathcal{D}$ is a cross system. For the remainder of this section, assume $\varphi : \omega \times \omega \to \omega - \{0\}$ is a fixed bijection which is monotonic in both variables, and which is in N_0.

4.51 Definition. Define, by recursion on positive $\rho < \omega_1$, for cross systems $\mathcal{D}$ of rank ρ, a *standard well-ordering of* $\overline{D}$.

1. For $\alpha \in S_1$, if $D = \{\alpha_n : n < \omega\}$ is the underlying set of a cross system $\mathcal{D}$ of rank 1, then the standard well-ordering of $\overline{D}$ has least element $d_0 = \alpha$, and for positive k, has kth element $d_k = \alpha_{k-1}$.
2. For $\rho > 1$, if $D = \bigcup\{D_n : n < \omega\}$ is the underlying set of a cross system $\mathcal{D}$ of rank ρ where D_n is the underlying set of $\mathcal{D}_n$ of the witnessing sequence of $\mathcal{D}$, then the standard well-ordering of $\overline{D}$ has least element $d_0 = \alpha$, and for positive $k = \varphi(n, j)$, has kth element $d_k = d_{n,j}$, where $d_{n,j}$ is the jth element of $\overline{D_n}$.

By some abuse of notation, we write d_n for the nth element of the standard well-ordering.

4.52 Lemma. *For all positive $\rho < \omega_1$ and all $\alpha \in S_\rho$, if $\mathcal{D}$ is a cross system of rank ρ for α and $\langle d_k : k < \omega \rangle$ is the standard well-ordering of $\overline{D}$, then for all positive $n < \omega$, there is some $m < n$ so that $d_n \in \mathrm{succ}_D(d_m)$.*

Proof. The proof is by induction on ρ over the recursive definition of standard well-orderings. ⊣

Proof of the Main Lemma 4.47. The proof is by induction on ρ. For $\rho = 0$, the lemma is trivial.

For the induction step, assume $\rho > 0$ and the lemma is true for all $\sigma < \rho$. Let $\alpha \in S_\rho$ and $F \in \mathcal{F}_\alpha$ be arbitrary. If $\rho = \sigma + 1$, then let $\rho_n = \sigma$ for all $n < \omega$. If ρ is a limit, then let $\langle \rho_n : n < \omega \rangle \in N_{\alpha+1}$ be a strictly increasing cofinal sequence with limit ρ, and assume $\rho_0 \geq 1$.

Now, for all $n < \omega$, $\alpha \in S_{\rho_n+1}$, so α is a limit of ordinals in S_{ρ_n} and $\alpha \in \tilde{S}_{\rho_n}$. Temporarily fix an enumeration of $\mathcal{F}_\alpha$ as $\{G_n : n < \omega\}$. By definition of $\tilde{S}_{\rho_n}$, $(S_{\rho_n} \cap F \cap G_0 \cap \cdots \cap G_n) \cap \alpha \notin I_\alpha$.

Define by recursion sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle \mathcal{D}_n : n < \omega \rangle$. To start, choose $\alpha_0 \in (S_{\rho_0} \cap F \cap G_0) \cap \alpha$ large enough so that $F, G_0 \in N_{\alpha_0}$. Then $F, G_0 \in \mathcal{F}_{\alpha_n}$. Use the induction hypothesis on ρ_0, α_0, $F_0' = F \cap G_0$ to find an f-canonical cross system $\mathcal{D}_0 \in N_{\alpha_0+1}$ of rank ρ_0 for α_0 so that $\overline{D}_0 \subseteq S_0 \cap F_0'$.

Continue, taking care to make sure the sequence of α_n's increases to α. If α_n has been defined, then choose $\alpha_{n+1} \in (S_{\rho_{n+1}} \cap F \cap G_0 \cap \cdots \cap G_{n+1} - (\alpha_n + 1)) \cap \alpha$ large enough so that $F, G_0, G_0, \ldots, G_{n+1} \in N_{\alpha_0}$. Then $F, G_0, \ldots, G_{n+1} \in \mathcal{F}_{\alpha_{n+1}}$. Use the induction hypothesis on ρ_{n+1}, α_{n+1}, $F_{n+1}' = F_n' \cap G_{n+1} \cap \omega_1 - (\alpha_{n+1} + 1)$ to find an f-canonical cross system $\mathcal{D}_{n+1} \in N_{\alpha_{n+1}+1}$ of rank ρ_{n+1} for α_{n+1} so that $\overline{D}_{n+1} \subseteq S_0 \cap F_{n+1}'$.

Also, since m is finite, there is an infinite subsequence of $\langle \alpha_n : n < \omega \rangle \in N_{\alpha+1}$ and an $i < m$ so that $i_{\mathcal{D}_n}(\alpha) = i$ for all n in the subsequence. By shrinking if necessary, we may assume, without loss of generality, that this subsequence is the entire sequence. Now $\langle D_n : n < \omega \rangle$ is a witnessing sequence for a cross set of rank ρ for α by construction. Hence $\langle \mathcal{D}_n : n < \omega \rangle$ is a witnessing sequence for a cross system of rank ρ for α.

Finally, as $N_\alpha, \alpha, \in N_{\alpha+1}$, and since S_ρ is amenable by Lemma 4.33, we may assume that $\langle \mathcal{D}_n : n < \omega \rangle$ is defined in $N_{\alpha+1}$.

Claim. *There is an infinite set $T \subseteq \omega$ with $T \in N_{\alpha+1}$ and a family $\{ C_n : n \in T \}$ so that C_n is a full subset of $\overline{D_n}$ for $n \in T$ and $f(\beta, \gamma) = i$ for all $\beta \in C_n$ and $\gamma \in C_p$ with $n, p \in T$ and $n < p$.*

The induction step of the Main Lemma follows from the claim by Lemma 4.46, as each C_n can be replaced by an f-canonical system $\mathcal{C}_n \in N_{\alpha_n+1}$ and $\langle \mathcal{C}_n : n \in T \rangle$ is the witnessing sequence of the desired f-canonical system of rank ρ for α.

To prove the claim, we will pick elements of $\{ \alpha \} \cup \bigcup \{ \overline{D}_n : n \in \omega \}$ according to a certain bookkeeping. We pick α first. Infinitely often we pick a new element n for T, larger than any element of T picked earlier. Our choice of n means we have picked the top point α_n of $\overline{D}_n$. For each point n of T, we promise that infinitely often we will pick an element of $\overline{D}_n$ according to the standard well-ordering of $\overline{D}_n$.

For notational convenience, let $n(\beta)$ denote that value of n with $\beta \in \overline{D}_n$.

Assume we have picked a finite non-empty set $A \subseteq \{\alpha\} \cup \bigcup \{ \overline{D_n} : n < \omega \}$ which satisfies the following condition:

$*(A)$: For any $n < p$, $\beta \in \overline{D}_n \cap A$ and $\xi \in \overline{D}_p \cap A$, $j_{\mathcal{D}_n}(\beta, \xi) = j_{\mathcal{D}_n}(\beta, \alpha) = i$.

We have to pick a new point γ for A so that the enlarged set still satisfies the condition $*(A \cup \{\gamma\})$.

For the first scenario, suppose we want to add a new α_p to A. That is, we want to add a new value p to T. Let

$$Z_0 = Z_0(A) = \bigcap \{ \{ \xi : j_{\mathcal{D}_{n(\beta)}}(\beta, \xi) = i \} : \beta \in A \}.$$

Note that Z_0 is in N_α and $\alpha \in Z_0$. As $\operatorname{succ}(\alpha)$ is reflecting, we can choose the desired $\alpha_p \in \operatorname{succ}(\alpha)$ as large as we want.

For the second scenario, assume we want to pick a β to add to A so that $\beta \in \overline{D}_p$ for some $p \in T$ where $\alpha_p \in A$ and so that $\beta \in \operatorname{succ}(\gamma)$ for some $\gamma \in A \cap \overline{D}_p$. There are three cases, $\alpha_p = \min(A \cap \operatorname{succ}_{\mathcal{D}}(\alpha))$, $\alpha_p = \max(A \cap \operatorname{succ}_{\mathcal{D}}(\alpha))$, and $\min(A \cap \operatorname{succ}_{\mathcal{D}}(\alpha)) < \alpha_p < \max(A \cap \operatorname{succ}_{\mathcal{D}}(\alpha))$. We sketch only the last, and leave the others to the reader. Let $A^- := A \cap \bigcup \{ \overline{D}_n : n < p \}$, and $A^+ := A \cap \bigcup \{ \overline{D}_n : n > p \}$, and define

$$Z^+ = Z^+(A) := \bigcap \{ \delta \in \operatorname{succ}_{\mathcal{D}_p}(\gamma) : j_{\mathcal{D}_p}(\delta, \xi) = i \wedge \xi \in A^+ \}.$$

Now Z^+ is a subset of $\operatorname{succ}_{\mathcal{D}_p}(\gamma)$ which is a reflecting subset of γ by the Reflection Lemma 4.41. Since by $*(A)$, $j_{\mathcal{D}_p}(\gamma, \xi) = i$ for $\xi \in A$, and A is finite, it follows that Z^+ is a reflecting subset of γ. Next define

$$Z^- = Z^-(A) := \bigcap \{ \xi < \omega_1 : j_{\mathcal{D}_n(\delta)}(\delta, \xi) = i \wedge \delta \in A^- \} .$$

By Lemma 4.50, $Z^- \in N_{\max A^- + 1}$. Since $\max A^- < \gamma$, it follows that $Z^- \in N_\gamma$. By $*(A)$, $\gamma \in Z^-$. Hence $Z^+ \cap Z^-$ is infinite and any element of $Z^+ \cap Z^-$ is a suitable choice for β.

Use the technique of "jumping around" and these two scenarios to intertwine the recursive definitions of T and of all the C_n's for $n \in T$. Specifically, use the standard well-ordering of α to define a sequence $\langle \eta_k : k < \omega \rangle$. At stage 0, pick $\eta_0 = \alpha$. Suppose η_ℓ has been defined for $\ell < k$. Look at d_k. If $d_k \in \operatorname{succ}_D(\alpha)$, then use the first scenario to choose $\eta_k \in \operatorname{succ}_D(\alpha)$. If $d_k \in \operatorname{succ}_D(\eta_\ell)$ for some $\ell < k$, then use the second scenario to choose $\eta_k \in \operatorname{succ}_D(\eta_\ell)$. Otherwise, set $\eta_k = \eta_{k-1}$. Finally, let $E = \{ \eta_k : k < \omega \}$.

Let $T = \{ p < \omega : (\exists k)(\eta_k = \alpha_p) \}$. Since the standard order lists all the successors of α, the set T is infinite and in $N_{\alpha+1}$. For $p \in T$, let $C_p = E \cap \overline{D}_p$. Temporarily fix $p \in T$. For any $\gamma \in C_p$, since $\alpha_p = d_\ell$ for some ℓ, and $\operatorname{succ}_{D_n}(\gamma)$ forms an infinite monotonic subsequence of $\{ d_k : k < \omega \}$, the set C_p has infinitely many successors of γ. Thus C_p is full. Therefore T and the sets $\{ C_p : p \in T \}$ are the ones required to prove the claim.

As noted above, the claim suffices to complete the induction step of the Main Lemma, so it follows. ⊣

5. The Milner-Rado Paradox and $\Omega(\kappa)$

Erdős and Rado considered Ramsey's Theorem to be a generalization of the pigeon-hole principle (for cardinals). In 1965, Milner and Rado [45] turned around this view, noting that the pigeon-hole principle is a partition relation with exponent 1, and that a partition relation with exponent 1 and ordinal resource and goal would be a pigeon-hole principle for ordinals.

A case in point of this approach is the easily checked family of partition relations $\kappa^n \to (\kappa^n)^1_\gamma$ for $\kappa \geq \omega$, $n < \omega$, and $\gamma < \operatorname{cf}(\kappa)$. Soon Milner and Rado discovered that basically nothing stronger is true.

5.1 Theorem (Milner-Rado [45]). *For all cardinals $\kappa \geq \omega$ and all $\alpha < \kappa^+$,*

$$\alpha \nrightarrow (\kappa^n)^1_{n<\omega}.$$

Proof. It is sufficient to prove

$$(*) \quad \kappa^\rho \nrightarrow (\kappa^n)^1_{n<\omega} \text{ for } \rho < \kappa^+.$$

Clearly we may assume $\kappa > \omega$. We prove (∗) by transfinite induction on ρ. We can write $\kappa^\rho = \bigcup_{\nu<\sigma} A_\nu$ with $A_0 < \cdots < A_\nu < \ldots$ and each $\operatorname{ot} A_\nu = \kappa^{\rho_\nu}$ for some $\rho_\nu < \rho$, where $\sigma = \operatorname{cf}(\rho)$ if $\operatorname{cf}(\rho) > 1$ and $\sigma = \kappa$ otherwise. By the induction hypothesis, each $A_\nu = \bigcup_{n<\omega} A_{\nu,n}$ where $\operatorname{ot} A_{\nu,n} < \kappa^n$ for $\nu < \sigma$, $n < \omega$. In the case of $\sigma = \omega$, define a witnessing partition $\kappa^\rho = \bigcup_{j<\omega} B_j$ where $B_j = A_{\nu,n}$ for $j = 2^\nu(2n+1)$. In the case of $\sigma > \omega$, let $B_0 := \emptyset$, $B_{n+1} := \bigcup\{A_{\nu,n} : \nu < \sigma\}$. Clearly $\kappa^\rho = \bigcup_{n<\omega} B_n$; and $\operatorname{ot} B_{n+1} \leq \sum_{\nu<\sigma} \kappa^n \leq \kappa^{n+1} < \kappa^\omega$. ⊣

We state one consequence of the above theorem giving further limitations on to positive relations (as discussed in Theorem 4.3).

5.2 Theorem. *For all cardinals $\kappa \geq \omega$, $\kappa^+ \not\rightarrow (\kappa^n)^2_{n<\omega}$.*

Proof. For $\alpha < \kappa^+$, use Theorem 5.1 to choose partitions $\alpha = \bigcup_{n<\omega} A^\alpha_n$ with $\operatorname{ot} A^\alpha_n < \kappa^n$ for each $n < \omega$. Define $f : [\kappa^+]^2 \to \omega$ as follows: for $\alpha < \beta < \kappa^+$, set $f(\alpha, \beta) = n + 1$ if and only if $\alpha \in A^\beta_n$. ⊣

The word *paradox* was used in reference to Theorem 5.1 because this result was so contrary to expectations. It turned out that the phenomena described in Theorem 5.1 is involved in many problems concerning uncountable cardinals, and often it leads to unexpected difficulties.

In this section we are trying to turn this tide and use the paradox in our favor. For the remainder of this section, let κ be a fixed infinite cardinal.

5.3 Definition. For $\alpha < \kappa^+$, call a partition $\alpha = \bigcup_{\gamma\in\Gamma} A_\gamma$ with $\Gamma < \kappa$ a *MR-decomposition of* α if there is a sequence $\langle n_\gamma : \gamma < \Gamma \rangle \in {}^\Gamma\omega$ such that $\operatorname{ot} A_\gamma = \kappa^{n_\gamma}$.

From Theorem 5.1 and the fact that any $\delta < \kappa^n$ is the finite sum of ordinals of the form $\kappa^m \cdot \nu$ where $m < n$ and $\nu < \kappa$, we get the following corollary.

5.4 Corollary. *Each $\alpha < \kappa^+$ has a MR-decomposition.*

Another way to put Definition 5.3 is that α has a MR-decomposition if there are sequences $\langle n_\gamma : \gamma < \Gamma \rangle \in {}^\Gamma\omega$ and functions $\Psi_\gamma : [\kappa]^{n_\gamma} \to \alpha$ for $\gamma < \Gamma < \kappa$ such that Ψ_γ is the canonical monotone map from $[\kappa]^{n_\gamma}$ ordered lexicographically into α.

The next definition from [20] is motivated by this formulation.

5.5 Definition. Call $\alpha < \kappa^+$ *codeable* if there are $\Gamma < \kappa$ and sequences $\langle n_\gamma : \gamma < \Gamma \rangle \in {}^\Gamma\omega$ and $\langle \Psi_\gamma : \gamma < \Gamma \rangle$ so that $\Psi_\gamma : [\kappa]^{n_\gamma} \to \alpha$ for $\gamma < \Gamma$ and for every $A \in [\kappa]^\kappa$,

$$\operatorname{ot} \bigcup_{\gamma<\Gamma} \Psi_\gamma``[A]^{n_\gamma} = \alpha.$$

5.6 Definition. Let $\Omega(\kappa)$ be defined as the least ordinal $\Omega \leq \kappa^+$ so that each $\alpha < \Omega$ is codeable.

Note that this definition from [20] is only interesting if κ is a large cardinal, say at least a Jonsson cardinal.

The following list of properties of $\Omega(\kappa)$ proved in [20] gives some sense of this ordinal for a measurable cardinal $\kappa > \omega$.

1. $\Omega(\kappa) < \kappa^+$;
2. $\Omega(\kappa)$ is closed under the operations of ordinal addition, multiplication, exponentiation, and taking fixed points of these operations;
3. $\Omega(\kappa)$ cannot be changed by (κ, ∞)-distributive forcing;
4. if $V \subseteq W$ and both V and W are models of "ZFC + κ is measurable", then $\Omega(\kappa)^V \leq \Omega(\kappa)^W$;
5. by using generic elementary embeddings in the situation of 4., it is possible to make $\Omega(\kappa)^V < \Omega(\kappa)^W$.

Moreover, $\Omega(\kappa)$ is big, e.g. if U is a normal ultrafilter on κ and ν is the least ordinal such that $L_\nu[U] \cap \kappa^{<\kappa} = L[U] \cap \kappa^{<\kappa}$, then $L[U] \models \Omega(\kappa) = \nu$. Since the statement $\delta < \Omega(\kappa)$ is upwards absolute, this implication shows that the value of $\Omega(\kappa)^V$ is at least as big as ν. Moreover ν is much bigger than, for example, the first $\eta > \kappa$ such that $L_\eta[U]$ is an admissible structure, but much to our regret, we must omit the proofs.

However, we have to confess that we know very little about the combinatorial properties involved in the definitions of $\Omega(\kappa)$. In fact, we do not know if $\Omega(\kappa)$ would become smaller if we requested that the mappings Ψ_γ be monotone.

6. Shelah's Theorem for infinitely many colors.

In this section we prove Shelah's Theorem 4.6, that $\lambda^+ \to (\kappa + \mu)^2_\mu$ for $\mu < \kappa = \mathrm{cf}(\kappa)$ and $\lambda = 2^{<\kappa}$, under the assumption that $\mu < \sigma \leq \kappa$ for some strongly compact cardinal σ. By Theorem 4.18 we may assume $\mu \geq \omega$.

First we need a lemma that was studied and proved independently in [20]. We say that $B \subseteq \lambda^+$ *has essential colors for* g, $\mathcal{I}$, where g is a 2-partition of λ^+ and $\mathcal{I}$ is a normal ideal on λ^+, if $B \notin \mathcal{I}$ and every $C \subseteq B$ with $C \notin \mathcal{I}$ satisfies $g``[C]^2 = g``[B]^2$.

6.1 Lemma (Reduction to essential colors). *Assume $\mu < \kappa = \mathrm{cf}(\kappa)$, and $\lambda := 2^{<\kappa}$. Further suppose that $g : [\lambda^+]^2 \to \mu$ is a 2-partition of λ^+ with μ colors, I is a normal ideal concentrating on S_{κ,λ^+}, and $A \subseteq \lambda^+$ is not in I.*

Then there are a subset $B \subseteq A$ and a normal ideal $J \supseteq I$, such that B has essential colors for g, J.

Proof. By the normality of I and Facts 3.2 we can choose $N \prec H(\lambda^{++})$ suitable for κ such that $g, I, A \in N$, $N \cap \lambda^+ = \alpha < \lambda^+$, $\alpha \in A$, and N satisfies the following condition:

$$(*) : \text{for all } C \in N, \text{ if } \alpha \in C \subseteq \lambda^+, \text{ then } C \notin I.$$

To see this situation may be assumed, choose an elementary chain $N_0 \prec \cdots \prec N_\alpha \prec H(\lambda^{++})$ as in Subsection 4.4 and use normality to see that

$$\{\, \alpha \in S_0 : (*) \text{ fails for some } C \,\} \in I.$$

To prove the lemma, define a decreasing sequence $\langle\, A_\xi : \xi < \kappa \,\rangle$ of subsets of λ^+ by recursion on $\xi < \kappa$. To start the recursion, let $A_0 := A$. Assume $0 < \xi < \kappa$ and A_ζ is defined for $\zeta < \xi$ in such a way that

$$(+) : A_\zeta \in N \text{ and } \alpha \in A_\zeta \subseteq \lambda^+, \quad \text{for } \zeta < \xi.$$

Put $A_\xi = \bigcap_{\zeta<\xi} A_\zeta$ in case ξ is a limit ordinal.

Suppose A_ζ has been defined, and set $\Gamma_\zeta = g``[A_\zeta]^2$. Let I_ζ be the normal ideal generated on A_ζ from

$$I \cap \mathcal{P}(A_\zeta) \cup \{\, x \subseteq A_\zeta : g``[x]^2 \subsetneqq \Gamma_\zeta \,\}\,.$$

If $A_\zeta \notin I_\zeta$, then set $A_{\zeta+1} = A_\zeta$. If $A_\zeta \in I_\zeta$, then it is a finite or diagonal union of elements of the generating set. We treat the case where there is a sequence $\mathcal{B}_\zeta = \langle B_{\zeta,\eta} : \eta < \lambda^+ \rangle$ such that $A_\zeta = \bigcup_{\eta<\lambda^+} B_{\zeta,\eta}$, and for $\eta < \lambda^+$, $B_{\zeta,\eta} \cap (\eta+1) = \emptyset$, and either $B_{\zeta,\eta} \in I$ or $g``[B_{\zeta,\eta}]^2 \subsetneqq \Gamma_\zeta$.

Then, by elementarity, there is a sequence $\mathcal{B}_\zeta \in N$ as described above. Moreover, $\alpha \in B_{\zeta,\eta}$ for some $\eta < \lambda^+$ with $\eta < \alpha$ and $\eta \in N$, and thus $B_{\zeta,\eta} \in N$ for this η. We set $A_{\zeta+1} = B_{\zeta,\eta}$ for this η. Note that in this case, $\alpha \in A_{\zeta+1} \notin I$ and $g``[A_{\zeta+1}]^2 \subsetneqq g``[A_\zeta]^2$. This defines the sequence $\langle A_\eta : \eta < \kappa \rangle$.

Since g maps pairs from λ^+ into μ, there are at most $\mu < \kappa$ many ζ with $A_\zeta \subsetneqq A_{\zeta+1}$. Let ζ be the least ordinal with $A_\zeta = A_{\zeta+1}$, and set $B := A_\zeta$. Then I_ζ is a proper ideal on B. The ideal J generated from $I \cup I_\zeta$ is normal, and $B \notin J$. So by definition of I_ζ, B has essential colors for g, J. ⊣

Given a 2-partition g, we say that y and z are *color equivalent over* x and write $y \equiv^g_x z$ if $x < y$, $x < z$, $\mathrm{ot}(y) = \mathrm{ot}(z)$, and the order isomorphism $\pi : x \cup y \to x \cup z$ has $\pi|x = \mathrm{id}$ and is color preserving: $g(\zeta, \eta) = g(\pi(\zeta), \pi(\eta))$.

6.2 Corollary. *For any 2-partition $g : [\lambda^+]^2 \to \mu$, and any normal ideal J, if B has has essential colors for g and J, then there is a set $C \subseteq B$ with $B - C \in J$ such that for all $\alpha \in C$, for all $x \in [\alpha]^{<\kappa}$, and for all $\gamma \in \Gamma := g``[B]^2$, the set $D(\alpha, x, \gamma)$ is J-positive, where*

$$D(\alpha, x, \gamma) := \{\, \beta \in C : \alpha < \beta \wedge \{\, \alpha \,\} \equiv^g_x \{\, \beta \,\} \wedge g(\alpha, \beta) = \gamma \,\}\,.$$

Proof. To see that the set B has the desired property, assume to the contrary that for all α in some J-positive set $X \subseteq B$, there are $x(\alpha) \in [\alpha]^{<\kappa}$ and $\gamma(\alpha) \in g``[B]^2$ such that the set $D(\alpha, x(\alpha), \gamma(\alpha)) \in J$. By normality and $\mathrm{cf}(\alpha) = \kappa$, there are $Y \subseteq X$ with $Y \notin J$ such that for some x, γ one has $x(\alpha) = x$, $\gamma(\alpha) = \gamma$ for all $\alpha \in Y$. Then for some $Z \subseteq Y$ with $Z \notin J$ the condition $\{\alpha\} \equiv^g_x \{\beta\}$ holds for all $\alpha, \beta \in Z$. If for each $\alpha \in Z$ the set $\{\, \beta \in Z : g(\alpha, \beta) = \gamma \,\} \in J$, then, because of the normality, for the set $W := \{\, \delta \in Z : \forall \beta \in \delta \cap Z\ (\, g(\beta, \delta) \neq \gamma \,)\,\}$ both $W \notin J$ and $\gamma \notin g``[W]^2$ would hold, contradicting the fact that B has essential colors for g, J. ⊣

The above lemma and corollary are to be used with different 2-partitions, and hence were stated in generality. Now fix a 2-partition $f : [\lambda^+]^2 \to \mu$ for which we seek a homogeneous set of type $\kappa + \mu$.

6.3 Lemma (Pulldown Lemma). *There is a subset $S_0 \subseteq S_{\kappa,\lambda^+}$ closed in S_{κ,λ^+} such that for all $\alpha \in S_0$, for all $x \in [\alpha]^{<\kappa}$, and for all $z \in [\lambda^+ - (\alpha + 1)]^{<\kappa}$, there is a $y \in [\alpha - \sup x]^{<\kappa}$ such that $y \equiv^f_x z$.*

Proof. By the facts listed in 3.2, there is an elementary chain $\langle N_\alpha : \alpha < \lambda^+ \rangle \prec H(\lambda^{++})$ with $\langle N_\alpha, \in \rangle \prec \langle N_\beta, \in \rangle \prec H(\lambda^{++})$ for $\alpha < \beta < \lambda^+$ and $f \in N_0$ and $S_0 = \{\, \alpha < \lambda^+ : N_\alpha \cap \lambda^+ = \alpha \,\}$.

Then Lemma 6.3 is true by reflection. ⊣

The Pulldown Lemma 6.3 does not say anything about the colors of edges that go between the sets y and z, while Corollary 6.2 detailed a situation in which any essential color may be pre-selected.

We apply Lemma 6.1 to f and the smallest normal ideal on λ^+, the non-stationary ideal, to get $B_0 \subseteq S_0$ and J_0, so J_0 is a normal ideal extending the non-stationary ideal, and B_0 has essential colors for f, J_0. We apply Corollary 6.2 to get $A_0 \subseteq B_0$ so that $B_0 - A_0 \in J_0$ and the other conditions of the corollary hold for all $\alpha \in A_0$. Then we choose $\alpha_0 \in A_0$, and put $T := A_0 - \alpha_0$.

6.4 Lemma. *There exists a function $h : T \times T \to \mu$ such that for all $x \in [\alpha_0]^{<\kappa}$ and $z \in [T]^{<\sigma}$ there is a $y \in [\alpha_0 - \sup x]^{<\sigma}$ such that*

(a). $y \equiv^g_x z$ via $\pi : x \cup y \to x \cup z$ and

(b). $g(\zeta, \zeta') = h(\pi(\zeta), \zeta')$ for all $\zeta \in y$, $\zeta' \in z$.

Proof. As σ is strongly compact is suffices to show that for every $Z \in [T]^{<\sigma}$ there exists a function $H : Z \times Z \to \mu$ as required.

Assume for the sake of contradiction that for every $H : Z \times Z \to \mu$ there is an $x_H \in [\alpha]^{<\sigma}$ such that for all $y \subseteq \alpha - \sup x_H$ satisfying (a), the function given by (b) is not H.

Let $x = \bigcup \{\, x_H :\ H : Z \times Z \to \mu \,\}$. Then $|x| < \sigma$ as $|x| \leq \mu^{|Z|} < \sigma$, since σ is strongly inaccessible.

By Lemma 6.3, there is a y satisfying (a). Then (b) defines a function $H : Z \times Z \to \mu$. By the definition of x, the set $x_H \subseteq x$ is a set on which the function defined by (b) for y is not H, and that is a contradiction. $\dashv$

Now we define $k : [\lambda^+]^2 \to \mu \times \mu$ for $u, v \in \lambda^+$ with $u < v$ by

$$k(u,v) = \langle f(u,v), h(v,u) \rangle.$$

Next apply Lemma 6.1 and Corollary 6.2 to k and the normal ideal J_0 and the set T.

6.5 Corollary. *We get a normal ideal $J_1 \supseteq J_0$, a non-empty set $\Gamma \subseteq \mu \times \mu$, and subsets $S_1 \subseteq B_1 \subseteq T$ with $B_1 \notin J_1$, $B_1 - S_1 \in J_1$ such that B_1 has essential colors for k, J_1, and for each $\alpha \in S_1$ and for each $x \in [\alpha]^{<\kappa}$ and $\langle \gamma, \delta \rangle \in \Gamma$ the set $E(\alpha, x, \langle \gamma, \delta \rangle)$ is J_1-positive, where*

$$E(\alpha, x, \langle \gamma, \delta \rangle) := \left\{ \beta \in S_1 : \alpha < \beta \wedge \{\alpha\} \equiv^k_x \{\beta\} \wedge k(\alpha, \beta) = \langle \gamma, \delta \rangle \right\}.$$

6.6 Lemma. *There is a subset $a \in [S_1]^{<\sigma}$ such that for every partition of a, say $a = \bigcup \{ a_\zeta : \zeta < \mu \}$, there is a $\zeta < \mu$ such that for every $\gamma < \mu$, there is a subset $b_{\zeta,\gamma}$ of a_ζ of type μ homogeneous for f in the color γ.*

Proof. Notice that since $S_1 \subseteq B_0 \subseteq S_0$ with $B \notin J_1 \supseteq J_0 \supseteq I$ for the non-stationary ideal I, we may assume that for the N suitable for κ in the proof of Lemma 6.1 and $\alpha = N \cap \lambda^+$, we have $\alpha \in S_1$. Hence S_1 has the property that any partition of it into μ pieces has a part A (chose the one with $\alpha \in A$) which contains a homogeneous subset of type $\kappa > \mu$ for every $\gamma \in f``[S_1]^2$, else just like in the proof of the Erdős-Rado Theorem 3.10 (apply it to the function that is 1 on pairs $f$ sends to $\gamma$ and 0 elsewhere), there would be a $B \subseteq A$ with $B \notin I$ and $\gamma \notin f``[B]^2$.

By the strong compactness of σ, there must be a set $a \subseteq S_1$ of size $< \sigma$ satisfying the same statement as S_1 about f, all partitions into μ parts and the existence of homogeneous subsets of type μ for all colors $\gamma \in f``[S_1]^2$. $\dashv$

We now describe the construction of the required homogeneous set.

Recall that immediately following Lemma 6.3 we chose α_0. Next choose a as in Lemma 6.1 Then choose $\alpha_1 \in S_1$ satisfying Corollary 6.5.

Then $\alpha_0 < a < \alpha_1$.

Define $a_{\gamma,\delta} := \{ u \in \alpha : g(u, \alpha_1) = \langle \gamma, \delta \rangle \}$.

By Lemma 6.6 there is a $\langle \gamma_0, \delta_0 \rangle \in \Gamma$ such that a_{γ_0,δ_0} contains a subset of type μ homogeneous for color γ for every γ, hence it contains a subset $b \subseteq \alpha_{\gamma_0,\delta_0}$ of type $\operatorname{ot} b = \mu$ homogeneous for f in color δ_0. This will be "our color" and b will be the "μ-part" of our set. We are going to construct the "λ-part" of the set by transfinite recursion on $\xi < \kappa$ as follows. Assume $\xi < k$ and we have constructed $X = X_\xi$ of order type ξ homogeneous for f in color δ_0 and so that all edges from x to $b \cup \{\alpha_1\}$ have color δ_0.

We now apply Corollary 6.2 to α_1, and $X \cup b$ and we obtain an $\alpha_2 \in S_1$, with $\alpha_1 < \alpha_2$ such that $\alpha_1 \equiv^k_{X \cup b} \alpha_2$ and $k(\alpha_1, \alpha_2) = \langle \gamma_0, \delta_0 \rangle$.

As a corollary of this we have $\delta_0 = f(u, \alpha_1) = f(u, \alpha_2)$ for $u \in X$ and $h(v, \alpha_1) = h(v, \alpha_2) = \delta_0$ for $v \in b$.

Apply Lemma 6.4 for α to X, $b \cup \{\alpha_1, \alpha_2\} \subseteq T$. We get $b' \cup \{\alpha'_1, \alpha'_2\}$. We claim that $X_{\xi+1} = X \cup \{\alpha'_2\}$ is homogeneous in color δ_0 and sends all edges to $b \cup \{\alpha_1\}$ of color δ_0.

Indeed $f(u, \alpha'_2) = f(u, \alpha_2) = \delta_0$ for $u \in X$ by the equivalence over X. For $v \in b$, we have $f(\alpha'_2, v) = h(\alpha_2, v) = h(\alpha_1, v) = \delta_0$. By choice of α_2, we have $k(\alpha_1, \alpha_2) = \langle g(\alpha_1, \alpha_2), h(\alpha_2, \alpha_1) \rangle = \langle \gamma_0, \delta_0 \rangle$. Hence $f(\alpha'_2, \alpha_1) = \delta_0$ also.

7. Singular Cardinal Resources

It should be clear to the attentive reader that neither the ramification method as described in Remark 2.4 nor its refinements discussed up to now can yield any specific partition results for a singular resource. To get such results the method of *canonization* was invented in [15].

7.1 Definition. Assume $f : [\kappa]^r \to \gamma$ is an r-partition of length γ of κ, and $\langle A_\nu : \nu < \mu \rangle$ is a sequence of disjoint subsets of κ. Then f is said to be *canonical on* $\langle A_\nu : \nu < \mu \rangle$ if $f(x) = f(y)$ for all $x, y \in A := \bigcup_{\nu<\mu} A_\nu$ whenever x, y are positioned the same way in the sequence, i.e. if

$$|x \cap A_\nu| = |y \cap A_\nu| \text{ for all } \nu < \mu.$$

The idea is that, for a singular cardinal κ, we want to find a sequence $\langle A_\nu : \nu < \mathrm{cf}(\kappa) \rangle$ with $|A_\nu| < \kappa$ for $\nu < \mathrm{cf}(\kappa)$, and $A := \bigcup \{ A_\nu : \nu < \mathrm{cf}(\kappa) \}$ of power κ such that f is canonical on $\langle A_\nu : \nu < \mathrm{cf}(\kappa) \rangle$ and use it to piece together large homogeneous sets. The following is the classical canonization theorem.

7.2 Theorem (General Canonization Lemma [15]). *Suppose that* $\tau \geq 2$ *is a cardinal,* $r \geq 1$ *is an integer,* $\langle \kappa_\xi : \xi < \mu \rangle$ *is a strictly increasing sequence of infinite cardinals with* $\kappa_0 \geq \tau^{|\mu|}$ *and* $\exp_{\binom{r}{2}}(\kappa_\xi) < \exp_{\binom{r}{2}}(\kappa_\eta)$ *for* $\xi < \eta < \mu$. *For any disjoint union* $A = \dot{\bigcup} \{ A_\nu : \nu < \mu \}$, *and any coloring* $f : [A]^r \to \tau$, *if* $|A_\nu| \geq \left(\exp_{\binom{r}{2}}(\kappa_\nu)\right)^+$ *for all* $\nu < \mu$, *then there are sets* $B_\nu \subseteq A_\nu$ *for* $\nu < \mu$ *so that* $|B_\nu| \geq {\kappa_\nu}^+$ *and the sequence* $\langle B_\nu : \nu < \mu \rangle$ *is canonical with respect to* f.

We are omitting the proof, since any reader with some experience in combinatorics should be able to reconstruct it, and since neither this proof nor the subsequent proofs fall into the line of the methods we are describing. We include canonization results because we think that no chapter on partition relations would be complete without them.

Here is the very first application of Theorem 7.2.

7.3 Theorem (Reduction Theorem). *Assume $\kappa > \mathrm{cf}(\kappa)$ is a strong limit cardinal. Then $\kappa \to (\kappa, \kappa_\nu)^2_{1 \le \nu < \gamma}$ if and only if $\mathrm{cf}(\kappa) \to (\mathrm{cf}(\kappa), \kappa_\nu)^2_{1 \le \nu < \gamma}$.*

Indeed, the next theorem is the only one obtained for a singular resource using a method different from canonization. The elementary proof of the theorem is left to the reader (see [14]).

7.4 Theorem (Erdős; Dushnik and Miller [9]). *For every infinite cardinal κ, $\kappa \to (\kappa, \omega)^2$.*

See also [14] for a proof. Added in Proof: The General Canonization Lemma implies Theorem 7.4 for singular strong limit κ and for $\mathrm{cf}\,\kappa > \omega$ it yields $\kappa \to (\kappa, \omega + 1)^2$. It has been a longstanding problem if this partition relation holds if we do not assume that κ is strong limit. Recently Saharon Shelah [55] proved this partition relation holds under the much weaker condition that $2^{\mathrm{cf}\,\kappa} < \kappa$.

Erdős, Hajnal and Rado in [15] pursued the idea of finding the right generalization of the form $\kappa \to (\kappa, \omega_1)^2$ for singular κ. The first possible case is $\kappa = \aleph_{\mathfrak{c}^+}$, where $\mathfrak{c} = 2^\omega$, and the Reduction Theorem 7.3 gives a positive answer in case κ is a strong limit. The very first question of the Erdős-Hajnal problem list [12] asks if this additional hypothesis is necessary. Shelah and Stanley in [61] and [62] proved that the partition relation $\kappa \to (\kappa, \omega_1)^2$ can be both false and true if κ is not a strong limit cardinal. A description of this deep result is beyond the scope of this section.

There is one more *canonization* result that we want to mention. It was isolated during the discussion of the ordinary partition relation in the book [14] that the following result should be true, and Shelah later proved it.

7.5 Theorem (Shelah [58]). *Assume that κ is a singular cardinal of weakly compact cofinality. If $\kappa < 2^{<\kappa}$ and $2^\rho < 2^{<\kappa}$ for $\rho < \kappa$, then*

$$2^{<\kappa} \to (\kappa)^2_2.$$

To prove this partition relation, Shelah worked out a new group of canonization results in [58]. We only state here one of the main results. Call a sequence of cardinals $\langle \kappa_\nu : \nu < \mu \rangle$ *exponentially increasing* if $\xi < \nu < \mu$ implies $2^{\kappa_\xi} < 2^{\kappa_\nu}$. A sequence of sets $\langle B_\nu : \nu < \mu \rangle$ is *weakly canonical* if $f(u) = f(v)$ whenever $u, v \in [B]^r$ ($B = \bigcup_{\nu < \mu} B_\nu$) and $|u \cap B_\nu| = |v \cap B_\nu| \le 1$ for every $\nu < \mu$. A set $F \subseteq \mathcal{P}(A)$ *sustains* A over κ if for every $X \subseteq A$ with $|X| = (2^\kappa)^+$, there is $Y \in F$ so that $Y \subseteq X$ and $|Y| = \kappa^+$.

7.6 Theorem (Shelah's Canonization Lemma [58]). *Suppose $\langle \kappa_\xi : \xi < \mu \rangle$ is an exponentially increasing sequence of infinite cardinals with $\kappa_0 \ge \tau, \mu, \omega$,* *for a cardinal* $\tau \geq 2$. *Then for any disjoint union* $A = \dot{\bigcup}\{ A_\nu : \nu < \mu \}$, *any sequence* $\langle F_\nu \subseteq \mathcal{P}(A_\nu) : \nu < \mu \rangle$, *and any coloring* $f : [A]^2 \to \tau$, *if* $|A_\nu| > 2^{\kappa_\nu}$ *and* F_ν *sustains* A_ν *for all* $\nu < \mu$, *then there is a sequence* $\langle B_\nu : \nu < \mu \rangle$ *weakly canonical with respect to* f *with* $|B_\nu| = {\kappa_\nu}^+$ *for all* $\nu < \mu$.

8. Polarized Partition Relations

Polarized partition relations were defined in the introduction. We do not have the space to give an orderly discussion of the problems and results on this partition relation. Rather, we will only give a few examples, where the method of elementary submodels described in the previous section can be resourcefully used. The first appearance in the literature of the use of elementary submodels for the proofs of polarized partition relations is the following theorem of Albin Jones which generalizes a result of Erdős, Hajnal and Rado [15] from 1965:

8.1 Theorem (A. Jones [30]). *Let* κ *be an infinite cardinal and* $\lambda = 2^{<\kappa}$. *Then the following polarized partition relation holds:*

$$\begin{pmatrix} \lambda^+ \\ \\ \lambda^+ \end{pmatrix} \to \begin{pmatrix} \lambda^+ & & \gamma & & \kappa+1 \\ & or & & , & \\ \gamma & & \lambda^+ & & \kappa+1 \end{pmatrix}^{1,1}.$$

In the remainder of this section, we apply the method of elementary submodels using the "method of double ramification".

8.1. Successors of weakly compact cardinals

The first example is chosen with an eye to a clean presentation of the method.

8.2 Theorem (Baumgartner and Hajnal [3]). *Suppose that* κ *is a weakly compact cardinal. Then*

$$\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \kappa \\ \kappa \end{pmatrix}^{1,1}_{\gamma} \qquad \textit{for } \gamma < \kappa.$$

Before going into the details of the proof, we give some historical remarks and state an open problem. In [26], Hajnal proved that for measurable κ, the following partition relations holds:

$$\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \alpha \\ \kappa \end{pmatrix}^{1,n}_{<\kappa} \qquad \text{for } n < \omega \text{ and } \alpha < \kappa^+.$$

In an early paper of Choodnovsky [6], it was claimed that

$$\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \alpha \\ \kappa \end{pmatrix}^{1,1}_{<\kappa} \qquad \text{for } \alpha < \kappa^+$$

remains valid for weakly compact κ, but no proof was given. Realizing that this claim was by no means obvious, both Kanamori [32] and Wolfsdorf [67] published proofs that the relation is true for two colors:

$$\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \alpha \\ \kappa \end{pmatrix}^{1,1}_{2} \qquad \text{for } \alpha < \kappa^+$$

Theorem 8.2 was generalized in the thesis of Albin Jones [27], [29], who proved, using elementary submodels, that for weakly compact cardinals κ,

$$\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \left(\begin{pmatrix} \alpha \\ \kappa \end{pmatrix}_m, \begin{pmatrix} \kappa^n \\ \kappa \end{pmatrix}_\gamma \right)^{1,1} \text{for } m,n < \kappa,\, \gamma < \kappa,\, \alpha < \kappa^+.$$

To the best of our knowledge, the following problem remains unsolved.

8.3 Question. Does the partition relation

$$\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \alpha \\ \kappa \end{pmatrix}^{1,1}_{\omega} \qquad \text{hold for all weakly compact } \kappa \geq \omega,\, \alpha \geq \kappa^\omega?$$

The rest of this subsection is devoted to the proof of Theorem 8.2 for $\kappa > \omega$. To that end, let $\kappa > \omega$ be a weakly compact cardinal, and let $f : \kappa^+ \times \kappa \to \gamma$ be a fixed partition. We outline background assumptions below, using work from earlier sections.

8.4 Definition. Let $\langle \langle N_\alpha, \in \rangle : \alpha < \kappa^+ \rangle$ be a sequence of elementary submodels of $H(\kappa^{++})$ satisfying 3.2 with $\lambda = \kappa^{<\kappa} = \kappa$ and $A = \{f\}$. Let $\langle I_\alpha : \alpha < \kappa^+ \rangle$ be the ideals defined in 3.4 and let

$$S_0 := \{ \alpha < \kappa^+ : \alpha(N_\alpha) = \alpha \ \wedge\ \mathrm{cf}(\alpha) = \kappa \ \wedge\ N_\alpha \text{ is suitable} \}$$

as defined in subsection 4.2. Note that for $\alpha \in S_0$, I_α is a κ-complete proper ideal, by 3.6.

8.5 Definition. Call $\mathcal{N} = \langle N_{\alpha,\xi} : \alpha < \kappa^+ \wedge\ \xi < \kappa \rangle$ a *double ramification system for $\langle N_\alpha : \alpha < \kappa^+ \rangle$ as in Definition 8.4* if for each $\alpha < \kappa^+$, the sequence $\langle N_{\alpha,\xi} : \xi < \kappa \rangle \in N_{\alpha+1}$ is an increasing continuous sequence of elementary submodels of N_α with $\bigcup \{ N_{\alpha,\xi} : \xi < \kappa \} = N_\alpha$ such that $|N_{\alpha,\xi}| < \kappa$ for $\xi < \kappa$.

We use the name *double ramification system* since, as we explained in the proof of the Erdős-Rado Theorem, the N_α's play the role of the ramification system of Erdős and Rado.

Just like in 3.2, using general facts about elementary submodels, and the uncountability and strong Mahloness of κ, we can see that there is a system satisfying the next definition.

8.6 Definition. Let $\mathcal{N} = \langle N_{\alpha,\xi} : \alpha < \kappa^+ \wedge\ \xi < \kappa \rangle$ be a double ramification system such that for each $\alpha \in S_0$ there is a $T^0_\alpha \subseteq \kappa$, with $T^0_\alpha \in \mathrm{Stat}(\kappa)$ satisfying the following conditions for all $\xi \in T^0_\alpha$:

1. $N_{\alpha,\xi} \cap \kappa = \xi > \gamma$;
2. ξ is a regular cardinal; and
3. $[N_{\alpha,\xi}]^{<\xi} \subseteq N_{\alpha,\xi}$.

Next we relativize certain important sets to the submodels of the double ramification system.

8.7 Definition. For each $\alpha \in S_0$ and $\xi \in T^0_\alpha$, define the following sets:

1. $X_{\alpha,\xi} := N_{\alpha,\xi} \cap \kappa^+$;
2. $I_{\alpha,\xi} := \{\, X \subseteq \xi : (\exists Y)(Y \subseteq \kappa \wedge Y \in N_{\alpha,\xi} \wedge \xi \notin Y \wedge X \subseteq Y) \,\}$;
3. $\hat{I}_{\alpha,\xi} := \{\, X \subseteq X_{\alpha,\xi} : (\exists Y)(Y \subseteq \kappa^+ \wedge Y \in N_{\alpha,\xi} \wedge \alpha \notin Y \wedge X \subseteq Y) \,\}$.

8.8 Lemma. *For $\alpha \in S_0$ and $\xi \in T^0_\alpha$, both $I_{\alpha,\xi}$ and $\hat{I}_{\alpha,\xi}$ are ξ-complete ideals, and $I_{\alpha,\xi}$ is proper.*

Proof. The first statement follows from the fact that $[N_{\alpha,\xi}]^{<\xi} \subseteq N_{\alpha,\xi}$. To see that $I_{\alpha,\xi}$ is proper, then just like in Lemma 3.6, assume $Z \subseteq \kappa$, $\xi \in Z$ and $Z \in N_{\alpha,\xi}$. Then $\sup Z \in N_{\alpha,\xi}$, hence $\sup Z = \kappa$ and $\sup Z \cap \xi = \xi$. This implies $\xi \notin I_{\alpha,\xi}$. ⊣

Note that $\hat{I}_{\alpha,\xi}$ is proper for many α and ξ as well (see 8.11 below).

Notation. For all $\nu < \gamma$, let

$f^{\downarrow}(\alpha;\nu) := \{\, \xi < \kappa : f(\alpha,\xi) = \nu \,\}$ for $\alpha < \kappa^+$, and
$f^{\uparrow}(\xi;\nu) := \{\, \alpha < \kappa^+ : f(\alpha,\xi) = \nu \,\}$ for $\xi < \kappa$.

8.9 Definition. For $\alpha \in S_0$ and $\xi \in T^0_\alpha$, let

$$a_{\alpha,\xi} := \left\{\, \nu < \gamma : f^{\downarrow}(\alpha;\nu) \cap \xi \notin I_{\alpha,\xi} \,\right\}.$$

Note that $a_{\alpha,\xi} \neq \emptyset$ by Lemma 8.8 and the fact that $\gamma < \xi$.

8.10 Lemma (Main Lemma). *There are subsets $a \subseteq \gamma$ and $S \subseteq S_0$ with $S \in \mathrm{Stat}(\kappa^+)$, and for each $\alpha \in S$, there is a subset $T_\alpha \subseteq T^0_\alpha$ with $T_\alpha \in \mathrm{Stat}(\kappa)$, so that $f(\alpha,\eta) \in a = a_{\alpha,\eta}$ for all $\alpha \in S$ and $\eta \in \bigcup\{\, T_\beta : \beta \in S \,\}$.*

Proof. First thin each T^0_α for $\alpha \in S_0$ to a stationary subset T^1_α so that for some a_α, one has $a_{\alpha,\xi} = a_\alpha$ for all $\xi \in T^1_\alpha$. Then thin S_0 to a stationary subset S_1 so that for some $a \subseteq \gamma$ and for all $\alpha \in S_1$, $a_\alpha = a$. We may assume without loss of generality that $\gamma < \xi$ for all $\xi \in T^1_\alpha$.

Notice that for all $\alpha \in S_1$ and all $\xi \in T^1_\alpha$, if $\nu \notin a$, then $f^{\downarrow}(\alpha;\nu) \cap \xi \in I_{\alpha,\xi}$. Hence, by the definition of $f^{\downarrow}$ and the ξ-completeness of $I_{\alpha,\xi}$, it follows that $\{\, \eta < \xi : f(\alpha,\eta) \notin a \,\} \in I_{\alpha,\xi}$. By Definition 8.7, for $\alpha \in S_1$ and $\xi \in T^1_\alpha$, we can choose sets $Y_{\alpha,\xi} \subseteq \kappa$ such that $\xi \notin Y_{\alpha,\xi} \in N_{\alpha,\xi}$ and $\{\, \eta < \xi : f(\alpha,\eta) \notin a \,\} \subseteq Y_{\alpha,\xi}$. Using Fodor's Theorem twice, we get $Y \subseteq \kappa$, $S \subseteq S_1$ with $S \in \mathrm{Stat}(\kappa^+)$, and $\langle\, T_\alpha \subseteq T^1_\alpha : \alpha \in S \,\rangle$ such that $T_\alpha \in \mathrm{Stat}(\kappa)$ for all $\alpha \in S$, and $Y_{\alpha,\xi} = Y$ for $\alpha \in S$ and $\xi \in T_\alpha$.

Consequently, for all $\xi \in \bigcup\{\, T_\beta : \beta \in S \,\}$, we have $\xi \notin Y$, since $\xi \notin Y_{\beta,\xi} = Y$. However, if $\alpha \in S$ and $\eta < \kappa$ are such that $f(\alpha,\eta) \notin a$, then for some $\xi \in T_\alpha$, one has $\eta \in Y_{\alpha,\xi} = Y$, so the theorem follows. ⊣

8.11 Corollary. *There is an $\alpha < \kappa^+$, so that for κ-many ξ, the following condition holds:*

$$(+)(\exists \nu < \gamma)(f^{\downarrow}(\alpha;\nu) \cap \xi \notin I_{\alpha,\xi} \wedge f^{\uparrow}(\xi;\nu) \cap X_{\alpha,\xi} \notin \hat{I}_{\alpha,\xi}).$$

Proof. Let α be such that $S \cap \alpha \notin I_\alpha$. Such an α must exist by Corollary 4.8. A standard argument shows that if $S \cap \alpha \notin I_\alpha$, then $W = \left\{\, \xi < \kappa : S \cap \alpha \cap X_{\alpha,\xi} \in \hat{I}_{\alpha,\xi} \,\right\}$ is non-stationary in κ. By Main Lemma 8.10, $f(\beta,\xi) \in a$ for $\xi \in T_\alpha$ and $\beta \in S \cap \alpha \cap X_{\alpha,\xi}$. Hence $f^{\uparrow}(\xi;\nu) \cap X_{\alpha,\xi} \notin \hat{I}_{\alpha,\xi}$ for some $\nu \in a$ and for every $\xi \in T_\alpha - W$. On the other hand, $f^{\downarrow}(\alpha;\nu) \cap \xi \notin I_{\alpha,\xi}$ for all $\nu \in a$ and for every $\xi \in T_\alpha$. ⊣

8.12 Lemma (Compactness Lemma). *Assume that for some $\alpha < \kappa^+$ there are κ-many ξ so that for some $A_\xi \subseteq X_{\alpha,\xi}$, $B_\xi \subseteq \xi$ with $\mathrm{ot}\, A_\xi = \mathrm{ot}\, B_\xi = \xi$, the set $A_\xi \times B_\xi$ is homogeneous for f. Then there are $A \subseteq \kappa^+$, $B \subseteq \kappa$ with $\mathrm{ot}\, A = \kappa + 1$ and $\mathrm{ot}\, B = \kappa$ such that $A \times B$ is homogeneous for f.*

Proof. Use the weak compactness of κ via its Π^1_1-indescribability. ⊣

After all these preliminaries, Theorem 8.2 now follows from Corollary 8.11, the Compactness Lemma 8.12 above, and the Reflection Lemma below.

8.13 Lemma. *Assume that for α as in Corollary 8.11 and for some $\nu < \gamma$, the ordinal ξ satisfies the formula (+) of 8.11. Then there are $A \subseteq X_{\alpha,\xi}$, $B \subseteq \xi$ with $\mathrm{ot}\, A_\xi = \mathrm{ot}\, B_\xi = \xi$ so that $A \times B$ is homogeneous for f in color ν.*

Proof. Let $\overline{A} := f^{\uparrow}(\xi;\nu) \cap X_{\alpha,\xi}$ and let $\overline{B} = f^{\downarrow}(\alpha;\nu) \cap \xi$. Since (+) holds for ν and ξ, we know that $\overline{B} \notin I_{\alpha,\xi}$ and $\overline{A} \notin \hat{I}_{\alpha,\xi}$. These last two statements imply the existence of the sets A, B as required. Indeed, we can define sequences $A = \{ a_\mu : \mu < \xi \} \subseteq \overline{A}$ and $B = \{ b_\mu : \mu < \xi \} \subseteq \overline{B}$ by transfinite recursion on $\mu < \xi$ so that for all $\mu', \mu'' < \xi$,

$$f(a_{\mu'}, b_{\mu''}) = \nu,$$
$$a_{\mu'} \in f^{\uparrow}(\xi;\nu),$$
$$b_{\mu''} \in f^{\downarrow}(\alpha;\nu).$$

At stage $\mu < \xi$, assume this has been done for $\mu', \mu'' < \mu$. First choose a_μ. Toward that end, let

$$Z_\mu^- := \left\{ \beta < \kappa^+ : f(\beta, b_{\mu''}) = \nu \text{ for all } \mu'' < \mu \right\}.$$

Then $\alpha \in Z_\mu^-$ since $b_{\mu''} \in f^{\downarrow}(\alpha;\nu)$ for all $\mu'' < \mu$. Since $f, \{ b_{\mu''} : \mu'' < \mu \} \in N_{\alpha,\xi}$, it follows that $Z_\mu^- \in N_{\alpha,\xi}$. So $Z_\mu^- \cap \overline{A} - \{ a_{\mu'} : \mu' < \mu \}$ is not in $\hat{I}_{\alpha,\xi}$, so we can choose α_μ from it.

Then choose b_μ similarly using $f^{\uparrow}(\xi;\nu)$ in the role of $f^{\downarrow}(\alpha;\nu)$ and $I_{\alpha,\xi}$ instead of $\hat{I}_{\alpha,\xi}$ and taking care to make $f(a_{\mu'}, b_\mu) = \nu$ for $\mu' \leq \mu$. ⊣

8.2. Successors of singular cardinals

In this subsection we investigate the following question.

8.14 Question. Assume κ is a singular strong limit cardinal and $\gamma < \kappa$. Under what circumstances does the following partition relation hold?

$$(*) \qquad \begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \kappa \\ \kappa \end{pmatrix}^{1,1}_{\gamma}$$

The problem was isolated in Problem 11 of [15], where it was asked if (∗) holds for $\kappa = \aleph_{\omega_1}$ under GCH. In the same paper, it was proved that (∗) holds provided $\mathrm{cf}(\kappa) = \omega$, but we omit the proof of this fact.

After about thirty years, a shocking partial result was proved by Saharon Shelah.

8.15 Theorem (Shelah [60]). *Assume κ is a singular strong limit cardinal of uncountable cofinality. Then (∗) holds if $2^\kappa > \kappa^+$.*

For another proof of this result, see Kojman [34]. A little extra information is contained in an unpublished result of M. Foreman, which we prove here using the result of Shelah.

8.16 Theorem (Foreman unpublished). *Suppose that κ is a singular strong limit cardinal in V and $(2^\kappa)^V > (\kappa^+)^V$. Then there is a κ-complete partial order P which satisfies the $(2^\kappa)^+$-chain condition so that*

$$V^P \models 2^\kappa = \kappa^+ \text{ and } \begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \kappa \\ \kappa \end{pmatrix}^{1,1}_\gamma \qquad \text{for } \gamma < \kappa.$$

Proof. We can choose for P the κ^+-complete Levy collapse of $2^{<\kappa}$ to κ^+. For every $p \in P$ and every name for a partition $\dot{f}$, we can define in V a decreasing sequence $\langle p_\alpha \mid \alpha < \kappa^+ \rangle$ of conditions and a function $g : \kappa^+ \times \kappa \to \gamma$ such that $p_0 = p$ and

$$\forall \alpha < \kappa^+ \; \forall \beta \leq \alpha \; \forall \xi < \kappa \; p_\alpha \Vdash \dot{f}(\beta, \xi) = g(\beta, \xi).$$

By Theorem 8.15, we can choose A, B such that $A \times B$ is homogeneous for g and $|A| = |B| = \kappa$. For some $\alpha < \kappa$, we have $A, B \subseteq \alpha$ and then

$$p_\alpha \Vdash \exists A \; \exists B \; (|A| = |B| = \kappa \wedge A \times B \text{ is homogeneous for } \dot{f})$$

Hence V^P satisfies the claim. ⊣

All other problems remain unsolved, even for $\gamma = 2$. For notational convenience, for the rest of this section let $\mu = \mathrm{cf}(\kappa)$. We may assume that $\mu > \omega$, and we will embark on a lengthy proof of a mild strengthening of the result of Shelah.

8.17 Theorem. *Suppose that κ is a singular strong limit cardinal of uncountable cofinality μ. Then $(**)$ holds if $2^\kappa > \kappa^+$:*

$$(**) \qquad \begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{pmatrix} \kappa + 1 \\ \kappa \end{pmatrix}^{1,1}_\gamma$$

The proof we are going to describe will be a double ramification, quite similar in structure to the proof of Theorem 8.2 and different from the simplified proof of Theorem 8.15 in Kojman [34].

8.18 Definition. Choose $\vec{\kappa} = \langle \kappa_\nu : \nu < \mu \rangle$ to be an increasing continuous sequence of cardinals satisfying the following properties:

1. $\sup \{ \kappa_\nu : \nu < \mu \} = \kappa$;
2. $\mu < \kappa_0$; and
3. $2^{\kappa_\nu} < \kappa_{\nu+1} = \mathrm{cf}(\kappa_{\nu+1})$ for $\nu < \mu$.

We use results of Shelah's pcf theory [59] to guarantee the existence of the sequence delineated in the next definition.

8.19 Definition. Choose $\vec{\lambda} = \langle \lambda_\nu : \nu < \mu \rangle$ to be an increasing sequence of regular cardinals with $\kappa_\nu < \lambda_\nu < \kappa$ for $\nu < \mu$ such that the product $\Pi := \prod_{\nu<\mu} \lambda_\nu$ satisfies

$$(\forall \{ \varphi_\alpha : \alpha < \kappa^+ \} \subseteq \Pi)(\exists \varphi \in \Pi)(\forall \alpha < \kappa^+)(\varphi_\alpha \prec \varphi)$$

where $\prec$ is the relation of eventual domination on Π.

We now choose a sequence of models to serve as the skeleton of a double ramification.

8.20 Definition. Let $A := \mu \cup \left\{ \mu, f, \vec{\kappa}, \vec{\lambda} \right\}$. Using 3.2, we can choose an increasing chain $\langle \langle N_\alpha, \in \rangle : \alpha < \kappa^+ \rangle$ of elementary submodels of $H(\kappa^{++})$ with $A \in N_0$ such that

$$S_0 := \{ \alpha < \kappa^+ : \alpha(N_\alpha) = \alpha > \kappa \ \wedge \ \mathrm{cf}(\alpha) = \mu \ \wedge \ N_\alpha \text{ is suitable for } \kappa \}$$

is a club in S_{μ,κ^+}. As in Definition 3.4, we define

$$I_\alpha := \{ X \subseteq \alpha^+ : \exists Y \ (Y \subseteq \kappa^+ \wedge Y \in N_\alpha \wedge \alpha \notin Y \wedge |X - Y| < \kappa) \},$$

and note that since κ is singular, the last condition may no longer be replaced by $X \subseteq Y$.

8.21 Facts. The following statements hold.

1. I_α is a μ-complete proper ideal for all $\alpha \in S_0$;
2. for every stationary $S \subseteq S_0$, there is some $\alpha \in S$ so that $S \cap \alpha \notin I_\alpha$;
3. for every $\alpha \in S_0$, every $X \in \mathcal{P}(\alpha) - I_\alpha$ and every $\tau < \kappa$, there is some $W \subseteq X$ with $|W| = \tau$ so that $W \in N_\alpha$.

Proof. The first item follows from Lemma 3.6, and the second from Corollary 4.8. To see that the third item holds, fix $\alpha \in S_0$, and assume $X \in \mathcal{P}(\alpha) - I_\alpha$. By the definition of I_α, we have $|X| \geq \kappa$. Let $\tau < \kappa$ be given. Since $\mathrm{cf}(\kappa) = \mu < \kappa$, there is a $\beta < \alpha$ with $|X \cap \beta| \geq \tau$. Since $N_\alpha \prec H(\kappa^{++})$ and $\beta \in N_\alpha$, there is some U in N_α with $|U| < \kappa$ and $|X \cap U| \geq \tau$. Then any $W \subseteq X \cap U$ with $|W| = \tau$ satisfies the requirement of the item since $|\mathcal{P}(U)| < \kappa$ and therefore $\mathcal{P}(U) \subseteq N_\alpha$. ⊣

For notational convenience, we use the same names for our double ramification system here as in the proof of Theorem 8.2.

8.22 Definition (Double ramification). For each $\alpha \in S_0$, we choose $<_\alpha$, a well-ordering of type κ of N_α. Choose $\mathcal{N} = \langle N_{\alpha,\nu} : \alpha < \kappa^+ \wedge \nu < \mu \rangle$ for the skeleton chosen above so that for $\alpha \in S_0$, the sequence $\langle N_{\alpha,\nu} : \nu < \mu \rangle$ is increasing, continuous and internally approachable and satisfies the following conditions:

1. $A \in N_{\alpha,0}$;

2. $\kappa_\nu \subseteq N_{\alpha,\nu}$, $|N_{\alpha,\nu}| = \kappa_\nu$, and $N_{\alpha,\nu}$ contains the νth section of $N_{\alpha,\nu}$ in the well-ordering $<_\alpha$ for each $\nu < \mu$.

Next we relativize certain important sets to the submodels of the double ramification system.

Notation. For each $\alpha \in S_0$, define the set $X_{\alpha,\nu} := N_{\alpha,\nu} \cap \lambda_\nu$ for $\nu < \mu$ and the function $\varphi_\alpha : \mu \to \kappa$ so that $\varphi_\alpha(\nu) := \sup X_{\alpha,\nu}$.

The following facts follow from Definition 8.19 of Π and $\vec{\lambda}$.

8.23 Lemma. *For all $\alpha \in S_0$, the function φ_α is in Π, and there is a function $\varphi \in \Pi$ which eventually dominates all the φ_α for $\alpha \in S_0$. That is, for each $\alpha \in S_0$, there is some $\nu_\alpha < \mu$, so that $\varphi_\alpha(\nu) < \varphi(\nu)$ for all ν with $\nu_\alpha \leq \nu < \mu$.*

For the remainder of this section, fix a function φ which eventually dominates all the φ_α for $\alpha \in S_0$, and let ν_α as above be the point at which domination sets in.

8.24 Definition. For $\alpha \in S_0$ and ν with $\nu_\alpha \leq \nu < \mu$, define

$$I_{\alpha,\nu} := \{\, X \subseteq \nu : \exists Y \ (Y \subseteq \lambda \wedge Y \in N_{\alpha,\nu} \wedge \varphi(\nu) \notin Y \wedge |X - Y| < \kappa_\nu)\,\}.$$

8.25 Lemma. *Let $\alpha \in S_0$ and ν with $\nu_\alpha \leq \nu < \mu$ be given. Then*

1. *$I_{\alpha,\nu}$ is a proper ideal;*

2. *for each $X \subseteq X_{\alpha,\nu}$ with $X \in I^+_{\alpha,\nu}$, there is a $W \subseteq X$ with $|W| = \kappa_\nu$ so that $W \in N_{\alpha,\nu+1}$.*

Proof. For the first item, note that the set $I_{\alpha,\nu}$ is an ideal because $N_{\alpha,\nu}$ is closed with respect to finite unions. To see that $X_{\alpha,\nu} \notin I_{\alpha,\nu}$, let $Z \in N_{\alpha,\nu}$ be a subset of λ_ν with $\varphi(\nu) \in Z$. It is enough to show $|Z \cap X_{\alpha,\nu}| \geq \kappa_\nu$. Now $Z \in N_{\alpha,\nu}$ and $\sup Z \in N_{\alpha,\nu}$. Hence $\sup Z = \lambda_\nu$. Thus there is a one-to-one function $g : \kappa_\nu \to Z$. Using the fact that κ_ν and λ_ν are in $N_{\alpha,\nu}$, by elementarity, there is a function $g \in N_{\alpha,\nu}$ like this. Using the fact that $\kappa_\nu + 1 \subseteq N_{\alpha,\nu}$, we get that $\operatorname{ran}(g) \subseteq N_{\alpha,\nu} \cap \lambda_\nu = X_{\alpha,\nu}$.

For the second item, there is a subset $W \subseteq X$ with $|W| = \kappa_\nu$ by Definition 8.24. Also, by Definition 8.22, we know that $X_{\alpha,\nu} \in N_{\alpha,\nu+1}$, $2^{\kappa_\nu} < \kappa_{\nu+1}$ and $\mathcal{P}(X_{\alpha,\nu}) \subseteq N_{\alpha,\nu+1}$. Therefore $W \in N_{\alpha,\nu+1}$ as required. ⊣

Recall the notation $f^{\downarrow}(\alpha; i)$ introduced after Lemma 8.8:

$$f^{\downarrow}(\alpha; i) := \{\, \xi < \kappa : f(\alpha, \xi) = i \,\} \text{ for } \alpha < \kappa^+,\ i < \gamma.$$

Using the facts that $\gamma, \omega < \mu$ and $2^\mu < \kappa$, we can show directly that

$$\begin{pmatrix} \kappa^+ \\ \mu \end{pmatrix} \to \begin{pmatrix} \mathrm{Stat}(\kappa^+) \\ \mathrm{Stat}(\mu) \end{pmatrix}^{1,1}_{\gamma}.$$

We get the next lemma by applying this partition relation to the coloring $f \circ \varphi$ of $\kappa^+ \times \mu$.

8.26 Lemma. *There are $S \subseteq S_0$, $T \subseteq \mu$, $\overline{\nu} < \mu$ and $i < \gamma$ such that $S \in \mathrm{Stat}(\kappa^+)$, $T \in \mathrm{Stat}(\mu)$, $\overline{\nu} \cap T = \emptyset$, $\varphi``T \subseteq f^{\downarrow}(\alpha; i)$ and $\nu_\alpha = \overline{\nu}$ for all $\alpha \in S$.*

We now prove our main claim.

8.27 Lemma (Main Claim). *There is an $\alpha \in S$ such that $S \cap \alpha \notin I_\alpha$ and*

$$\{\, \nu \in T : f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} \notin I_{\alpha,\nu} \,\} \in \mathrm{Stat}(\mu).$$

Proof. By Corollary 4.8, it is sufficient to see that

$$\{\, \alpha \in S : \{\, \nu \in T : f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} \notin I_{\alpha,\nu} \,\} \in \mathrm{Stat}(\mu) \,\} \in \mathrm{Stat}(\kappa^+).$$

Let $T_\alpha := \{\, \nu \in T : f^{\downarrow}(\alpha, i) \cap X_{\alpha,\nu} \in I_{\alpha,\nu} \,\}$ for $\alpha \in S$. Assume by way of contradiction that for some $S' \in \mathrm{Stat}(\kappa^+) \cap \mathcal{P}(S)$, one has $T_\alpha \in \mathrm{Stat}(\mu)$ for all $\alpha \in S'$.

For $\alpha \in S'$, $\nu \in T_\alpha$, choose $Y_{\alpha,\nu}$ satisfying the following conditions: $Y_{\alpha,\nu} \subseteq \lambda_\nu$, $Y_{\alpha,\nu} \in N_{\alpha,\nu}$, $\varphi(\nu) \notin Y_{\alpha,\nu}$, and $|f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} - Y_{\alpha,\nu}| < \kappa_\nu$. For each $\alpha \in S'$, by Fodor's Theorem, the sets $Y_{\alpha,\nu}$ stabilize on a stationary subset of T_α. That is, for each $\alpha \in S'$, there are $T'_\alpha \subseteq T_\alpha$ with $T'_\alpha \in \mathrm{Stat}(\mu)$, Y_α and $\rho_\alpha < \kappa$ such that $Y_{\alpha,\nu} = Y_\alpha$ and $|f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} - Y_{\alpha,\nu}| \leq \rho_\alpha$ for $\nu \in T'_\alpha$ and

$$Y_\alpha \cap \{\, \varphi(\nu) : \nu \in T'_\alpha \,\} = \emptyset.$$

Note that $\bigcup \{\, X_{\alpha,\nu} : \nu \in T'_\alpha \,\} = \kappa$, hence

$$|f^{\downarrow}(\alpha; i) - Y_\alpha| \leq \rho_\alpha.$$

Now, using Fodor's Theorem again, Y_α stabilizes on a stationary subset of S'. That is, there are $T' \in \mathrm{Stat}(\mu)$, Y and ρ such that for some $S'' \in \mathrm{Stat}(\kappa^+) \cap \mathcal{P}(S')$, one has $T'_\alpha = T'$, $Y_\alpha = Y$ and $\rho_\alpha = \rho$ for all $\alpha \in S''$.

Now choose two elements $\alpha', \beta' \in S''$ with $\alpha' < \beta'$, and let $\nu' \in T'$ be such that $\beta' \in N_{\alpha',\nu'}$ and $\kappa_{\nu'} > \rho$. Since $\alpha' \in S'' \subseteq S$ and $\nu' \in T' \subseteq T$, it follows that $f(\beta', \varphi(\nu')) = i$ by Lemma 8.26. In other words, $\varphi(\nu') \in f^{\downarrow}(\beta'; i)$. However, $f^{\downarrow}(\beta'; i) \in N_{\alpha',\nu'}$, hence

$$f^{\downarrow}(\beta'; i) \cap X_{\alpha',\nu'} \notin I_{\alpha',\nu'}.$$

This last fact contradicts the inequality $|f^{\downarrow}(\beta'; i) - Y| < \rho$ and the lemma follows. $\dashv$

To finish the proof of Theorem 8.17 using the Main Claim 8.27, we want to define sequences $\langle A_\xi : \xi < \mu \rangle$ with $A_\xi \subseteq \kappa$ and $\langle B_\xi : \xi < \mu \rangle$ with $B_\xi \subseteq S_0$ so that the sets are pairwise disjoint, $|A_\xi| = |B_\xi| = \kappa_\xi$, $A_\xi, B_\xi \in N_{\alpha,\nu_\xi}$ for some $\nu_\xi \in T^0$, where $T^0 := \{ \nu \in T : f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} \notin I_{\alpha,\nu} \}$ is the set defined in the Main Claim 8.27, and f is constantly i on the set

$$\bigcup_{\xi<\mu} B_\xi \cup \{ \alpha \} \times \bigcup_{\xi<\mu} A_\xi.$$

To carry out an induction of length μ to define the desired sequences, we only need the following lemma.

8.28 Lemma. *Assume $A, B \in N_{\alpha,\nu}$ for some $\nu \in T^0$, $B \subseteq S$, $\rho < \kappa$, and f is homogeneous of color i on $(B \cup \{ \alpha \}) \times A$. Then the following two statements hold.*

1. *There is $C \in [\kappa - (A \cup B)]^\rho$ with $C \subseteq \bigcap \{ f^{\downarrow}(\beta; i) : \beta \in B \cup \{ \alpha \} \}$ so that for some $\nu' \in T^0$ with $\kappa_{\nu'} > \nu$, one has $C \in N_{\alpha,\nu'}$.*
2. *There is $D \in [S - (A \cup B)]^\rho$ with $A \subseteq \bigcap \{ f^{\downarrow}(\beta; i) : \beta \in D \}$ so that for some $\nu' \in T^0$ with $\kappa_{\nu'} > \nu$, one has $D \in N_{\alpha,\nu'}$.*

Proof. For the first item, choose $\nu' \in T^0$ with $\nu' > \nu$ and $\kappa_{\nu'} > \rho$. By the definition of S, we know $f(\beta, \varphi(\nu')) = i$ for $\beta \in B \cup \{ \alpha \}$. By the Main Claim 8.27, we know that $f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu'} \notin I_{\alpha,\nu'}$. Let $Z = \bigcap \{ f^{\downarrow}(\beta; i) : \beta \in B \}$. Then $Z \in N_{\alpha,\nu'}$ and $\varphi(\nu') \in Z$. Hence $|Z \cap f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu'}| \geq \rho$ by Lemma 8.25, and we can choose a subset of this intersection for C.

For the second item, the set $Z := \bigcap \{ f^{\uparrow}(\eta; i) : \eta \in A \}$ is in $N_{\alpha,\nu}$ and $\alpha \in Z$. Since $S \cap \alpha \notin I_\alpha$, we can choose a suitable D by Facts 8.21. ⊣

9. Countable Ordinal Resources

9.1. Some history

In this section we look at ordinal partition relations of the form $\alpha \to (\beta, m)^2$ for limit ordinals α and β of the same cardinality. The goal m will be taken to be finite, since if $\pi : \alpha \to |\alpha|$ is a one-to-one mapping, then the partition defined on pairs $x < y < \alpha$ by

$$f(x,y) = \begin{cases} 0, & \text{if } x < y \text{ and } \pi(x) < \pi(y) \\ 1, & \text{if } x < y \text{ and } \pi(x) > \pi(y) \end{cases}$$

shows that $\alpha \not\to (|\alpha| + 1, \omega)^2$.

This particular branch of the partition calculus dates back to the 1950's, in particular to to the seminal paper of Erdős and Rado [19] which introduced the partition calculus for linear order types and to the paper of Specker [63], in which he proves the following theorem.

9.1 Theorem (Specker [63]). *The following partition relations hold:*

1. $\omega^2 \to (\omega^2, m)^2$ *for all* $m < \omega$.

2. $\omega^n \not\to (\omega^n, 3)^2$ *for all* $3 \leq n < \omega$.

The finite powers of ω are all *additively indecomposable (AI)*, since they cannot be written as the sum of two strictly smaller ordinals. It is well-known that the additively indecomposable ordinals are exactly those of the form ω^γ (see Exercise 5 on page 43 of Kunen [36], [37]). We will focus on additively indecomposable α and β. There are additional combinatorial complications for decomposable ordinals.

For notational convenience in discussions of $\alpha \to (\beta, m)^2$, call α the *resource*, β the 0-*goal* and m the 1-*goal*.

For a specified countable 0-goal β and finite 1-goal m, it is possible to determine an upper bound for the resource α needed to ensure that the positive partition relation holds. In particular, Erdős and Milner showed $\omega^{1+\mu m} \to (\omega^{1+\mu}, 2^m)^2$. This result dates back to 1959 and a proof appeared in Milner's thesis in 1962. See also pages 165-168 of [66] where the proof is given via the following stepping-up result:

9.2 Theorem. *Suppose* γ, δ *are countable and* k *is finite. If* $\omega^\gamma \to (\omega^{1+\delta}, k)^2$, *then* $\omega^{\gamma+\delta} \to (\omega^{1+\delta}, 2k)^2$.

9.3 Corollary (Erdős and Milner [16]). *If* $m < \omega$ *and* $\mu < \omega_1$, *then* $\omega^{1+\mu\cdot\ell} \to (\omega^{1+\mu}, 2^\ell)^2$.

The partition calculus for finite powers of ω is largely understood via the results below of Nosal. Her work built on 9.3 and earlier work by Galvin (unpublished), Hajnal, Haddad and Sabbagh [24], Milner [43].

9.4 Theorem (Nosal [47], [48]).

1. *If* $1 \leq \ell < \omega$, *then* $\omega^{2+\ell} \to (\omega^3, 2^\ell)^2$ *and* $\omega^{2+\ell} \not\to (\omega^3, 2^\ell + 1)^2$.

2. *If* $1 \leq \ell < \omega$ *and* $4 \leq r < \omega$, *then* $\omega^{1+r\cdot\ell} \to (\omega^{1+r}, 2^\ell)^2$ *and* $\omega^{r+r\cdot\ell} \not\to (\omega^{1+r}, 2^\ell + 1)^2$.

Some progress has been made for the case in which the goal is ω^4. Nosal showed in her thesis that $\omega^6 \not\to (\omega^4, 3)^2$, which is sharp, since $\omega^7 \to (\omega^4, 4)^2$ by Corollary 9.3. Darby (unpublished) has shown that $\omega^9 \not\to (\omega^4, 5)^2$.

9.2. Small Counterexamples

In this section we look at partition relations of the form $\alpha \not\to (\alpha, m)^2$ for limit ordinals α and $m < \omega$.

In the previous section, we noted that E. Specker proved that $\omega^n \not\to (\omega^n, 3)^2$. In the 1970's, Galvin used *pinning*, defined below, to exploit the counterexample $\omega^3 \not\to (\omega^3, 3)^2$ to the full.

9.5 Definition. Suppose α and β are ordinals. A mapping $\pi : \alpha \to \beta$ is a *pinning map of α to β* if $\operatorname{ot} X = \alpha$ implies $\operatorname{ot} \pi``X = \beta$ for all $X \subseteq \alpha$. We say α *can be pinned to* β, in symbols, $\alpha \to \beta$, if there is a pinning map of α to β.

9.6 Theorem (Galvin [22]). *For all countable ordinals $\beta \geq 3$, if β is not AI and $\alpha = \omega^\beta$, then $\alpha \not\to (\alpha, 3)^2$.*

The first countable ordinal not covered by the Specker and Galvin results mentioned so far is ω^ω. Chang showed that $\omega^\omega \to (\omega^\omega, 3)^2$ and Milner modified his proof to work for all $m < \omega$.

9.7 Theorem (Chang [5]; Milner; see also [39], [66]). *For all $m < \omega$,*

$$\omega^\omega \to (\omega^\omega, m)^2.$$

Chang's original manuscript was about 90 pages long, and he received \$250 from Erdős for this proof, one of the largest sums Erdős had paid to that time. Paul Erdős continued to focus attention on partition relations of the form $\alpha \to (\alpha, m)^2$ through offering money. In 1985, he [11] offered \$1000 for a complete characterization of those countable α for which $\alpha \to (\alpha, 3)^2$.

9.8 Definition. Any ordinal α can be uniquely written as the sum of AI ordinals, $\alpha = \alpha_0 + \cdots + \alpha_k$ with $\alpha_0 \geq \cdots \geq \alpha_k$. This sum is called the *additive normal form* (ANF) of α, and in this case, we say *the ANF of α has $k+1$ summands.* The summand α_k is called the *final summand.* The *initial part of the ANF of α* is $\alpha_0 + \cdots + \alpha_{k-1}$ if $k > 0$ and, for notational convenience, is 0 if α is AI.

An AI ordinal α is *multiplicatively indecomposable (MI)* if it is cannot be written as a product $\gamma \cdot \delta$ where γ, δ are AI and $\alpha > \gamma \geq \delta$. Any AI ordinal α can be written uniquely as a product of MI ordinals $\alpha = \alpha_0 \cdot \ldots \cdot \alpha_k$ with $\alpha_0 \geq \cdots \geq \alpha_k$. This product is called the *multiplicative normal form* (MNF) of α, and in this case, we say *the MNF of α has $k+1$ factors.* The factor $\widehat{\alpha} := \alpha_k$ is called the *final factor.* The *initial part of the MNF of α* is $\overline{\alpha} := \alpha_0 + \cdots + \alpha_{k-1}$ if $k > 0$ and, for notational convenience, is $\overline{\alpha} := 1$ if α is MI.

Note that if $\alpha = \omega^\beta$, then α is MI exactly when β is AI. Thus Galvin's result (Theorem 9.6) may be rephrased to say that for all countable ordinals $\alpha > \omega^2$, if α is not MI, then $\alpha \not\to (\alpha, 3)^2$. In the 1990's, Carl Darby [7] and Rene Schipperus [54], [52] working independently, came up with new families of counterexamples for MI ordinals α. Larson [40] built on their work to improve one of the results obtained by both of them.

9.9 Theorem.

1. *(Darby) If $\beta = \omega^{\alpha+1}$ and $m \to (4)^3_{2^{32}}$, then $\omega^{\omega^\beta} \not\to (\omega^{\omega^\beta}, m)^2$.*

2. *(Darby; Schipperus; Larson) If* $\beta \geq \gamma \geq 1$*, then* $\omega^{\omega^{\beta+\gamma}} \not\to (\omega^{\omega^{\beta+\gamma}}, 5)^2$.

3. *(Darby; Schipperus) If* $\beta \geq \gamma \geq \delta \geq 1$*, then* $\omega^{\omega^{\beta+\gamma+\delta}} \not\to (\omega^{\omega^{\beta+\gamma+\delta}}, 4)^2$.

4. *(Schipperus) If* $\beta \geq \gamma \geq \delta \geq \varepsilon \geq 1$*, then* $\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}} \not\to (\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}}, 3)^2$.

We plan to sketch a proof that there is some finite k so that $\omega^{\omega^2} \not\to (\omega^{\omega^2}, k)^2$, using the basic approach developed by Darby and some of his construction lemmas. Surprisingly, the partition counterexamples developed by Darby and Schipperus were the same, even if their approaches to uniformization were at least cosmetically different.

Rather than working directly with the ordinals, we use collections of finite increasing sequences from ω under the lexicographic ordering. Since our sequences are increasing, we will identify them with the set of their elements.

We write $\mathbf{s}^\frown\mathbf{t}$ for the concatenation of the two sequences under the assumption that the last element of $\mathbf{s}$ is smaller than the first element of $\mathbf{t}$, in symbols $\mathbf{s} < \mathbf{t}$.

We extend the notion of concatenation from individual sequences to sets of sequences by setting

$$S^\frown T := \{\, \mathbf{s}^\frown\mathbf{t} \mid \mathbf{s} \in S \wedge \mathbf{t} \in T \wedge \mathbf{s} < \mathbf{t} \,\}.$$

9.10 Definition. Define sets G_α for $\alpha = \omega^\ell$ by recursion on $1 \leq \ell < \omega$.

$$G_\omega := \{\, \langle m \rangle^\frown \langle k_1, k_2, \ldots, k_m \rangle \mid m < k_1 < k_2 < \cdots < k_m < \omega \,\}$$

$$G_{\omega^{k+1}} := \bigcup \left\{ \{\langle m \rangle\}^\frown \overbrace{G_{\omega^k}{}^\frown \cdots {}^\frown G_{\omega^k}}^{m \text{ copies}} \mid m < \omega \right\}$$

Given a collection of sequences S and a particular sequence t, write $S(t) := \{\, s \in S \mid t \sqsubseteq s \,\}$ for the set of extensions of t in S.

9.11 Lemma. *For* $1 \leq \ell, m, p < \omega$, $\operatorname{ot} G_{\omega^\ell}(\langle m \rangle) = \left(\omega^{\omega^{\ell-1}}\right)^m$, $\operatorname{ot} G_{\omega^\ell} = \omega^{\omega^\ell}$*, and*

$$\operatorname{ot}\left(\overbrace{G_{\omega^\ell}{}^\frown \cdots {}^\frown G_{\omega^\ell}}^{p \text{ copies}}\right) = \left(\omega^{\omega^\ell}\right)^p.$$

Proof. First observe that $\operatorname{ot} G_\omega(\langle m \rangle) = \omega^m$ for all $1 \leq m < \omega$ and $\operatorname{ot} G_\omega = \omega^\omega$. Next notice that for subsets S and $T \subseteq [\omega]^{<\omega}$ which have indecomposable order types and which have arbitrarily large first elements, the order type of the concatenation $S^\frown T$ is the product of the order types $(\operatorname{ot} T) \cdot (\operatorname{ot} S)$. Then use induction on ℓ, m, and p. ⊣

9.12 Remark. Darby [7, Definition 2.8] defines G_α for all $\alpha < \omega_1$ so that $\operatorname{ot} G_\alpha = \omega^\alpha$ using a nice ladder system to assign to each limit ordinal an increasing cofinal sequence of type ω. In particular, for $\alpha = \overline{\alpha} \cdot \omega$ where $\overline{\alpha}$ is an AI ordinal, the cofinal sequence is $\alpha_m = \overline{\alpha} \cdot m$.

Our main interest is in G_α for α AI. We defined G_{ω^k} for $k < \omega$ in Definition 9.10. If $\alpha = \overline{\alpha} \cdot \omega$ where $\overline{\alpha}$ is an AI ordinal, then

$$G_\alpha = \bigcup \left\{ \{\langle m \rangle\}^\frown \overbrace{G_{\overline{\alpha}}{}^\frown \cdots {}^\frown G_{\overline{\alpha}}}^{m \text{ copies}} \mid m < \omega \right\}.$$

If $\alpha \geq \omega^\omega$ is an AI ordinal not of the form $\alpha = \overline{\alpha} \cdot \omega$, then the cofinal sequence is a strictly increasing sequence $\langle \alpha_m : m < \omega \rangle$ of AI ordinals and G_α is the union of $\{ \langle m \rangle \}^\frown G_{\alpha_m}$.

Recall we write $\mathbf{s} \sqsubseteq \mathbf{t}$ to indicate that $\mathbf{s}$ is an *initial segment* of $\mathbf{t}$, and $\mathbf{s} \sqsubset \mathbf{t}$ to indicate it is a *proper initial segment.*

9.13 Definition. For any collection of increasing sequences $S \subseteq [\omega]^{<\omega}$, let S^* denote the collection of initial segments of elements of S. For any $\mathbf{s} \in S^*$, let $S(\mathbf{s}) := \{ \mathbf{t} \in S \mid \mathbf{s} \sqsubseteq \mathbf{t} \}$ be the set of all extensions of $\mathbf{s}$ that are in S.

9.14 Definition (See Definition 3.1 of [7]). Suppose $\omega < \alpha = \overline{\alpha} \cdot \widehat{\alpha} < \omega_1$ is AI but not MI with initial part $\overline{\alpha}$ and final factor $\widehat{\alpha}$. Call a non-empty sequence $\mathbf{p} \in G^*_\alpha$ a *level prefix of* G_α if $\operatorname{ot} G_\alpha(\mathbf{p}) = \omega^\gamma$ where the final summand in the ANF of γ is $\overline{\alpha}$.

The next lemma is of particular interest when $\mathbf{s}$ is a level prefix.

9.15 Lemma (See Lemma 2.9 of [7]). *Suppose $\gamma \leq \alpha < \omega_1$ where the ANF of γ is $\gamma = \gamma_0 + \gamma_1 + \cdots + \gamma_k$ for $k > 0$. Further suppose that $\mathbf{s} \in G^*_\alpha \setminus \{\emptyset\}$. If $\operatorname{ot} G_\alpha(\mathbf{s}) = \omega^\gamma$, then $G_\alpha(\mathbf{s}) = \{\mathbf{s}\}^\frown G_{\gamma_k}{}^\frown \ldots {}^\frown G_{\gamma_0}$.*

Proof. We only prove this in the special case where $\alpha = \overline{\alpha} \cdot \omega$ and $\gamma = \overline{\alpha} \cdot n$. In this case, $\mathbf{s}$ has an extension in $G_\alpha(\langle m \rangle) = \{\langle m \rangle\}^\frown G_{\overline{\alpha}} \ldots {}^\frown G_{\overline{\alpha}}$ for $m = \min \mathbf{s}$ by Definition 9.10 or Remark 9.12. Let $\mathbf{t} \sqsubseteq \mathbf{s}$ be the longest initial segment of $\mathbf{s}$ for which $G_\alpha(\mathbf{t})$ is the concatenation of $\{\mathbf{t}\}$ with some finite number of copies of $G_{\overline{\alpha}}$. There must be such a $\mathbf{t}$ since $\langle m \rangle$ has this property. If $\mathbf{s} = \mathbf{t}$, then we are done. So assume by way of contradiction that $\mathbf{u} = \mathbf{s} \setminus \mathbf{t} \neq \emptyset$. By the maximality of $\mathbf{t}$, it follows that $\mathbf{u} \in G^*_{\overline{\alpha}} \setminus G_{\overline{\alpha}}$. Since $\mathbf{u} \neq \emptyset$, $G_{\overline{\alpha}}(\mathbf{u})$ has order type δ for some $\delta < \omega^{\overline{\alpha}}$ with $\delta > 1$. Let r be the number of copies of $G_{\overline{\alpha}}$ in the decomposition of $G_\alpha(\mathbf{t})$. If $r = 1$, then $G_\alpha(\mathbf{s}) = \{\mathbf{t}\}^\frown G_{\overline{\alpha}}(\mathbf{u})$ has order type $\delta < \omega^{\overline{\alpha}}$. If $r > 1$, then $G_\alpha(\mathbf{s})$ is the concatenation of $\{\mathbf{t}\}^\frown G_{\overline{\alpha}}(\mathbf{u})$ with $r-1$ copies of $G_{\overline{\alpha}}$, so has order type $\omega^{\overline{\alpha} \cdot (r-1)} \cdot \delta$, by the argument of Lemma 9.11. In both cases, since $\delta \neq 1$ and $\delta \neq \omega^{\overline{\alpha}}$, we have a contradiction to the assumption that $\operatorname{ot} G_\alpha(\mathbf{s}) = \omega^{\overline{\alpha} \cdot n}$. $\dashv$

9.16 Definition (See Definition 3.1 of [7]). Suppose the MNF of $\alpha < \omega_1$ has at least four factors. Call $\mathbf{t} \in G^*_\alpha$ a *sublevel prefix of* G_α if there are a level prefix $\mathbf{p}$ for G_α and a level prefix $\mathbf{q}$ for $G_{\overline{\alpha}}$ so that $\mathbf{t} = \mathbf{p}^\frown\mathbf{q}$. Call $\mathbf{u} \in G^*_\alpha$ a *sub-sublevel prefix of* G_α if there are a sublevel prefix $\mathbf{t}$ for G_α and a level prefix $\mathbf{r}$ for $G_{\overline{\overline{\alpha}}}$ so that $\mathbf{u} = \mathbf{t}^\frown\mathbf{r}$.

If we look at a pair $s \leq_{\text{lex}} t$ from G_α, if s and t are disjoint as sets, then they partition one another into convex segments. That is, s and t can be expressed as concatenations, $s = s_0{}^\frown s_1{}^\frown \dots {}^\frown s_{n-1}({}^\frown s_n)$ and $t = t_0{}^\frown t_1{}^\frown \dots {}^\frown t_{n-1}$ where $s_0 < t_0 < s_1 < t_1 < \cdots < s_{n-1} < t_{n-1} (< s_n)$.

The next definition uses Definition 9.16 to identify four types of segments used in the proofs of the negative partition relations (2)-(4) of Theorem 9.9.

9.17 Definition. Suppose the MNF of $\alpha < \omega_1$ has at least four factors. Further suppose that $\mathbf{s} \in G_\alpha$ has been decomposed into a convex partition $\mathbf{s} = \mathbf{s}_0{}^\frown\mathbf{s}_1{}^\frown \dots {}^\frown\mathbf{s}_n$ where $\mathbf{s}_0 < \mathbf{s}_1 < \cdots < \mathbf{s}_n$.

1. Call $\mathbf{s}_i$ a $\square$-*segment of* $\mathbf{s}$ if $i = 0$ or $i = n$ or there are a level prefix $\mathbf{t}$ of G_α and $\mathbf{a} \in G_{\overline{\alpha}}$ so that $\mathbf{s}_0{}^\frown \dots {}^\frown\mathbf{s}_{i-1} \sqsubset \mathbf{t} \sqsubset \mathbf{s}_0{}^\frown \dots {}^\frown\mathbf{s}_{i-1}{}^\frown\mathbf{s}_i \sqsubseteq \mathbf{t}^\frown\mathbf{a}$.
2. Call $\mathbf{s}_i$ a $\triangle$-*segment of* $\mathbf{s}$ if it is not a $\square$-segment of $\mathbf{s}$ and there are a sublevel prefix $\mathbf{u}$ of G_α and $\mathbf{b} \in G_{\overline{\overline{\alpha}}}$ so that $\mathbf{s}_0{}^\frown \dots {}^\frown\mathbf{s}_{i-1} \sqsubset \mathbf{u} \sqsubset \mathbf{s}_0{}^\frown \dots {}^\frown\mathbf{s}_{i-1}{}^\frown\mathbf{s}_i \sqsubseteq \mathbf{u}^\frown\mathbf{b}$.
3. Call $\mathbf{s}_i$ a $\blacksquare$-*segment of* $\mathbf{s}$ if it is not a $\square$ or $\triangle$-segment of $\mathbf{s}$ and there are a sub-sublevel prefix $\mathbf{u}$ of G_α and $\mathbf{c} \in G_{\overline{\overline{\overline{\alpha}}}}$ so that $\mathbf{s}_0{}^\frown \dots {}^\frown\mathbf{s}_{i-1} \sqsubset \mathbf{v} \sqsubset \mathbf{s}_0{}^\frown \dots {}^\frown\mathbf{s}_{i-1}{}^\frown\mathbf{s}_i \sqsubset \mathbf{v}^\frown\mathbf{c}$.
4. Call $\mathbf{s}_i$ a $\bullet$-*segment of* $\mathbf{s}$ there are a sub-sublevel prefix $\mathbf{u}$ of G_α and $\mathbf{c} \in G_{\overline{\overline{\overline{\alpha}}}}$ so that $\mathbf{v} \sqsubset \mathbf{s}_0{}^\frown \dots {}^\frown\mathbf{s}_{i-1}$ and $\mathbf{s}_0{}^\frown \dots {}^\frown\mathbf{s}_{i-1}{}^\frown\mathbf{s}_i \sqsubseteq \mathbf{v}^\frown\mathbf{c}$.

For simplicity, we include an example for which only $\square$-segments are needed to illustrate the technique. We have chosen to give an example that is easy to discuss rather than an optimal one.

9.18 Proposition. *The following partition relation holds:* $\omega^{\omega^2} \not\to (\omega^{\omega^2}, 6)^2$.

The remainder of this section is devoted to the proof of Proposition 9.18. We define a graph Γ on $G = G_{\omega_2}$ below. Then in Lemma 2, we show it has no 1-homogeneous set of size 6. After considerably more work, in Lemma 9.31, we show it has no 0-homogeneous subset of order type ω^{ω^2}. These two lemmas complete the proof.

9.19 Definition. Let $G = G_{\omega^2}$. Call a coordinate x of $\mathbf{x} \in G$ a *box coordinate* if it is either the minimum or the maximum of $\mathbf{x}$ or if $x = \min \mathbf{x} - \mathbf{p}$ for some level prefix $\mathbf{p} \sqsubseteq \mathbf{x}$. Define a graph $\Gamma : [G]^2 \to 2$ by $\Gamma(\mathbf{x}, \mathbf{y}) = 1$ if and only if there are convex partitions

$$\mathbf{x} = X_0{}^\frown X_1{}^\frown X_2{}^\frown X_3{}^\frown X_4 \text{ and } \mathbf{y} = Y_0{}^\frown Y_1{}^\frown Y_2{}^\frown Y_3$$

with $X_0<Y_0<X_1<Y_1<X_2<Y_2<X_3<Y_3<X_4$ so that all of X_0, X_2, X_4 are $\Box$-segments of $\mathbf{x}$, Y_0, Y_3 are $\Box$-segments of $\mathbf{y}$, and none of X_1, X_3, Y_1, Y_2 have box coordinates of $\mathbf{x}$, $\mathbf{y}$, respectively.

For notational convenience, let $\gamma^-(\mathbf{x},\mathbf{y}) = \max Y_1$, $\gamma^+(\mathbf{x},\mathbf{y}) = \min Y_2$, $\delta^-(\mathbf{x},\mathbf{y})$ be the largest box coordinate of Y_0, and $\delta^+(\mathbf{x},\mathbf{y})$ be the smallest box coordinate of Y_3. The graphical display below shows how the two sequences are interlaced and which have box coordinates if $\Gamma(\mathbf{x},\mathbf{y}) = 1$.

$$\begin{array}{ccccccccc} \boxed{X_0} & & X_1 & & \boxed{X_2} & & X_3 & & \boxed{X_4} \\ & \boxed{Y_0} & & Y_1 & & Y_2 & & \boxed{Y_3} & \end{array}$$

9.20 Lemma. *The graph Γ has no 1-homogeneous set of size six.*

Proof. The proof starts with a series of claims which delineate basic properties of the partition.

Claim A. *Suppose $\mathbf{x} < \mathbf{y}$, $\Gamma(\mathbf{x},\mathbf{y}) = 1$.*

1. *There is a box coordinate $x \in \mathbf{x}$ with $\min \mathbf{y} < x < \max \mathbf{y}$.*
2. *For any box coordinate $x \in \mathbf{x}$ with $\min \mathbf{y} < x < \max \mathbf{y}$, the inequalities $\gamma^-(\mathbf{x},\mathbf{y}) < x < \gamma^+(\mathbf{x},\mathbf{y})$ hold.*
3. *There is no sequence $x < y < x' \in \mathbf{x}$ where $\min \mathbf{y} < x \in \mathbf{x}$, $x' < \max \mathbf{y}$ and y is a box coordinate of $\mathbf{y}$.*

Proof. Use the diagram above to verify these basic properties. ⊣

Claim B. *Suppose $\{\mathbf{x},\mathbf{y},\mathbf{z}\}_< \subseteq G$ is 1-homogeneous for Γ. If $\Box x \in \mathbf{x}$, $\Box y \in \mathbf{y}$, and $\Box z \in \mathbf{z}$ are box coordinates and $\min \mathbf{z} < \Box x, \Box y < \max \mathbf{z}$, then either $\Box x, \Box y < \Box z$ or $\Box z < \Box x, \Box y$.*

Proof. Suppose the hypothesis holds but the conclusion fails. Then either (a) $\Box x < \Box z < \Box y$ or (b) $\Box y < \Box z < \Box x$. Note that $\min \mathbf{y} < \min \mathbf{z} < \Box x$ and $\Box x < \max \mathbf{z} < \max \mathbf{y}$, since $\mathbf{y} < \mathbf{z}$. By Claim A(2), $\gamma^-(\mathbf{x},\mathbf{y}) < \Box x < \gamma^+(\mathbf{x},\mathbf{y})$. Use the definition of Γ to find $x^-, x^+ \in \mathbf{x}$ such that $\delta^-(\mathbf{x},\mathbf{y}) < x^- < \gamma^-(\mathbf{x},\mathbf{y})$ and $\gamma^+(\mathbf{x},\mathbf{y}) < x^+ < \delta^+(\mathbf{x},\mathbf{y})$. If (a) holds, then either $\Box x < \Box z < x^+$ or $\gamma^+(\mathbf{x},\mathbf{y}) < \Box z < \Box y$ is a sequence that contradicts Claim A(3). If (b) holds, then either $\Box y < \Box z < \gamma^-(\mathbf{x},\mathbf{y})$ or $x^- < \Box z < \Box x$ is a sequence that contradicts Claim A(3). Thus the above claim follows. ⊣

Claim C. *Suppose $\{\mathbf{x},\mathbf{y},\mathbf{z}\}_< \subseteq G$ is 1-homogeneous for Γ. If $\Box x \in \mathbf{x}$, $\Box y \in \mathbf{y}$ are box coordinates with $\min z < \Box x, \Box y < \max z$, then some coordinate z of $\mathbf{z}$ lies between $\Box x$ and $\Box y$.*

Proof. For the first case, suppose $\Box x < \Box y$. In this case, let $z = \gamma^+(\mathbf{x}, \mathbf{z})$. Then $z \in \mathbf{z}$ and by Claim A, $\Box x < z$. By definition of Γ, there is some $x' \in \mathbf{x}$ with $z < x' < \max \mathbf{z}$. Since $\mathbf{y} < \mathbf{z}$, it follows that $x' < \max \mathbf{y}$, so $x' < \delta^+(\mathbf{x}, \mathbf{y}) \leq \Box y$. By transitivity, $\Box x < z < \Box y$. The second case for $\Box y < \Box x$ is left to the reader with the hint that $z = \gamma^-(\mathbf{x}, \mathbf{z})$ works. ⊣

Now prove the lemma from the claims. Assume by way of contradiction that $U = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}_< \subseteq G$ is 1-homogeneous for Γ. Use Claim A to choose box coordinates $\varepsilon_0 \in \mathbf{a}$, $\varepsilon_1 \in \mathbf{b}$, $\varepsilon_2 \in \mathbf{c}$, $\varepsilon_3 \in \mathbf{d}$, $\varepsilon_4 \in \mathbf{e}$, so that $\min \mathbf{f} < \varepsilon_i < \max \mathbf{f}$. Let $ijk\ell$ be a permutation of 0123 so that $\varepsilon_i < \varepsilon_j < \varepsilon_k < \varepsilon_\ell$. Use Claim C to choose coordinates $e', e'' \in \mathbf{e}$ and $f' \in \mathbf{f}$ with $\varepsilon_i < e' < \varepsilon_j < f' < \varepsilon_k < e'' < \varepsilon_\ell$. By Claim B, either (a) $\varepsilon_4 < \varepsilon_i$ or (b) $\varepsilon_\ell < \varepsilon_4$. Choose coordinate $f'' \in \mathbf{f}$ between ε_4 and the appropriate one of ε_i and ε_ℓ.

Let $\mathbf{x}, \mathbf{y} \in U$ be such that $\varepsilon_i \in \mathbf{x}$ and $\varepsilon_\ell \in \mathbf{y}$. By Claim A, $\delta^-(\mathbf{x}, \mathbf{f}) < \gamma^-(\mathbf{x}, \mathbf{f}) < \varepsilon_i$ and $\varepsilon_\ell < \gamma^+(\mathbf{y}, \mathbf{f}) < \delta^+(\mathbf{y}, \mathbf{f})$.

Let $\mathbf{e} = E_0{}^\frown E_1{}^\frown E_2{}^\frown E_3{}^\frown E_4$, $\mathbf{f} = F_0{}^\frown F_1{}^\frown F_2{}^\frown F_3$ be the partition that witnesses $\Gamma(\mathbf{e}, \mathbf{f}) = 1$. Note that $\varepsilon_4 \in E_2$.

If (a) holds, then $\delta^+(\mathbf{y}, \mathbf{f}) \in F_3$, and

$$\varepsilon_4 < f'' < e' < f' < e'' < \delta^+(\mathbf{y}, \mathbf{f}).$$

However, this inequality contradicts the definition of g, since there are only two blocks between E_2 and F_3. If (b) holds, then $\delta^-(\mathbf{x}, \mathbf{f}) \in F_0$, and

$$\delta^-(\mathbf{x}, \mathbf{f}) < e' < f' < e'' < f'' < \varepsilon_4.$$

This inequality also contradicts the definition of g, since there are only two blocks between F_0 and E_2. In either case we have reached the contradiction required to prove the lemma. ⊣

Now we turn to the task of showing that every subset $X \subseteq G$ of order type ω^{ω^2} includes a pair $\{\mathbf{x}, \mathbf{y}\}_< \subseteq X$ so that $\Gamma(\mathbf{x}, \mathbf{y}) = 1$. The first challenge is to guarantee that when we build a segment of one of $\mathbf{x}$ and $\mathbf{y}$, we will be able to extend it starting above the segment of the other that we will have constructed in the meanwhile. To that end, we introduce *β-prefixes* and *maximal β-prefixes.*

9.21 Definition. Suppose $\alpha < \omega_1$. Call a sequence $\mathbf{s} \in G^*_\alpha$ a *β-prefix* of $W \subseteq G_\alpha$ if $\operatorname{ot} W(\mathbf{s}) = \beta$, and a *maximal β-prefix* if no proper extension is a β-prefix.

9.22 Lemma (Galvin; see Lemma 4.5 in [7]). *Suppose $\mathbf{s} \in G^*_\alpha$ and β is AI. If $W \subseteq G_\alpha$ has $\operatorname{ot} W(\mathbf{s}) \geq \beta$, then there is an extension $\mathbf{t} \sqsupseteq \mathbf{s}$ so that $\mathbf{t}$ is a maximal β-prefix for W.*

The proof of the above lemma depends on the fact that the sequences in G_α are well-founded under extension. We use the next lemma for sequences $\mathbf{r}$ which are either maximal ω^2-prefixes or maximal ω^3-prefixes.

9.23 Lemma. *Suppose $\delta < \beta \leq \omega^\alpha$ for AI δ and β. Further suppose $W \subseteq G_\alpha$ and $\mathbf{r}$ is a maximal β-prefix for W. Then $\mathbf{r}$ has infinitely many one point extensions $r^\frown\langle p\rangle \in W^*$ with $\operatorname{ot} W(r^\frown\langle p\rangle) \geq \delta$. Also, for any sequence $\mathbf{s}$, there is a sequence $\mathbf{t}$ so that $\mathbf{s} < \mathbf{t}$, $\mathbf{r}^\frown\mathbf{t} \in W^*$, and $\mathbf{r}^\frown\mathbf{t}$ is a maximal δ-prefix for W.*

Proof. Since $\mathbf{r}$ is a maximal β-prefix for W, $\operatorname{ot} W(\mathbf{r}^\frown\langle\, p\,\rangle) < \beta$ for all $p < \omega$. Consequently, since β is AI, it follows that $\sum_{q<p<\omega} \operatorname{ot} W(\mathbf{r}^\frown\langle\, p\,\rangle) = \beta$ for all $q < \omega$. Since $\sum_{q<p<\omega} \gamma_p \leq \delta$ if each $\gamma_p < \delta$, it follows that for infinitely many $p < \omega$, $W(\mathbf{r}^\frown\langle\, p\,\rangle)$ has order type $\geq \delta$. Thus given $\mathbf{s}$, there is $p > \max \mathbf{s}$ with $\operatorname{ot} W(\mathbf{r}^\frown\langle\, p\,\rangle) \geq \delta$. In particular, $W(\mathbf{r}^\frown\langle\, p\,\rangle) \neq \emptyset$. To complete the proof, apply Lemma 9.22 to get $\mathbf{t} \sqsupseteq \langle\, p\,\rangle$ so that $\mathbf{r}^\frown\mathbf{t}$ is a maximal δ-prefix. ⊣

In our construction of $\mathbf{x}$, $\mathbf{y}$, we must be able to iterate the process of extending to a level prefix. To that end, we introduce the notion of *levels.*

9.24 Definition (See Def. 5.2 of [7]). Suppose α is AI but not MI and $\mathbf{q}$ is a level prefix of G_α. The *level of W prefixed by* $\mathbf{q}$ is the set

$$L(W,\mathbf{q}) := \{\, \mathbf{a} \in G_{\overline{\alpha}} \mid W(\mathbf{q}^\frown\mathbf{a}) \neq \emptyset \,\}\,.$$

A non-empty sequence $\mathbf{s} \in G^*_\alpha \setminus G_\alpha$ *ends in the level of W prefixed by* $\mathbf{q}$ if there is some $\mathbf{a} \in L(W,\mathbf{q})$ so that $\mathbf{q} \sqsubseteq \mathbf{s} \sqsubset \mathbf{q}^\frown\mathbf{a}$.

Next we state without proof a series of lemmas from Darby [7] that lead up to Lemma 9.29. The interested reader can fill in the proofs for the case where $\alpha = \omega^\ell < \omega^\omega$.

9.25 Lemma (See Def. 4.6 and Lemma 4.7 of [7]). *Suppose $\delta \leq \gamma \leq \alpha < \omega_1$, where δ, γ are AI and $\gamma \cdot \delta \leq \alpha$. If $\mathbf{s} \in G^*_\alpha$ is a maximal $\gamma \cdot \delta$-prefix for $W \subseteq G_\alpha$, then the following set has order type δ:*

$$W(\beta,\mathbf{s}) := \{\, \mathbf{p} \in G^*_\alpha \mid \mathbf{s} \sqsubseteq \mathbf{p} \text{ and } \mathbf{p} \text{ is a maximal } \gamma\text{-prefix for } W \,\}\,.$$

9.26 Lemma (See Lemma 5.5 of [7]). *Suppose $\alpha < \omega_1$ is AI but not MI, $\mathbf{q}$ is a level prefix of G_α and $W \subseteq G_\alpha$. If $\mathbf{s}$ ends in level $L(W,\mathbf{q})$ and $\operatorname{ot} W(\mathbf{s}) \geq \omega^{\overline{\alpha}\cdot n}$, then for any $\gamma < \overline{\alpha}$, there is an $\mathbf{a} \in L(W,\mathbf{q})$ so that $\mathbf{s} \sqsubset \mathbf{q}^\frown\mathbf{a}$ and $\operatorname{ot} W(\mathbf{q}^\frown\mathbf{a}) \geq \omega^{\overline{\alpha}(n-1)+\gamma}$.*

9.27 Lemma (See Lemma 5.6 of [7]). *Suppose $\alpha < \omega_1$ is AI but not MI, $W \subseteq G_\alpha$ and every level of W has order type $\leq \omega^\delta$. If $\mathbf{s} \in G^*_\alpha$ and $\operatorname{ot} G_\alpha(\mathbf{s}) = \omega^{\overline{\alpha}\cdot\beta}$, then $\operatorname{ot} W(\mathbf{s}) \leq \omega^{\delta\cdot\beta}$.*

9.28 Lemma (See Lemma 5.7 of [7]). *Suppose $\alpha < \omega_1$ is AI but not MI, $W \subseteq G_\alpha(\langle m \rangle)$ and $\operatorname{ot} W > \omega^\gamma$. Then for any δ so that $\delta \cdot m < \gamma$, there is a level of W of order type $> \omega^\delta$.*

The following lemma of Darby, mildly rephrased since the general definition of G_α has been omitted, is the key to constructing pairs 1-colored by any generalization of the graph Γ to a Γ_α defined for $\alpha = \overline{\alpha} \cdot \omega$, since it allows one to plan ahead: one takes a sufficiently large set, thins it to something tractable, dives into a large level to work within, knowing that on exit from the level, one will have a large enough set of extensions to continue according to plan.

9.29 Lemma (See Lemma 5.9 of [7]). *Suppose α is AI but not MI, $0 < m < \omega$ and $\operatorname{ot} G_\alpha(\langle m \rangle) = \omega^{\overline{\alpha} \cdot \beta}$. Further suppose $W \subseteq G_\alpha(\langle m \rangle)$ and $\operatorname{ot} W \geq \omega^{\overline{\alpha} \cdot n + \varepsilon}$ where $\varepsilon \leq \overline{\alpha}$ and $0 < n < \omega$, and assume δ is such that $\delta \cdot \beta < \varepsilon$. Then there is a set $U \subseteq W$ and a level prefix $\mathbf{q}$ so that $U = U(\mathbf{q})$, $\operatorname{ot} L(U, \mathbf{q}) > \omega^\delta$ and $\operatorname{ot} U(\mathbf{q}^\frown \mathbf{a}) \geq \omega^{\overline{\alpha} \cdot (n-1) + \varepsilon}$ for all $\mathbf{a} \in L(U, \mathbf{q})$.*

Here our focus is on ω^{ω^k} for finite k, that is, on $\alpha = \omega^k$. In this case, $G_\alpha(\langle m \rangle)$ has order type $\omega^{\omega^{k-1} \cdot m}$, so the β of the previous lemma is simply m. The following weaker version of the above lemma suffices for our purposes.

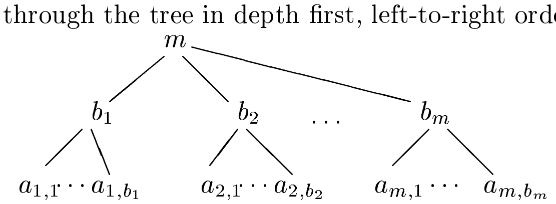
9.30 Lemma. *Suppose $\alpha = \overline{\alpha} \cdot \omega$ is AI but not MI, $0 < n \leq m < \omega$, and $W \subseteq G_\alpha(\langle m \rangle)$ has order type $\geq \omega^{\overline{\alpha} \cdot n}$. Further assume δ is such that $\delta \cdot m < \overline{\alpha}$. Then there is a set $U \subseteq W$ and a level prefix $\mathbf{q}$ so that $U = U(\mathbf{q})$, $\operatorname{ot} L(U, \mathbf{q}) > \omega^\delta$ and $\operatorname{ot} U(\mathbf{q}^\frown \mathbf{a}) \geq \omega^{\overline{\alpha} \cdot (n-1)}$ for all $\mathbf{a} \in L(U, \mathbf{q})$.*

9.31 Lemma. *Suppose $W \subseteq G_{\omega^2}$ has order type ω^{ω^2}. Then there is a pair $\mathbf{x}, \mathbf{y}$ from W so that $\Gamma(\mathbf{x}, \mathbf{y}) = 1$.*

Proof. We revisit the set G_{ω^2} to better understand how it is constructed by unraveling the recursive construction. A typical element σ is

$$\langle m \rangle^\frown \langle b_1 \rangle^\frown \langle a_1^1, \ldots, a_{b_1}^1 \rangle^\frown \langle b_2 \rangle^\frown \langle a_1^2, \ldots, a_{b_2}^2 \rangle \ldots {}^\frown \langle b_m \rangle^\frown \langle a_1^m, \ldots, a_{b_m}^m \rangle .$$

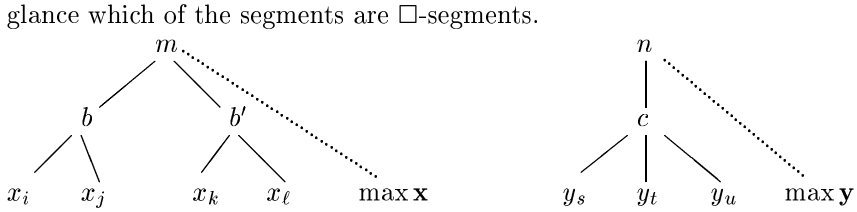
Notice that the initial element, m, tells how many levels there will be, and each level starts with a box coordinate, b_i, which determines the order type of the level, ω^{b_i}. To make the identification of the various types of elements visually immediate, we fold the sequence σ into a tree, with the initial element at the top, the box coordinates as immediate successors, and the remaining coordinates as terminal nodes. To rebuild the sequence from the tree, one walks through the tree in depth first, left-to-right order.



Use Lemmas 9.22, 9.23, and 9.30 to build $\mathbf{x} = X_0{}^\frown X_1{}^\frown X_2{}^\frown X_3{}^\frown X_4$ and $\mathbf{y} = Y_0{}^\frown Y_1{}^\frown Y_2{}^\frown Y_3$ one convex segment at a time so that

$$X_0 < Y_0 < X_1 < Y_1 < X_2 < Y_2 < X_3 < Y_3 < X_4$$

For notational convenience, we plan to let $i < j < k < \ell$ be such that $\max X_0 = x_i$, $\max X_1 = x_j$, $\max X_2 = x_k$, $\max X_3 = x_\ell$. Similarly, we plan to let $s < t < u$ be such that $\max Y_0 = y_s$, $\max Y_1 = y_t$, $\max Y_2 = y_u$. In addition it will be convenient to write b for the largest box coordinate of X_0, b' for the largest box coordinate of X_2, and $c = \delta^-(\mathbf{x}, \mathbf{y})$ for the largest box coordinate of Y_0. Here is a pair of subtrees of the trees we get by folding the sequences we build for $\mathbf{x}$ and $\mathbf{y}$, that include only the critical coordinates named above, together with $\max \mathbf{x}$, $\max \mathbf{y}$. These subtrees highlight the relationships between the critical coordinates, and allow one to see at a glance which of the segments are $\Box$-segments.



Observe that since G is the union of $G(\langle 0 \rangle), G(\langle 1 \rangle), G(\langle 2 \rangle), \ldots$, it follows that for $\beta < \omega^2$, there are infinitely many $m_\beta < \omega$ with $\operatorname{ot} W \cap G(\langle m_\beta \rangle) \geq \omega^\beta$. We start our construction by choosing m so that $U_0 := W \cap G(\langle m \rangle)$ has order type at least $\omega^{\omega \cdot 4}$.

Next we apply Lemma 9.30 to find a set $U_1 \subseteq U_0$ and a level prefix $\mathbf{p}$ so that $U_1 = U_1(\mathbf{p})$, $\operatorname{ot} L(U_1, \mathbf{p}) > \omega^5$, and $\operatorname{ot} U_1(\mathbf{p}{}^\frown\mathbf{a}) \geq \omega^{\omega \cdot 3}$ for all $\mathbf{a} \in L(U_1, \mathbf{p})$. Apply Lemma 9.22 to get $\mathbf{u}$, a maximal ω^4 prefix in $L(U_1, \mathbf{p})$. Then $b = \min \mathbf{u}$ is the box coordinate of our diagram. We set $X_0 = \mathbf{p}{}^\frown\mathbf{u}$ and note that $\max \mathbf{u} = x_i$ on our diagram.

Choose $n > x_i$ so that $V_0 := W \cap G(\langle n \rangle)$ has order type at least $\omega^{\omega \cdot 4}$. Continue as in the previous step. Use Lemma 9.30 to find a set $V_1 \subseteq V_0$ and a level prefix $\mathbf{q}$ so that $V_1 = V_1(\mathbf{q})$, $\operatorname{ot} L(V_1, \mathbf{q}) > \omega^7$, and $\operatorname{ot} V(\mathbf{q}{}^\frown\mathbf{a}) \geq \omega^{\omega \cdot 3}$ for all $\mathbf{a} \in L(V_1, \mathbf{q})$. Let $\mathbf{v}$ be a maximal ω^6 prefix in $L(V_1, \mathbf{q})$. Then $c = \min \mathbf{v}$ is the box coordinate of our diagram. We set $Y_0 = \mathbf{q}{}^\frown\mathbf{v}$ and note that $\max \mathbf{v} = y_s$ on our diagram.

By Lemma 9.23, there is a sequence X_1 with $Y_0 < X_1$ so that $\mathbf{u}{}^\frown X_1$ is a maximal ω^3 prefix in $L(U_1, \mathbf{p})$. Note that $X_0{}^\frown X_1$ is not a level prefix nor is any one point extension.

By Lemma 9.23, there is a sequence Y_1 with $X_1 < Y_1$ so that $\mathbf{v}{}^\frown Y_1$ is a maximal ω^5 prefix in $L(V_1, \mathbf{q})$.

By Lemma 9.23, the sequence $\mathbf{u}{}^\frown X_1$ has infinitely many one point extensions in $L(U_1, \mathbf{p})^*$. By choosing a suitable one point extension and then extending it into $L(U_1, \mathbf{p})$, we find $\mathbf{w}$ so that $Y_1 < \mathbf{w}$ and $\mathbf{u}{}^\frown X_1{}^\frown\mathbf{w} \in$

$L(U_1, \mathbf{p})$. By choice of U_1 and $\mathbf{p}$, we know $\operatorname{ot} U_1(\mathbf{p}^\frown(\mathbf{u}^\frown X_1{}^\frown \mathbf{w})) \geq \omega^{\omega\cdot 3}$. Use Lemma 9.30 to find $U_2 \subseteq U_1(\mathbf{p}^\frown(\mathbf{u}^\frown X_1{}^\frown \mathbf{w}))$ and a level prefix $\mathbf{p}'$ so that $U_2 = U_2(\mathbf{p}')$, $\operatorname{ot} L(U_2, \mathbf{p}') > \omega^5$, and $\operatorname{ot} U_2(\mathbf{p}'^\frown \mathbf{a}) \geq \omega^{\omega\cdot 3}$ for all $\mathbf{a} \in L(U_2, \mathbf{p}')$. Then $\mathbf{p}^\frown(\mathbf{u}^\frown X_1{}^\frown \mathbf{w}) \sqsubseteq \mathbf{p}'$. Apply Lemma 9.22 to get $\mathbf{u}'$, a maximal ω^4 prefix in $L(U_2, \mathbf{p}')$. Then $b' = \min \mathbf{u}'$ is another box coordinate in our diagram. Then $\mathbf{p}'^\frown \mathbf{u}'$ is not a level prefix of U_2, nor is any one point extension of it a level prefix. We set $X_2 = \mathbf{p}' \setminus (X_0{}^\frown X_1)$, and note that $\max X_2 = \max \mathbf{u}' = x_k$ on our diagram.

By Lemma 9.23, there is a sequence Y_2 with $X_2 < Y_2$ so that $\mathbf{v}^\frown Y_1{}^\frown Y_2$ is a maximal ω^4 prefix in $L(V_1, \mathbf{q})$.

By Lemma 9.23, there is a sequence X_3 with $Y_2 < X_3$ so that $\mathbf{u}'^\frown X_3$ is a maximal ω^3 prefix in $L(U_2, \mathbf{p}')$.

By Lemma 9.23, the sequence $\mathbf{v}^\frown Y_1{}^\frown Y_2$ has infinitely many one point extensions in $L(V_1, \mathbf{q})^*$. Hence by first choosing a suitable one point extension and then extending it into $L(V_1, \mathbf{q})$, and finally extending it into V_1, we can find Y_3 so that $X_3 < Y_3$ and $\mathbf{y} = Y_0{}^\frown Y_1{}^\frown Y_2{}^\frown Y_3 \in V_1 \subseteq W$.

By Lemma 9.23, the sequence $\mathbf{u}'^\frown X_3$ has infinitely many one point extensions in $L(U_2, \mathbf{p}')^*$. Hence by first choosing a suitable one point extension and then extending it into $L(U_2, \mathbf{p}')$, and finally extending it into U_2, we can find X_4 so that $Y_3 < X_4$ and $\mathbf{x} = X_0{}^\frown X_1{}^\frown X_2{}^\frown X_3{}^\frown X_4 \in U_2 \subseteq W$.

By construction, X_0, X_2, X_4 and Y_0, Y_3 are all $\square$-segments, while X_1, X_3 and Y_1, Y_2 have no box coordinates. Thus $\mathbf{x}, \mathbf{y}$ witnesses the fact that W is not a 0-homogeneous set for Γ. ⊣

Lemmas 9.20 and 9.31 show that Γ is a witness to $\omega^{\omega^2} \nrightarrow (\omega^{\omega^2}, 6)^2$. The coloring can easily be generalized to ω^{ω^α} where α is decomposable, since it was described using only box segments and segments without box coordinates. Hence the proof of Lemma 9.20 carries through for these generalizations. In the proof of Lemma 9.31, we have taken advantage of the fact that $\alpha = 2$ is a successor ordinal, but use of lemmas from Darby's paper allow one to modify the given construction suitably.

The proof of the previous lemma gives some evidence for the following remark.

9.32 Remark. We have the following heuristic for building pairs. Suppose σ is a list of specifications of convex segments detailing which have box, triangle, bar (or dot) coordinates and which do not. If the first two and last two segments are to be box segments, then for any ordinal α of sufficient decomposability for the description to make sense, there is a disjoint pair $\mathbf{x}, \mathbf{y} \in G_{\omega^\alpha}$ so that the sequence of convex segments they create fits the description.

For the actual construction, one needs to iterate the process of taking levels and look at the approach taken carefully.

10. A positive countable partition relation

The previous section focused on countable counterexamples. Here we survey positive ordinal partition relations of the form $\alpha \to (\alpha, m)^2$ for countable limit ordinals α and sketch the proof of one of them.

Carl Darby [7] and Rene Schipperus [54], [52] working independently, extended Chang's positive result for ω^ω and $m = 3$ to larger countable limit ordinals.

10.1 Theorem. (Chang for $\beta = 1$ (see Theorem 9.7); Darby for $\beta = 2$ [7]; Schipperus for $\beta \geq 2$ [54]) *If the additive normal form of $\beta < \omega_1$ has one or two summands, then $\omega^{\omega^\beta} \to (\omega^{\omega^\beta}, 3)^2$.*

Recall that Erdős [11] offered \$1000 for a complete characterization of the countable ordinals α for which $\alpha \to (\alpha, 3)^2$. It is not difficult to show that additively decomposable ordinals fail to satisfy this partition relation. Recall that additively indecomposable ordinals are powers of ω. Specker showed that finite powers of ω greater than ω^2 fail to satisfy it. Galvin showed (see Theorem 9.6) that additively decomposable powers of ω greater than ω^2 fail to satisfy it. Thus attention has been on indecomposable powers of ω, $\alpha = \omega^{\omega^\beta}$, that is, the countable ordinals that are multiplicatively indecomposable. Schipperus (see Theorem 9.9) showed that if the additive normal form of β has at least four summands, then $\alpha \not\to (\alpha, 3)^2$. Thus to complete the characterization of which countable ordinals α satisfy this partition relation it suffices to characterize it for ordinals of the form $\alpha = \omega^{\omega^\beta}$ where the additive normal form of β has exactly three summands. We list below the first open case.

10.2 Question. Does $\omega^{\omega^3} \to (\omega^{\omega^3}, 3)^2$?

In light of Theorem 9.9, Darby and Larson have completed the characterization of the set of $m < \omega$ for which $\omega^{\omega^2} \to (\omega^{\omega^2}, m)^2$ with the following result.

10.3 Theorem (Darby and Larson [8]). $\omega^{\omega^2} \to (\omega^{\omega^2}, 4)^2$.

We complete this subsection with a sketch of the Schipperus proof that $\omega^{\omega^\omega} \to (\omega^{\omega^\omega}, 3)^2$, using somewhat different notation than he used originally. The sketch will be divided into seven subsections:

1. representation of ω^{ω^ω} as a collection $\mathcal{T}(\omega)$ of finite trees;
2. analysis of *node labeled* trees;
3. description of a two-player game $\mathcal{G}(h, N)$ for h a 2-partition of $\mathcal{T}(\omega)$ into 2 colors and $N \subseteq \omega$ infinite;

4. uniformization of play of the game $\mathcal{G}(h, N)$ via constraint on the second player to a *conservative style* of play determined by an infinite set $H \subseteq N$ and a bounding function b;

5. construction of a three element 1-homogeneous set when the the first player has a winning strategy for all games in $\mathcal{G}(h, N)$ in which the second player makes conservative moves;

6. construction of an almost 0-homogeneous set of order type ω^{ω^ω} when the first player has no such strategy;

7. completion of the proof.

10.1. Representation

Recall that, by convention, we are identifying a finite set of natural numbers with the increasing sequence of its members. The trees we have in mind for our representation are subsets of $[\omega]^{<\omega}$ which are trees under the subset relation, and the subset relation is the same as the end-extension relation when the subsets are regarded as increasing sequences.

In the proof that the coloring Γ had no independent subset of order type ω^{ω^2}, we found it convenient to fold an element

$$\mathbf{x} = \langle\, m, n_1, a_1^1, \ldots, a_{n_1}^1, n_2, a_1^2, \ldots, a_{n_2}^2, \ldots, n_m, a_1^m, \ldots, a_{n_m}^m \,\rangle$$

of G_{ω^2} into a tree with root $\langle\, m \,\rangle$, immediate successors $\langle\, m, n_i \,\rangle$ and terminal nodes $\langle\, m, n_i, a_j^i \,\rangle$. Then we could walk through the tree, node by node, so that the maximum element of each node continually increased along the walk, just as the elements of $\mathbf{x}$ increase.

We already have representations of ω^{ω^β} from the previous section as sets of increasing sequences under the lexicographic ordering. The definition of those sets is recursive, so we fold these sets up into trees recursively. Specifically, the next definition uses the representations of G_{ω^β} detailed in Definition 9.10 and Remark 9.12.

10.4 Definition. Define by recursion on $\beta \leq \omega$ a sequence of *folding maps*, $F_\beta : G_{\omega^\beta} \to \mathcal{T}$:

1. For $\tau = \langle\, k \,\rangle \in G_{\omega^0} = G_1$, set $F_0(\tau) := \{\langle\, k \,\rangle\}$.
2. For $\tau = \langle\, m \,\rangle^\frown \sigma_1{}^\frown \sigma_2{}^\frown \ldots {}^\frown \sigma_m \in G_{\omega^{n+1}}$, set

$$F_{n+1}(\tau) := \{\langle\, m \,\rangle\} \cup \bigcup \{\{\langle\, m \,\rangle\}^\frown F_n(\sigma_i) : 1 \leq i \leq m\}\,.$$

3. For $\tau = \langle\, m \,\rangle^\frown \sigma \in G_{\omega^\omega}$, set $F_\omega(\tau) := \{\langle\, m \,\rangle\} \cup \{\langle\, m \,\rangle\}^\frown F_m(\sigma)$.

Let $\mathcal{T}(\beta)$ be the range of F_β.

Prove the following lemmas by induction on β.

10.5 Lemma. *For each $\beta \leq \omega$, the mapping F_β is one-to-one and $\tau = \bigcup F_\beta(\tau)$. Thus, $<_{\text{lex}}$ on G_{ω^β} induces an order $<$ on $\mathcal{T}(\beta)$.*

10.6 Lemma. *For all $\beta \leq \omega$ and all infinite $H \subseteq \omega$, the collection of sequences in $G_{\omega^\beta} \cap [H]^{<\omega}$ has order type ω^{ω^β}, and hence so does the collection of trees in $\mathcal{T}(\beta, H) := \mathcal{T}(\beta) \cap \mathcal{P}([H]^{<\omega})$.*

Let $\mathcal{T}$ be the collection of all finite trees $(T, \sqsubseteq)$ of increasing sequences with the property that if $s, t \in T$ and as sets, $s \subseteq t$, then as sequences, $s \sqsubseteq t$. Identify each $t \in T \in \mathcal{T}$ with the set of its elements. Then $\sqsubseteq$ and $\subseteq$ coincide, so this identification permits one to use set operations on the nodes of T.

10.7 Lemma. *For all $\beta < \omega_1$, for all $T \in \mathcal{T}(\beta)$, the following conditions are satisfied:*

1. *(transitivity) $s \sqsubset t \in T$ implies $s \in T$;*
2. *(closure under intersection) for all $s, t \in T$, $s \cap t$ is an initial segment of both s and t;*
3. *(rooted) $(T, \sqsubseteq)$ is a rooted tree with $\emptyset \notin T$.*
4. *(node ordering) for all $s \neq t$ in T, exactly one of the following holds:*
   - *(a) $s \sqsubset t$,*
   - *(b) $t \sqsubset s$,*
   - *(c) $s \leq_{\text{lex}} t$ and $s < t - (s \cap t)$,*
   - *(d) $t \leq_{\text{lex}} s$ and $t < s - (s \cap t)$;*

10.8 Definition. For all $\beta < \omega_1$, for all $T \in \mathcal{T}(\beta)$, order the nodes of T by $u < v$ if and only if $u \sqsubset v$ or $u <_{\text{lex}} v$.

10.9 Lemma. *For all $\beta < \omega_1$, for all non-empty initial segments S, T of trees in $\mathcal{T}(\beta)$, $\bigcup S \sqsubset \bigcup T$ if and only if $S \sqsubset T$.*

Proof. By Lemma 10.7, if $\emptyset \neq S \sqsubset T \sqsubseteq T' \in \mathcal{T}(\beta)$, then $\bigcup S \sqsubset \bigcup T$. For $\beta = 0$, the reverse implication is trivially true, and for $\beta > 0$, it is true by definition of the fold map and the induction hypothesis. ⊣

10.10 Definition. For all $\beta \leq \omega$, define $e_\beta : [\omega]^{<\omega} \to \{-1\} \cup (\beta + 2)$ by recursion:

$$\begin{aligned} e_\beta(\emptyset) &= \beta + 1; \\ e_\beta(\sigma ^\frown \langle m \rangle) &= \begin{cases} -1 & \text{if } e_\beta(\sigma) \leq 0, \\ e_\beta(\sigma) - 1 & \text{if } e_\beta(\sigma) > 0 \text{ successor}, \\ \max(\sigma) & \text{if } e_\beta(\sigma) = \omega \text{ limit}. \end{cases} \end{aligned}$$

We refer to $e_\beta(x)$ as the *ordinal of* x.

Use induction on β, the definition of F_β, and the previous lemma to prove the next lemma.

10.11 Lemma. *For all $\beta \leq \omega$, for all $T \in \mathcal{T}(\beta)$, for all $t \in T$, $e_\beta(t) \geq 0$, and if $e_\beta(t) > 0$, then t has a proper extension $u \in T$.*

The following consequence of the recursive nature of Definition 10.10 is useful in induction proofs.

10.12 Lemma. *For all $\beta \leq \omega$, for all $\langle m \rangle ^\frown \tau \in [\omega]^{<\omega}$, $e_\beta(\langle m \rangle) = \beta$ and if $\tau \neq \emptyset$ and $\gamma = e_\beta(\langle m, \max \tau \rangle) \geq 0$, then $e_\beta(\langle m \rangle ^\frown \tau) = e_\gamma(\tau)$.*

10.13 Definition. Suppose $T \in \mathcal{T}$. For all $t \in T$, let $\sharp(t, T)$ be the number of successors of t in T.

10.14 Lemma. *For all $\beta \leq \omega$, for all $T \in \mathcal{T}(\beta)$, for all $t \in T$,*

$$\sharp(t,T) = \begin{cases} 0, & \text{if } e_\beta(t) = 0, \\ 1, & \text{if } e_\beta(t) = \omega \text{ is a limit}, \\ \max t, & \text{if } e_\beta(t) \text{ is a successor}, \end{cases}$$

10.15 Lemma. *For all $\beta \leq \omega$, for all $T \subseteq [\omega]^{<\omega}$, $T \in \mathcal{T}(\beta)$ if and only if T satisfies the four conclusions of Lemma 10.7, and for all $t \in T$, $e_\beta(t) \geq 0$ and $\sharp(t, T)$ has the value specified in Lemma 10.14.*

Proof. By Lemmas 10.7, 10.11, and 10.14, if $(T, \sqsubseteq) \in \mathcal{T}(\beta)$, then it satisfies the given list of conditions.

To prove the other direction, work by induction on β to show that if $T \subseteq [\omega]^{<\omega}$ satisfies the given conditions for β, then $\bigcup T \in G_{\omega^\beta}$ and $T = F_\beta(\bigcup T) \in \mathcal{T}(\beta)$. ⊣

10.16 Definition. For $0 < \beta \leq \omega$ and $\emptyset \neq S \sqsubset T \in \mathcal{T}(\beta)$, the *critical node of* S, in symbols $\operatorname{crit}(S)$, is the largest $s \in S$ with $\sharp(s, S)$ smaller than the value predicted in Lemma 10.14. For notational convenience, let $\operatorname{crit}(\emptyset) = \emptyset$, and set $\operatorname{crit}(T) = \emptyset$ for $T \in \mathcal{T}(\beta)$.

The next lemma shows why the name was chosen.

10.17 Lemma. *For $0 \leq \beta \leq \omega$ and $S \sqsubseteq T \in \mathcal{T}(\beta)$, if $t := \min(T - S)$, then $t = \operatorname{crit}(S)^\frown \langle \max t \rangle$.*

Proof. Let $m < \omega$ be such that $\langle m \rangle \in T$. Then $\langle m \rangle$ is the least element of T. If $S = \emptyset$, then $t = \langle m \rangle = \operatorname{crit}(S)^\frown \langle \max t \rangle$ and the lemma follows. Otherwise, $\langle m \rangle$ must be in S, and because it is the root of T, $\langle m \rangle \sqsubset t := \min(T - S)$. Let $r = t - \{\max t\}$. Then $\langle m \rangle \sqsubseteq r \sqsubset t$, $\sharp(r, S) < \sharp(r, T)$, so

r is an element of S with $\sharp(r,S)$ smaller than the value specified in Lemma 10.14.

If $p \in T$ and $p <_{\text{lex}} t$, then $p \in S$, since $S \sqsubset T$ and $T = \min(T - S)$. Moreover, if $p <_{\text{lex}} t$ and $p \sqsubset q \in T$, then $q <_{\text{lex}} t$. Hence if $p <_{\text{lex}} t$, then $\sharp(p,S) = \sharp(p,T)$ takes on the value specified in Lemma 10.14. Thus $\text{crit}(S) \sqsubset t$, so $\text{crit}(S) \sqsubseteq r$. It follows that $r = \text{crit}(S)$ and $t = \text{crit}(S)^\frown \langle \max t \rangle$ as required. ⊣

10.18 Lemma. *For all $\beta \leq \omega$, the set of initial segments of trees in $\mathcal{T}(\beta)$ is well-founded under $\sqsubset$.*

Proof. The proof is by induction on β. For $\beta = 0$, the lemma is clearly true, since the longest possible sequences are those of the form $\emptyset, \langle m \rangle$ for some $m < \omega$.

Next suppose the lemma is true for $k < \omega$ and $\beta = k+1$. Let S_0, S_1, ... be an arbitrary $\sqsubset$-increasing sequence, and without loss of generality, assume it has at least two trees in it. Then there is some $m < \omega$ so that $\langle m \rangle \in S_1$. By the definition of the fold map F_k, it follows that for $i > 1$, the tree S_i satisfies $\bigcup S_i = \langle m \rangle^\frown \sigma_{i,1}{}^\frown \ldots {}^\frown \sigma_{i,n_i}$ for some $n_i \leq m$, where $F_k(\sigma_{i,j}) \in \mathcal{T}(k)$ for $j < n_i$, and for some $\sigma' \sqsupseteq \sigma_{i,n_i}$, $F_k(\sigma') \in \mathcal{T}(k)$, so $\sigma_{i,n_i} = \bigcup T_i$ for T_i an initial segment of a tree in $\mathcal{T}(k)$. If $i < \ell$ and T_i, T_ℓ are such that $n_i = n_\ell$, then for $j < n_i$, $\sigma_{i,j} = \sigma_{\ell,j}$. Thus by the induction hypothesis, for each n with $1 \leq n \leq m$, there can be at most finitely many trees in the sequence with $n_i = n$. Hence the sequence must be finite, and the lemma is true for $\beta = k+1$.

The proof for $\beta = \omega$ is similar, since for all initial segments S of trees in $\mathcal{T}(\omega)$, either $S = \emptyset$, $S = \{\langle m \rangle\}$, or $S = \{\langle m \rangle\}^\frown S'$ for some $m < \omega$ and some S' which is an initial segment of a tree in $\mathcal{T}(m)$. The details are left to the reader.

Therefore, by induction, the lemma holds for all $\beta \leq \omega$. ⊣

10.2. Node labeled trees

A typical proof of a positive partition relation for a countable ordinal for pairs includes a uniformization of an arbitrary 2-partition into 2 colors, but only for those pairs for which some easily definable additional information is also uniformized. We will introduce node labelings to provide that extra information, but before we do so, we examine convex partitions of disjoint trees and the partition nodes that determine them.

10.19 Definition. For trees S^0, S^1 from $\mathcal{T}(\beta)$ with $\bigcup S^0 \cap \bigcup S^1 = \emptyset$, call $t \in S^\varepsilon$ a *partition node* if $t < \max S^\varepsilon$ and there is some $u \in S^{1-\varepsilon}$ with $\max t < \max u < \min(\bigcup S^\varepsilon - (1 + \max t))$.

For notational convenience, write $T(\emptyset, t]$ for the initial segment of T consisting of all nodes $s \leq t \in T$, and, for $t < u$ in T, write $T(t,u]$ for

$\{ s \in T : t < s \leq u \}$. With this notation in hand, we can state the lemma below justifying the label *partition nodes*. This lemma follows from Lemmas 10.7 and 10.9.

10.20 Lemma. *Suppose S^0, S^1 are in $\mathcal{T}(\beta)$ and $\bigcup S^0 \cap \bigcup S^1 = \emptyset$. Further suppose $t^0_0, t^0_1, \ldots, t^0_{k-1} \in S^0$ and $t^1_0, t^1_1, \ldots, t^1_{\ell-1} \in S^1$ are the partition nodes of these trees if any exist. Set $t^0_{-1} = t^1_{-1} = \emptyset$, $t^0_k = \max S^0$, $t^0_\ell = \max S^1$. Then every node of S^ε is in one and only one $S^\varepsilon(t^\varepsilon_{i-1}, t^\varepsilon_i]$, and the sets $\sigma^\varepsilon_i = \bigcup S^\varepsilon(t^\varepsilon_{i-1}, t^\varepsilon_i] - t^\varepsilon_{i-1}$ satisfy*

$$\sigma^0_0 < \sigma^1_0 < \sigma^0_1 < \sigma^1_1 < \ldots \sigma^0_{\ell-1} < \sigma^1_{\ell-1} (< \sigma^0_{k-1}).$$

Now we introduce node labelings. For simplicity, this concept is given a general form.

10.21 Definition. Suppose $\beta \leq \omega$ and $N \subseteq \omega$ is infinite. For any initial segment $S \sqsubseteq T \in \mathcal{T}(\beta)$, a function C is a *node labeling of S into N* if $C : S \to [N]^{<\omega}$ satisfies $\max C(s) < \max s$ for all $s \in S$ with $C(s) \neq \emptyset$.

We carry over from $\mathcal{T}(\beta)$ the notions of extension, complete tree and trivial tree. In particular, call (T, D) a *(proper) extension of* (S, C), in symbols, $(S, C) \sqsubset (T, D)$, if $S \sqsubset T$ and $D|S = C$. Call (T, D) *complete (for β)* if $T \in \mathcal{T}(\beta)$; call it *trivial* if $(T, D) = (\emptyset, \emptyset)$.
Call a pair S, T from $\mathcal{T}(\beta)$ *local* if S and T have a common root; otherwise it is global. Similarly, call (S, C), (T, D) *local* if S, T is local and otherwise call it global.

10.22 Definition. A pair $((S^0, C^0), (S^1, C^1))$ is *strongly disjoint* if (a) either $S^0 = \emptyset = S^1$ or $\left(\bigcup S^0 \cup \operatorname{ran} C^0\right) \cap \left(\bigcup S^1 \cup \operatorname{ran} C^1\right) = \emptyset$ and (b) for all $s, t \in S^0 \cup S^1$, whenever $\max s < \max t$ and $C^\varepsilon(t) \neq \emptyset$, then also $\max s < \min C^\varepsilon(t)$.

10.23 Definition. Call a pair $((S^0, C^0), (S^1, C^1))$ of node labeled trees *clear* if $S^0 < S^1$, $((S^0, C^0), (S^1, C^1))$ is strongly disjoint, all partition nodes $t \in S^0 \cup S^1$ are leaf nodes ($e_\beta(t) = 0$), and if for all $\varepsilon < 2$ and all $s \in S^\varepsilon$,

- $C^\varepsilon(s) = \emptyset$ if $e_\beta(s) = 0$;
- $C^\varepsilon(s) = \{ \sharp(s, S^\varepsilon(\emptyset, t]) : s \sqsubset t \in S^\varepsilon \text{ is a partition node} \}$ if $e_\beta(s)$ is a successor ordinal;
- $C^\varepsilon(s) = \{ e_\beta(t) : s \sqsubset t \in S^\varepsilon \,\&\, |C^\varepsilon(t)| > 1 \}$ if $e_\beta(s) = \omega$ is a limit ordinal.

Call a pair S^0, S^1 of trees from $\mathcal{T}(\beta)$ *clear* if it is local or if it is global and there are node labelings C^0, C^1 with $((S^0, C^0), (S^1, C^1))$ clear.

For $\beta > \omega$, the value of the node labeling for s with $e_\beta(s)$ limit is more complicated to describe.

Notice that for $2 \leq \beta \leq \omega$, if $(S^0, C^0), (S^1, C^1)$ is a global clear pair and neither C^0 nor C^1 is constantly the emptyset, then all initial segments of partition nodes are identifiable: they are the root of the tree, successor nodes whose node label is non-empty, and nodes of ordinal 0 whose immediate predecessor has non-empty node label that identifies it as a successor which is a partition node.

From the definition of *clear*, if u is a partition node of one of a pair of trees, say (S, C) then for each initial segment s whose ordinal $e_\omega(s)$ is a successor, the node label $C(s)$ must have as a member the number of immediate successors of s which are less than or equal to u in the lexicographic order. If we index the immediate successors of s in S in increasing lexicographic order starting with 1, then this value is the *index* of the immediate successor of s which is an initial segment of u. This analysis motivates the next definition.

10.24 Definition. Consider a node labeled tree (S, C) with root $\langle m \rangle$. A non-root node t of (S, C) is a *prepartition node* if for all $s \sqsubset t$ with $e_\beta(s)$ a successor ordinal, $\sharp(s, S(\emptyset, t]) \in C(s)$, and if $e_\omega(s) \in C(\langle m \rangle)$ whenever $\beta = \omega$ and $|C(s)| > 1$ The root is a prepartition node if $S \in \mathcal{T}(0)$ or $C(\langle m \rangle) \neq \emptyset$ or (S, C) has a non-root prepartition node. Call (S, C) *relaxed* if $S \notin \mathcal{T}(0)$ and $\max S$ is a prepartition node of ordinal 0.

Node labeled trees, clear pairs, prepartition nodes and relaxed initial segments are used in the game introduced in the next section.

10.3. Game

In this section we develop the game $\mathcal{G}(h, N)$ in which two players collaborate to build a pair of node labeled trees.

Here is a brief description of the game. Player I, the architect, plays specifications for Player II, the builder, telling him (a) which tree to extend, (b) whether to complete the tree or to build it to the next decision point, and (c) what the size of the node label of the next node to be constructed is, if it is not already determined. In turn, the builder extends the designated tree by a series of steps, adding a node and node label at each step using elements of N, until he reaches the next decision point on the given tree, if he has been so directed, or until he completes the tree. The architect wins if the pair $((S, C), (T, D))$ created at the end of the play of the game is a global clear pair with $h(S, T) = 1$; otherwise the builder wins.

Before giving a detailed description of the general game, as a warmup exercise, consider a 2-partition h into 2 colors, an infinite set N, and the game $\mathcal{G}_0(h, N)$ in which the architect plays the strategy σ_0 directing the builder to complete the first tree and then complete the second tree. The builder can use a fold map to fold an initial segment of N into a tree S and assign the constantly $\emptyset$ node labeling C to create his first response, (S, C). Then he can fold a segment of N starting above $\bigcup S$ into a tree T and assign the constantly $\emptyset$ node labeling D to create his second response, (T, D). By construction, the pair $((S, C), (T, D))$ is clear, since there are no partition nodes, so $\{ S, T \}$ is a clear global pair. If all pairs $\{X, Y\}$ of trees created using nodes from N in this game have $h(X, Y) = 1$, then playing another game, starting with (T, D) as the initial move of the builder and ending with (U, E), one builds a triple $\{ S, T, U \}$ each pair of which h takes to color 1. Thus if σ_0 is a winning strategy for the architect, then the architect can arrange for a triangle to be constructed.

As a second warmup exercise, consider a 2-partition h into 2 colors, an infinite set N with $0 \notin N$, and the game $\mathcal{G}_1(h, N)$ in which the architect plays the strategy σ_1 directing the builder to build the first tree to the next decision point starting from a root node whose node label has 0 elements, to start and complete the second tree, and then to complete the first tree.

In response to the architect's first set of specifications, the builder uses the least element n_0 of N to build the root, $\langle n_0 \rangle$ and gives it the empty set as node label. He then uses the next two elements of N, namely n_1 and n_2 by setting $\langle n_0, n_2 \rangle$ as the immediate successor of the root with node label $C_0(\langle n_0, n_2 \rangle) = \{n_1\}$. He continues with successive elements of N, extending the critical node of the tree create to that point, giving the new node an empty label unless the node to be created is the successor of a prepartition node whose index is the sole element of the node label of the prepartition node, in which case he extends and labels it as he did the successor of the root. He continues until he has created and labeled a prepartition node u whose ordinal is $e_\omega(u) = 0$, and the pair (S_0, C_0) he has built is his response.

In response to the architect's second set of specifications, the builder uses elements of N larger than any used so far to build a tree T in $\mathcal{T}(\omega)$ and gives it the constantly $\emptyset$ labeling. Then he responds to the final set of specifications of the architect by completing S_0 to S in $\mathcal{T}(\omega)$ and extending C_0 to C with all new nodes receiving empty node labels.

In the brief description of the game, the architect was allowed to direct the builder to stop at the next decision point. The decision point is either when a partition node has been created and it is time to switch to the other tree or when the next node to be created is permitted to have a node label whose size is greater than 2. Notice that if the architect switches trees after the builder has created a prepartition node with ordinal 0, then that node becomes a partition node.

10.25 Definition. A *decision node* of (S, C) is a prepartition node t with ordinal $e_\omega(t)$ such that either $e_\omega(t) = 0$ or $e_\omega = \ell + 1$ is a successor ordinal with $\ell \in C(t|1)$, t is the critical node of S and $1 + \sharp(t, S)$ is an element of $C(t)$.

In the game $\mathcal{G}_0(h, N)$, the final pair of trees S, T had the property that $\min \bigcup S < \min \bigcup T$ and $\max \bigcup S < \max \bigcup T$. Call such a pair an *outside* pair. In the game $\mathcal{G}_1(h, N)$, the final pair of trees S, T had the property that $\min \bigcup S < \min \bigcup T$ and $\max \bigcup S > \max \bigcup T$. Call such a pair an *inside* pair.

10.26 Definition. Suppose $N \subseteq \omega$ is infinite and h is a 2-partition of $\mathcal{T}(\omega)$ into 2 colors. Then $\mathcal{G}(h, N)$ is a two player game played in rounds. Player I is the architect who issues specifications, and Player II is the builder whose creates or extends one of a given pair of trees in round ℓ to $((S_\ell, C_\ell), (T_\ell, D_\ell))$. Note that if the second tree has not been started in round ℓ, then $T_\ell = D_\ell = \emptyset$.

The architect's moves: In the initial round, the architect declares the type of pair to be produced, either inside or outside. In round ℓ, the architect specifies the tree to be created or extended (first or second), specifies whether the extension is to completion with all new nodes receiving empty labels or to the point at which a decision node is created and labeled (completion or decision), and specifies the size of the label for the next node to be created. In her initial move, the architect must specify the first tree be created. She may not direct the builder to extend a tree which is complete.

The builder's moves: In round ℓ, the builder creates or extends the specified tree through a series of steps in which he adds one node and its label using elements of N larger than any used to that point. If he has been directed to continue to completion, he does so while assigning the empty set node label to all new nodes. Otherwise he adds nodes one at a time, until he creates the first decision node. He adds a node after determining the size of the node label, and choosing the node label, since all elements of the node label must be smaller than the single point used to extend the critical node. The size of the label of the first node to be created is specified by the architect's move. Otherwise, the builder determines if the node will be a prepartition node with non-zero ordinal. If so, its node label has one element and otherwise its node label is empty.

Stopping condition: Play stops at in round ℓ if both trees are complete.

Payoff set: The architect wins if both S_ℓ and T_ℓ are complete, the pair is inside or outside as specified at the onset, the pair $((S_\ell, C_\ell), (T_\ell, D_\ell))$ is a global clear pair and $h(S_\ell, T_\ell) = 1$; otherwise, the builder wins.

We are particularly interested in this game when we have a fixed 2-partition, $h : [\mathcal{T}(\beta)]^2 \to 2$, but the game may be modified to work with 2-partitions into more colors. This game may also be modified to require the builder to use an initial segment of an infinite sequence from N specified by the architect in her move or be modified to start with a specified pair of node labeled trees.

10.27 Lemma. *Suppose $N \subseteq \omega$ is infinite and h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors. Then every run of $\mathcal{G}(h, N)$ stops after finitely many steps.*

Proof. Use Lemma 10.18. ⊣

10.4. Uniformization

In this subsection, we prove the key dichotomy in which one or the other player has a winning strategy, at least up to some constraints on the play. Basically, we build a tree out of the plays of the game, show it is well-founded, and use recursion on the tree to define an infinite subset $H \subseteq \omega$ so that plays where the builder uses sufficiently large elements of H are uniform enough to allow us to prove the dichotomy.

10.28 Definition. Suppose $N \subseteq \omega$ is infinite, and h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors. Let $\mathcal{S}(N)$ be the set of sequences of consecutive moves in the game $\mathcal{G}(h, N)$, including the empty sequence.

10.29 Lemma. *For infinite $N \subseteq \omega$, $(\mathcal{S}(N), \sqsubset)$ is a rooted, well-founded tree.*

Proof. The root is the empty sequence. End-extension clearly is a tree order on $\mathcal{S}(N)$, and $\sqsubset$ is well-founded since every game is finite. ⊣

The basic idea for the builder is to use elements from a specified set and to always start high enough.

10.30 Definition. Suppose N is an infinite set with $1 < \min N$ and no two consecutive integers in N. Then a function $b : \mathcal{S}(N) \to \omega$ is a bounding function if $b(\emptyset) = 0$, and if $s \sqsubseteq t$, then $b(s) \leq b(t)$.

Use a bounding function and an infinite set to delineate *conservative* moves for the builder.

10.31 Definition. Suppose $H \subseteq N \subseteq \omega$ is infinite with $1 < \min N$ that b is a bounding function. If $\vec{R}$ is a position in the game $\mathcal{G}(h, N)$ ending with a move by the architect, then a move $((S_\ell, C_\ell), (T_\ell, D_\ell))$ for the builder is *conservative for b and H* if all new nodes and node labels are created using elements of H greater than $b(\vec{R})$.

10.32 Lemma (Ramsey Dichotomy). *Suppose $N \subseteq \omega$ is infinite, and h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors. Then there is an infinite subset $H \subseteq N$ and a bounding function b so that $1 < \min H$, no two consecutive integers are in H, and the following statements hold:*

1. *for every position $\vec{R} \in \mathcal{S}(N)$ ending in a play for the architect, there is a conservative (for b and H) move for the builder; and*

2. *either the architect has a strategy σ by which she wins $\mathcal{G}(h,N)$ if the builder plays conservatively, or the builder wins every run of $\mathcal{G}(h,N)$ by playing conservatively (for b and H).*

Before we tackle the proof of the dichotomy, we introduce some preliminary definitions and lemmas.

10.33 Definition. Call a set $B \subseteq [\omega]^{<\omega}$ *thin* if no u from B is a proper initial segment of any other v from B. Call B a *block for* $N \subseteq \omega$ if for every infinite set $H \subseteq N$, there is exactly one $u \in B$ which is an initial segment of H. Call it a *block* if it is a block for ω.

Note that if B is a block, then it is thin. A major tool of the proof of the dichotomy is the following theorem.

10.34 Theorem (Nash-Williams Partition Theorem). *Let $N \subseteq \omega$ be infinite. For any finite partition of a thin set $c : W \to n$, there is an infinite set $M \subseteq N$ so that c is constant on $W|M$.*

For a proof see [46] or [23]. The terminology *thin* comes from [23].

Here are some easy examples of blocks.

10.35 Lemma. *The families $\{\emptyset\}$, and $[\omega]^k$ for $k < \omega$ are blocks.*

10.36 Lemma. *Suppose $w \subseteq \omega$ is an increasing sequence, and $B \subseteq [\omega]^{<\omega}$ is thin. Then there is at most one initial segment u of w with $u \in B$. If B is a block, then there is exactly one such initial segment.*

10.37 Lemma. *Suppose $H \subseteq N \subseteq \omega$ is infinite, h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors, and b is bounding function. For every position $\vec{R} \in \mathcal{S}(N)$ ending in a move by the architect, there is some $k \geq b(\vec{R})$ and a block $\mathcal{B}(\vec{R})$ for $H - k$ such that for all $B \in \mathcal{B}(\vec{R})$, the builder can build his responding move using all elements of B.*

Proof. Recall the architect may not direct the builder to extend a complete tree, so if the architect has just moved, the tree she directs the builder to extend is not complete. Thus the builder's individual steps are specified up to the choice of elements of N, and his stopping point is determined by his individual steps. Hence the set of sequences of new elements used is thin. Moreover, for any infinite increasing sequence w from H above $b(\vec{R})$ and above the largest element of N used in prior moves, the builder can create a move using an initial segment of w. Therefore the set of possible moves is a block. $\dashv$

At this point we are prepared to prove the main result of this section.

Proof of Ramsey Dichotomy 10.32. Without loss of generality, assume $1 < \min N$ and N has no two consecutive elements, since otherwise one can shrink N to an infinite set for which these conditions hold. These conditions assure that no decision node is an immediate successor of another decision node.

Let ρ^* be the rank of $\mathcal{S}(N)$. Use recursion on $\mu \le \rho^*$ to define a sequence $\langle M_\mu \subseteq N : \mu < \rho^* \rangle$ and a valuation $v : \mathcal{S}(N) \to 2$.

For $\mu = 0$, the sequences $\vec{R}$ of rank 0 are ones in which the last move completes the play of the game. Let $M_0 = N$, and define $v(\vec{R}) = 0$ on a sequence of rank 0 if the game ends with a win for the architect and $v(\vec{R}) = 1$ otherwise.

Next suppose that $0 < \mu < \rho^*$, and v has been defined on all nodes of rank less than μ. Enumerate all the nodes of rank μ as $\vec{R}^0_\mu, \vec{R}^1_\mu, \ldots$ and let M^{-1}_μ be $M_{\mu-1}$ if μ is a successor ordinal and let M^{-1}_μ be a diagonal intersection of a sequence M_ν for a set of ν cofinal in μ otherwise.

Extend v to the nodes of rank μ and define sets M^i_μ by recursion. For the first case, suppose $\vec{R}^i_\mu$ ends with a move for the builder, and set $M^i_\mu = M^{i-1}_\mu$. If there is some move a^i_μ with $\vec{R}^i_\mu {}^\frown \langle a^i_\mu \rangle \in \mathcal{S}(N)$ and $v(\vec{R}^i_\mu {}^\frown \langle a^i_\mu \rangle) = 1$, then set $v(\vec{R}^i_\mu) = 1$, and otherwise set $v(\vec{R}^i_\mu) = 0$.

For the second case, assume $\vec{R}^i_\mu$ ends with an move for the architect. Let $\mathcal{B}(\vec{R}^i_\mu)$ be the block of Lemma 10.37 for the set M^{i-1}_μ and the position $\vec{R}^i_\mu$. Define $c : \mathcal{B}(\vec{R}^i_\mu) \to 2$ by $c(d) = v(\vec{R}^i_\mu {}^\frown \langle P(d) \rangle)$ where $P(d)$ is the unique approved move for the builder whose new elements are created using exactly the elements of d. Apply the Nash-Williams Partition Theorem 10.34 to c to get an infinite set $M^i_\mu \subseteq M^{i-1}_\mu$ and let $v(R^i_\mu)$ be the constant value of c on $\mathcal{B}(\vec{R}^i_\mu)$ restricted to M^i_μ.

Continue by recursion as long as possible, extending v to all nodes of rank μ. If there are only finitely many of them, let M_μ be M^i_μ where $\vec{R}^i_\mu$ is the last one. If there are infinitely many, let M_μ be a diagonal intersection of the sets M^i_μ.

Since every non-empty sequence of moves in the game $\mathcal{G}(N)$ extends the empty sequence, this root of $\mathcal{S}(N)$ has the largest rank of any element of $\mathcal{S}(N)$, namely rank $\rho^* - 1$. Let $H = M_{\rho^*-1}$. Let $v(\emptyset)$ be 1 if there is some move a by the architect so that $v(\langle a \rangle) = 1$, and set $v(\emptyset) = 0$ otherwise.

Define b on $\mathcal{S}(N)$ by recursion. Let $b(\vec{R}) = 2$ for all $\vec{R} \in \mathcal{S}(N)$ with $|\vec{R}| \le 1$. Continue by recursion on $|\vec{R}|$. For notational convenience, let $\vec{R}^-$ be obtained from $\vec{R} \in \mathcal{S}(N) - \{\emptyset\}$ by omission of the last entry. If $b(\vec{R}^-)$ has been defined and the last move in $\vec{R}$ is $\mathcal{B}_\ell = ((S_\ell, C_\ell), (T_\ell, D_\ell))$ for the builder, then let $b(\vec{R})$ be the least b greater than $b(\vec{R}^-)$ and any element of $\bigcup(S_\ell \cup \operatorname{ran} C_\ell \cup T_\ell \cup \operatorname{ran} D_\ell)$. If $b(\vec{R}^-)$ has been defined, the last move in $\vec{R}$ is a_ℓ for the architect, and $\vec{R} = \vec{R}^i_\mu$, then let $b(\vec{R})$ be the least b greater than $b(\vec{R}^-)$ so that for all d in the restriction of $\mathcal{B}(\vec{R}^i_\mu)$ to subsets of H with $\min d > b$, there is a conservative move for the builder for position $\vec{R}$ with new elements d. The existence of a value for $b(\vec{R})$ in this latter case follows from the fact that $H \subseteq^* M^i_\mu$ by construction, and by Lemma 10.37.

Since all $\vec{R}$ in $\mathcal{S}(N)$ are finite, this recursion extends b to all of $\mathcal{S}(N)$. This definition of H and b guarantees that the builder can always respond with conservative moves to plays of the architect.

If $v(\emptyset) = 1$, then the strategy for the architect is to keep $v(\vec{R}) = 1$. Given the definition of v, the architect will always succeed, as long as the builder moves conservatively with H and b. If $v(\emptyset) = 0$, and the builder always moves conservatively with H and b, then he will win, again by the recursive definition of v and the definition of winning the game. ⊣

10.5. Triangles

For this section we assume that $h : [\mathcal{T}(\omega)]^2 \to 0$ is fixed and that an infinite set $H \subseteq \omega$ and a bounding function b are given so that the architect has a winning strategy σ for games of $\mathcal{G}(h, H)$ in which the builder plays conservatively for b and H. The goal is to outline how one uses the strategy of the architect to construct a triangle.

10.38 Lemma. *Suppose σ is a strategy for the architect with which she wins $\mathcal{G}(h, N)$ if the builder moves conservatively for H, b. Then there is a three element 1-homogeneous set for h.*

Proof. Consider the possibilities for $\sigma(\emptyset)$. The architect must declare the pair to be built will be inside or outside, the initial move is to complete the first tree or construct it to a decision point and must declare the size d of the node label of the initial node constructed. We construct our triangles by playing multiple interconnected games in which the architect uses σ, the builder plays conservatively for H and b, and plays sufficiently large that his plays work in all the relevant games. While technically we should report a pair of node labeled trees for each play of the builder, for simplicity, we frequently only mentioned the one just created or modified.

Case 1: Using σ, the architect specifies the builder constructs a complete tree in her initial move.

Then the architect must call for an outside pair and must set $d = 0$, since otherwise the pair constructed will not be clear. The builder responds via conservative play with a complete tree (S, C) whose node labeling is constantly the emptyset. The strategy σ must then specify that the builder constructs a second complete tree whose initial node has a node label of size 0. The builder responds via conservative play with a complete tree (T, D) whose node labeling is constantly $\emptyset$. Since σ is a winning strategy, $h(S, T) = 1$.

Next the architect shifts to the game where the builder has responded to the opening move with (T, D), applies the strategy σ, to which the builder responds with (U, E), a (third) complete tree whose node labeling is constantly $\emptyset$ starting sufficiently large for this response to be appropriate for the game where the builder has responded to the opening move with (S, C). Since σ is a winning strategy, $h(T, U) = 1 = h(S, U)$. Thus $\{ S, T, U \}$ is the required triangle.

Case 2: Using σ, the architect declares the pair will be an inside pair, and specifies the initial node label size $d = 0$ and that the builder constructs to a decision node.

The proof in this case is similar to the last, with the architect starting one game to which the builder responds with a first tree (S_0, C_0) where the decision node is a prepartition node of ordinal zero, since no levels were coded for introducing decision nodes with successor ordinals. Thus the next play for the architect is to direct the builder to create a complete tree all of whose nodes are labeled by $\emptyset$.

The architect stops moving on the first game and, using σ, starts a new game, directing the builder to start high enough that the tree constructed could be the beginning of his response in the first game. The builder responds with a tree (T_0, D_0) where the decision node is a prepartition node of ordinal zero The architect continues this game using σ and the builder responds with a complete tree (U, E) all of whose nodes are labeled with $\emptyset$. After the architect and builder each move a final time on this game, the builder has created a complete tree (T, D) extending (T_0, D_0). Since σ is a winning strategy, $h(T, U) = 1$.

Now return to the first game: the builder plays (T, D') where D' is the constantly emptyset node labeling; The architect uses σ to respond and requires the builder to construct high enough that his response works in the game where the builder plays (U, E) as well as the one where the builder plays (T, D'). Since σ is a winning strategy, $h(S, T) = h(S, U)$, Thus $\{ S, T, U \}$ is the required triangle.

Case 3: Using σ, the architect declares the pair will be an outside pair, and specifies the initial node label size $d = 0$ and that the builder constructs to a decision node.

The proof in this case is similar to the last, so only the list of substrees to be constructed is given. Start with (S_0, C_0) and (T_0, D_0) as responses to the first two moves of the architect in the first game. Next build (U_0, E_0) and (S, C) as second and third moves in a game where (S_0, C_0) is the first move, and (U_0, E_0) is started high enough to be a reponse in the game starting with (T_0, D_0). Finally build (T, D) and (U, E) in the game starting with responses (T_0, D_0) and (U_0, E_0) and continuing high enough that play using (S, C) in the appropriate games is conservative.

In the remaining two cases, we use σ and conservative play for the builder to create trees S, T, U with node labelings (S, C^1) , (S, C^2), (T, D^0), (T, D^1), (U, E^0) and (U, E^1) through plays $\mathcal{G}_{0,1}$, $\mathcal{G}_{0,2}$, $\mathcal{G}_{1,2}$ of the game $\mathcal{G}(h, H)$. We pay special attention to the creation of the initial segments up to the first partition nodes for each pair and to the terminal segments, after the last partition nodes. We refer to the remainder of the run as "the mid-game".

Case 4: Using σ, the architect declares the pair will be an inside pair, and specifies the initial node label size $d > 0$ and directs the builder to construct the first tree to a decision node.

We start by displaying a schematic overview of the construction:

$$S \quad T \quad U \quad \boxed{T \quad U} \quad \boxed{S \quad U} \quad U \quad \boxed{S \quad T} \quad T \quad S$$

Next we outline the steps to be taken.

1. Choose from H codes for d levels for S and U; choose d larger levels for S and T; start the initial segment of S with respect to T; continue it to get the initial segment of S with respect to U (the difference is in the node labelings only), and apply σ to the results to determine the sizes d', d'' of node labels for the roots of T, U in $\mathcal{G}_{0,1}$, $\mathcal{G}_{0,2}$, respectively.

2. Choose d' levels for T's interaction with U; choose d larger levels for T's interaction with S; start the initial segment of T with respect to S; continue it to get the initial segment of T with respect to U; and apply σ to determine the size d''' of the node label of the root of U for $\mathcal{G}_{1,2}$.

3. Choose d''' levels for U's interaction with T; choose d'' larger levels for U's interaction with S; start the initial segment of U with respect to S; continue it to get the initial segment of U with respect to T.

4. Play the mid-game of $\mathcal{G}_{1,2}$ to the call for the completion of U.

5. The initial segments of T and U with respect to S are complete, so update the node labelings C^0 and C^1.

6. Play the mid-game of $\mathcal{G}_{0,2}$ until the architect calls for the completion of S. In particular, play until U is complete.

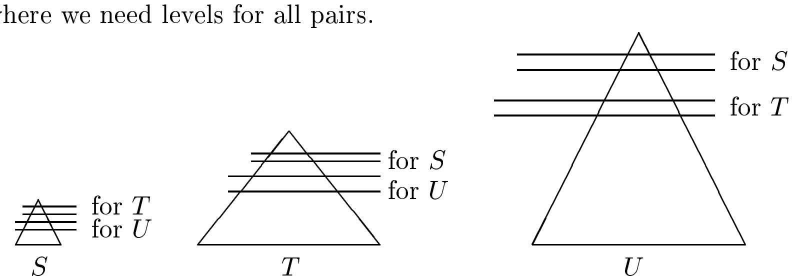
7. Update the node labeling E^1 for U by labeling all the new nodes by the empty set.

8. Complete the play of the game $\mathcal{G}_{0,1}$, starting by extending the part of S created in the play of the mid-game $\mathcal{G}_{0,2}$. Such a start is possible, since the levels of S for interaction with T are larger than those for interaction with U.

9. Update the node labelings C^2 for S and D^2 for T by labeling all the new nodes by the empty set.

Care must be taken to direct the builder to start high enough that all moves in the tree plays of $\mathcal{G}(h, H)$ are conservative. Since the construction of the initial segments calls for introducing levels, we describe the first such step in greater detail.

We know that we will need to choose *levels* for splitting of S with respect to T and U, and for splitting T with respect to U. Depending on the strategy σ, we may need to choose levels for the splitting of T with respect to S and for the splitting of U with respect to S and T. Here is a picture of the approach we plan to take on these splitting levels, in the general case where we need levels for all pairs.



To start the construction, choose $2d+1$ elements from H above $b(\langle \sigma(\emptyset) \rangle)$ ending in m^0, and use them to define $C^1(\langle m^0 \rangle)$ and $C^2(\langle m^0 \rangle)$ with $C^2(\langle m^0 \rangle) < C^1(\langle m^0 \rangle)$.

Start playing a game $\mathcal{G}_{0,1}$ where the architect starts with $R_0^{0,1} = \sigma(\emptyset)$ and the builder must use the elements of $C^1(\langle m^0 \rangle)$ and m^0 to start his initial move, $R_1^{0,1}$. Continue to play until the architect's last move $R_p^{0,1}$ before directing the builder to switch to the second tree. One can identify this point in the run of the game, since it is the first time the architect has stopped on a node, call it v_0, whose level is one more than $\min(C^1(\langle m^0 \rangle))$. Let (S_{p-1}^1, C_{p-1}^1) be the tree paired with $(\emptyset, \emptyset)$ by the builder in his last move.

Let C^2 be the node labeling of S_{p-1}^1 with the value of $C^2(\langle m^0 \rangle)$ specified above, with the empty set assigned for nodes which are not initial segments of v_0, and for initial segments of v_0 longer than the root, are the singletons needed to guarantee that v_0 is a prepartition node. Then the architect directs the builder to extend this node labeled tree to a response $R_1^{0,2}$ to $\sigma(\emptyset)$ in the second game $\mathcal{G}_{0,2}$. The two players continue the game until the architect, in $R_q^{0,2}$, directs the builder to switch to the second tree to start with a node label of size d'' and to go to a decision node. Such a move is the only one that will lead to a clear pair. Let (S_{q-1}^2, C_{q-1}^2) be the tree played by the builder in his previous move.

Return to game $\mathcal{G}_{0,1}$ and require the builder to respond to $R_p^{0,1}$ with (S_{p+1}^1, C_{p+1}^1) for $S_{p+1}^1 = S_{q-1}^2$ and C_{p+1}^1 the node labeling where all new nodes that are not initial segments of the largest node are labeled with the empty set and initial segments of the largest node are labeled minimally so that it is a prepartition node. Let d' be the size of the node label for the root of the second tree determined by the architect's use of σ in response to this move of the builder.

The remaining details are left to the reader. The careful reader will note that there is one possibility in which the architect initially calls for $d = 1$, specifies a node label of size 2 at the first decision node, and after the completion of the first full segment, calls for an empty node label for the root of the second tree. The construction proceeds as above but is simpler, so these details are also left to the reader.

As in the previous cases, since σ is a winning strategy for the architect, the set $\{ S, T, U \}$ we have constructed is the required triangle.

Case 5: Using σ, the architect declares the pair will be an outside pair, and specifies the initial node label size $d > 0$ and directs the builder to construct the first tree to a decision node.

This case is substantially like the previous one, so we give the schematic below to guide the reader and a few comments on how to move from one section to the next.

$$S \quad T \quad \boxed{S \quad T} \quad U \quad \boxed{S \quad U} \quad S \quad \boxed{T \quad U} \quad T \quad U$$

We start by building initial segments of S and T. We begin by choosing d small levels for the interaction of S with T and d larger levels for the interaction of S with U. We start to build the initial segment of S with respect to its convex partition by U, then extend that start to build the initial segment of S with respect to its convex partition by T. We obtain the size d' of the root node label of the second tree in $\mathcal{G}_{0,1}$ by applying σ, choose d' small levels for the interaction of T with S, and d larger levels for the interaction of T with U. We start building the initial segment of T with respect to U, then extend it to the initial segment of T with respect to S.

We play the mid-game of $\mathcal{G}_{0,1}$ until the architect calls for the completion of S. In the process we have completed the initial segments of S and T with respect to U, so we update C^2 and D^2, and apply σ to the current state of play of $\mathcal{G}_{0,2}$ to find d'' and to the current state of play of $\mathcal{G}_{1,2}$ to find d'''.

We choose d'' smaller levels for the interaction of U with respect to S and d''' larger levels for the interaction of U with respect to T. We start building the initial segment of U with respect to T, then extend it to the initial segment of U with respect to S.

We play the mid-game of $\mathcal{G}_{0,2}$ until the builder has completed the construction of S and the architect has called for the completion of U. In the process we have completed the initial segment of U with respect to T, and the final segment of S with respect to T so we update E^1 and and C^1.

Then we play the mid-game of $\mathcal{G}_{1,2}$ and complete the play of that game with the final segments of T and U. Finally, we update D^0 and E^0 on the new elements of T and U which complete the games $\mathcal{G}_{0,1}$ and $\mathcal{G}_{0,2}$.

As in the previous cases, since σ is a winning strategy for the architect, the set $\{ S, T, U \}$ we have constructed is the required triangle. $\dashv$

10.6. Free Sets

Our next goal is the construction of a subset of $\mathcal{T}(\omega)$ of order type ω^{ω^ω} which is 0-homogeneous for global pairs.

Recall the characterization of subsets of G_ω of order type at least ω^s that dates back to the late 1960's or early 1970's. (see [43], [42], [66]).

10.39 Definition. A non-empty set $S \subseteq \{ \sigma \in G_\omega : \min \sigma = n \}$ is *free above coordinate* k if for every $\mathbf{x} = \langle x_0, x_1, \ldots, x_n \rangle \in S$, there is an infinite set $N \subseteq \omega$ so that for each $x' \in N$, the set of extensions of $\langle x_0, x_1, \ldots, x_k, x' \rangle$ in S is non-empty. The set S is *free in s coordinates* if there are s coordinates above which it is free.

10.40 Lemma (see Lemma 7.2.2 of [66]). *A set $S \subseteq \{ \sigma \in G_\omega : \min \sigma = n \}$ has* $\operatorname{ot}(S) \geq \omega^s$ *if and only if there is a subset $V \subseteq S$ so that V is free in s coordinates.*

We would like to adapt this idea to sets of node labeled trees from $\mathcal{T}(\beta)$. By an abuse of notation, write $t \in (T, D) \in X$ to mean that $t \in T$ for some $(T, D) \in X$. The next definition facilitates our discussion. Recall that $e_\beta(s)$ is the ordinal of s.

10.41 Definition. For $\beta \leq \omega$ and any $s \in (S, C) \in \mathcal{T}^*(\beta)$, call s a *signal node* if either $|C(s)| > 1$ or $e_\beta(s)$ limit and $|C(s)| = 1$.

Recall Definition 10.24 of relaxed initial segments of trees in $\mathcal{T}(\beta)$. The first three parts of the next definition guarantee that locally Γ-free sets have nice regularity properties, and the last three guarantee (1) signal nodes are introduced whenever there is no constraint, (2) signal nodes are given large node labels, and (3) there are arbitrarily large starts for extensions of relaxed initial segments of trees in the collection. The definition of Γ-free from locally Γ-free guarantees that there are arbitrarily large new starts for trees as well.

10.42 Definition. Suppose $\beta \leq \omega$ and $0 \notin \Gamma \in [\beta + 1]^{<\omega}$. A non-empty set X of node labeled trees from $\mathcal{T}(\beta)$ is *locally Γ-free for β* if the following conditions are satisfied:

1. (commonality) if $\beta > 0$, then every tree in X has a proper relaxed initial segment and every local pair from X has a common proper relaxed initial segment and otherwise is disjoint;

2. (conformity) if $r \in (S, C) \in X$ and $k \in C(r) \neq \emptyset$, then there is some relaxed $(T, D) \sqsubseteq (S, C)$ so that $r \sqsubset \max T$ and if the ordinal of r is a successor, then $\sharp(r, T) = k$.

3. (Γ-signality) for any signal node $r \in (S, C) \in X$, either $e_\beta(r) \in \Gamma$ or for some $p \sqsubset r$ with $e_\beta(p) = \omega$, there is $k \in C(p)$ so that $e_\beta(r) = k$.

4. (Γ-forecasting) for any relaxed $(S, C) \sqsubseteq (T, D) \in X$, if $\gamma_i \in \Gamma$, then there is some signal node $r \sqsubset \max S$ with $e_\beta(r) = \gamma_i$; and if $p \sqsubset \max S$ is a signal node, $k \in C(p)$, and $e_\beta(p) = \omega$ is a limit ordinal, then there is some signal node $r \sqsubset \max S$ with $e_\beta(r) = k$;

5. (signal size) for any signal node $r \in (S, C) \in X$, the inequality $|C(r)| < \max r$ holds, and $\max t < \max r$ implies $\max t < |C(r)|$ for all $t \in (T, D) \in X$.

6. (push-up) for every $k < \omega$ and every relaxed initial segment $(T, D) \sqsubset (U, E) \in X$, there is some complete extension $(V, F) \sqsupset (T, D)$ in X whose new elements start above k, i.e. $k < \min(\bigcup V \cup \operatorname{ran} F - \bigcup T \cup \operatorname{ran} D)$.

We say X is Γ*-free for* β if it is locally Γ-free for β, and for all $k < \omega$, there is some $\langle m \rangle \in (S, C) \in X$ such that $k < |C(\langle m \rangle)|$ if $\beta \in \Gamma$ and $k < m$ otherwise.

By an abuse of notation, for a collection X of node labeled trees from $\mathcal{T}(\beta)$, we let $\operatorname{ot} X = \operatorname{ot}\{ S : \exists C (S, C) \in X \}$.

10.43 Lemma. *For all* $\beta \leq \omega$*, for all* $0 \notin \Gamma \in [\beta + 1]^{<\omega}$*, if* X *is* Γ*-free for* β*, then* $\operatorname{ot} X \geq \zeta(\beta, \Gamma)$ *where*

$$\zeta(\beta, \Gamma) := \begin{cases} \omega & \text{if } \beta = 0, \\ \omega^2 & \text{if } \beta > 0 \text{ and } \Gamma = \emptyset, \\ \omega^{\omega^{|\Gamma|}} & \text{if } \beta > 0 \text{ and } \omega \notin \Gamma \neq \emptyset, \text{ and} \\ \omega^{\omega^\omega} & \text{otherwise.} \end{cases}$$

Proof. Relaxed trees, especially with a specified node as an initial segment of the max, play an important role in the definition of free and locally free. Here is some notation to facilitate the discussion. For any set X of node labled trees, define $X(t) := \{ (T, D) \in X : t \in (T, D) \}$.

10.44 Claim. *If* X *is* Γ*-free for* $\beta = 0$ *and* $0 \notin \Gamma \subseteq 1$*, then* $\operatorname{ot} X \geq \omega$.

Proof. Since $0 \notin \Gamma \subseteq 1$, it follows that $\Gamma = \emptyset$. Since any Γ-free for $\beta = 0$ set X has arbitrarily large roots, it must have order type at least ω. $\dashv$

For $1 \leq \beta \leq \omega$, $\Gamma \subseteq \beta + 1$, Y a set of node labeled trees from $\mathcal{T}(\beta)$ and $m < \omega$, define $\rho(\beta, \Gamma, Y, m) := 0$ unless $Y(\langle m \rangle) \neq \emptyset$ is locally Γ-free for β and there is some $(S, C) \in Y$ with $\langle m \rangle \in (S, C)$, and in the latter case, set

$$\rho(\beta, \Gamma, Y, m) := \begin{cases} 1, & \text{if } \Gamma = \emptyset \\ \omega^{\omega^{\ell}}, & \text{if } \Gamma \neq \emptyset \text{ and } \beta = \max \Gamma \text{ limit}, \\ \omega^{\omega^{\mu} \cdot \ell}, & \text{otherwise} \end{cases}$$

where, for non-empty Γ, $\ell := |C(s)| - 1$ for s the least signal node of (S, C), $\mu := |\Gamma| - 1$. This function is well-defined, since if $Y(\langle m \rangle) \neq \emptyset$ is locally Γ-free for β with Γ non-empty, then all elements of $Y(\langle m \rangle)$ have a proper relaxed initial segment in common with (S, C) which must include the least signal node of (S, C).

Let $*(\beta, \Gamma)$ be the following statement.

> $*(\beta, \Gamma)$: For all locally Γ-free for β sets Y, if $\langle m \rangle \in (S, C) \in Y$, then $\operatorname{ot} Y(\langle m \rangle) \geq \rho(\beta, \Gamma, Y, m)$.

10.45 Claim. *For all $\beta \geq 1$ and $0 \notin \Gamma \subseteq \beta + 1$, if X is Γ-free for β and $*(\beta, \Gamma)$ holds, then $\operatorname{ot} X \geq \zeta(\beta, \Gamma)$.*

Proof. Use induction on n to prove the claim for subsets $\Gamma \subseteq \omega$ of size n.

To start the induction, consider subsets of size 0. If X is $\emptyset$-free for $\beta \geq 1$, then by definition, $X(\langle m \rangle)$ is non-empty for infinitely many m, and by commonality and push-up, $\operatorname{ot} X(\langle m \rangle) \geq \omega$, so $\operatorname{ot} X \geq \omega^2 = \zeta(\beta, \emptyset)$.

Next assume the claim is true for subsets of size k and that $n = k + 1$. If X is Γ-free for $\beta \geq 1$ and $0 \notin \Gamma \subseteq \beta + 1$ satisfies $\omega \notin \Gamma$ and $|\Gamma| = k + 1$, then there are arbitrarily large ℓ for which there are $m \in (S, C) \in X$ with $\ell < |C(\langle m \rangle)|$ if $\beta \in \Gamma$ and with $\ell < m$ otherwise. In the latter case, by Γ-forecasting and by signal size, there are arbitrarily large ℓ for which the first signal node $s \in (S, C) \in X$ has $\ell < |C(s)|$. Since $*(\beta, \Gamma)$ holds, it follows that there are arbitrarily large $\ell < m$ with $\operatorname{ot} X(\langle m \rangle) \geq \omega^{\omega^{k} \ell}$ for $k = |\Gamma| - 1$, hence $\operatorname{ot} X \geq \omega^{\omega^{k+1}} = \zeta(\beta, \Gamma)$ as desired.

Therefore by induction, the claim holds for all finite subsets $\Gamma \subseteq \omega$.

To complete the proof, consider Γ with $\omega \in \Gamma$. Then $\beta = \omega$. Suppose X is Γ-free for ω and $\omega \in \Gamma$. Then the root node of every tree in X is a signal node. Also X has arbitrarily large values for $|C(\langle m \rangle)|$ by the definition of Γ-free for $\beta = \omega \in \Gamma$. Hence from $*(\omega, \Gamma)$ it follows that $\operatorname{ot} X(\langle m \rangle) \geq \omega^{\omega^{\ell}}$ for $\ell = |C(\langle m \rangle)| - 1$, so $\operatorname{ot} X = \omega^{\omega^{\omega}} = \zeta(\omega, \Gamma)$ as required. ⊣

10.46 Claim. *For all $\beta \geq 1$ and $0 \notin \Gamma \subseteq \beta + 1$, the statement $*(\beta, \Gamma)$ holds.*

Proof. Suppose Y is locally $\emptyset$-free for $\beta \geq 1$ and $\langle m \rangle \in (S, C) \in Y$. Then by commonality and push-up, $\operatorname{ot} Y(\langle m \rangle) \geq \omega$, so $*(\beta, \emptyset)$ holds.

Use induction on β to show that for all non-empty $0 \notin \Gamma \subseteq \beta+1$, the statement $*(\beta,\Gamma)$ holds. For the basis case, $\beta = 1$, the only case to be considered is $\Gamma = \{1\}$. Suppose Y is locally $\{1\}$-free and $\langle m \rangle \in (S,C) \in Y$. Then $\langle m \rangle$ is a signal node, and $Z := \{\bigcup T : (T,D) \in Y(\langle m \rangle)\}$ is free in $|C(\langle m \rangle)|$ coordinates in the sense of Definition 10.39 by conformity and push-up. Thus Z has order type $\omega^{|C(\langle m \rangle)|}$ by Lemma 10.40. Hence $Y(\langle m \rangle)$ has this order type as well, so $*(1,\{1\})$ holds.

For the induction step, assume $*(\beta')$ is true for all β' with $1 \le \beta' < \beta$. Suppose Γ is non-empty with $0 \notin \Gamma \subseteq \beta+1$, Y is locally Γ-free for β and $\langle m \rangle \in (S,C) \in Y$. It follows that $Y(\langle m \rangle)$ is also locally Γ-free for β. Let (S^-,C^-) be the minimal proper relaxed initial segment of (S,C), required by commonality. Then (S^-,C^-) is a common initial segment of all trees in Y. Let $\langle m,m^- \rangle$ be the unique initial segment of $\max S^-$ of length 2.

Case 1: $\max\Gamma < \beta$ or $\max\Gamma = \beta = \omega$.

For each $(T,D) \in Y$, the *derived tree* $(\hat{T},\hat{D})$ is defined by $\hat{t} \in \hat{T}$ if and only if $\langle m,m^- \rangle \sqsubseteq \langle m \rangle ^\frown \hat{t} \in T$, and $\hat{D}(\hat{t}) = D(\langle m \rangle ^\frown \hat{t})$.

Let Z be the collection of derived trees. Note that $\langle m^- \rangle$ is an element of every tree in Z. Let $\beta' = \beta - 1$ and $\Gamma' = \Gamma$ if β is finite, and let $\beta' = m$ and $\Gamma' = (\Gamma - \{\omega\}) \cup C(\langle m \rangle)$ otherwise. Then $Z = Z(\langle m^- \rangle)$ is locally Γ'-free for β'. Also, $\operatorname{ot} Y(\langle m \rangle) \ge \operatorname{ot} Z(\langle m^- \rangle)$, so in this case, the desired inequality follows by the induction hypothesis.

Case 2: $\Gamma = \{\zeta+1\}$.

Consider the set $E \subseteq \mathcal{T}(1)$ of $\langle m,k_1,k_2,\ldots,k_m \rangle$ such that there is $(T,D) \in Y$ such that for all $1 \le i \le m$, $\langle m,k_i \rangle \in T$. By conformity and push-up, the set E is free in $\ell = |C(\langle m \rangle)| - 1$ many coordinates, so it has order type ω^ℓ, by Lemma 10.40. Thus $\operatorname{ot}(Y(\langle m \rangle)) \ge \operatorname{ot} E = \omega^\ell = \rho(\beta,\gamma,Y,m)$ as required.

Case 3: $\zeta+1 \in \Gamma \ne \{\zeta+1\}$.

Notice that every tree (T,D) in $Y(\langle m \rangle)$ may be thought of as a collection of m node labeled trees from $\mathcal{T}(\zeta)$ extending from the root $\langle m \rangle$.

Call an initial segment (T,D) of a tree in $Y(\langle m \rangle)$ *large* if $\max T$ is a prepartition node with ordinal 0 such that $\sharp(s,T) = \max C(s)$ for all proper $s \sqsubset t$ with $|s| > 1$. Every element of $Y(\langle m \rangle)$ has exactly $|C(\langle m \rangle)|$ many large initial segments.

Let $\Gamma' = \Gamma - \{\zeta+1\}$ and set $\mu = |\Gamma'|$. Fix attention on a large (T,D) for which $\sharp(\langle m \rangle,T) < \max C(\langle m \rangle)$, and let k be the least element of $C(\langle m \rangle)$ greater than $\sharp(\langle m \rangle,T)$. Let $E(T,D)$ be the set of initial segments (T',D') of elements of Y extending (T,D) to a tree with root $\langle m \rangle$ extended by exactly k subtrees from $\mathcal{T}(\zeta)$. Then $E(T,D)$ has order type ω^{ω^μ}, since the collection of trees that occur for the kth slot are Γ'-free for ζ. In fact the set of maximal large initial segments of these trees also has order type ω^{ω^μ}, since each has exactly ω extensions in $E(T,D)$ and ω^{ω^μ} is multiplicatively indecomposable. From this analysis, it follows that $\operatorname{ot} Y(\langle m\rangle) \geq \omega^{\omega^{\mu}\cdot\ell}$, where $\ell = |C(\langle m\rangle)| - 1$, so $*(\beta, \Gamma)$ holds in this final case.

Therefore by induction on β, the claim follows. ⊣

Now the lemma follows from Claims 10.44, 10.45 and 10.46. ⊣

10.47 Lemma. *Suppose h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors and $N \subseteq \omega$ is infinite with $1 < \min N$ and no two consecutive integers are in N. Further suppose a bounding function b and $H \subseteq N$ infinite are such that the builder wins every run of $\mathcal{G}(h, N)$ by playing conservatively for b and H. Then there is a set $Y \subseteq \mathcal{T}(\omega)$ of order type $\omega^{\omega^{\omega}}$ so that $h(S,T) = 0$ for all global pairs from Y.*

Proof. We will use recursion to build a $\{\omega\}$-free for ω set X with the property that every global pair $((S,C),(T,D))$ from X has a coarsening $((S,C'),(T,D'))$ which is a final play in a run of $\mathcal{G}(h,N)$ in which the builder plays conservatively for b and H. (By a *coarsening*, we mean that $C'(s) \subseteq C(s)$ and $D'(t) \subseteq D(t)$ for all $s \in S$, $t \in T$.) Since the builder wins the game, $h(S,T) = 0$ for such pairs. Thus $Y = \{\, S : (\exists C)((S,C) \in X) \,\}$ is the desired set, since, by Lemma 10.43, Y has order type $\omega^{\omega^{\omega}}$.

To start the recursion, let X_0 be the set with only $(\emptyset, \emptyset)$ in it. For positive $j < \omega$, we enumerate the node labeled trees in $\bigcup_{i<j} X_i$ which are proper initial segments, starting with $(\emptyset,\emptyset) = (S'_{j,0}, C'_{j,0})$ and ending with (S'_{j,n_j}, C'_{j,n_j}). Speaking generally, in stage j, for each $k \leq n_j$, we consider the kth initial segment, $(S'_{j,k}, C'_{j,k})$, use moves of the architect and builder in $\mathcal{G}(h,H)$ to create a relaxed or complete extension, $(S_{j,k}, C_{j,k})$, using elements of H larger than anything mentioned up to that point. Then we let X_k be the set of all $(S_{j,k}, C_{j,k})$ for $k \leq n_j$.

A simple induction shows that that there are only finitely many proper initial segments to be considered in each stage and they fall into at most three types: *trivial* (i.e. $(\emptyset,\emptyset)$), *ready for completion* (i.e. a relaxed initial segment (T,D) such that for all $s \subseteq \max T$ whose ordinal is a successor, $\sharp(s,T) = \max D(s)$), or relaxed but not ready for completion.

In stage j, for the trivial initial segment, one starts $\mathcal{G}(h,H)$ at the beginning. Otherwise, for the kth initial segment, one continues a game in which the first tree is $(S'_{j,k}, C'_{j,k})$ and the second tree is the relaxed initial segment constructed to extend $(\emptyset,\emptyset)$ in this stage, namely $(S_{j,0}, C_{j,0})$.

In the games played, the architect uses the following strategy. She always directs the builder to create or extend the first tree. If the architect is making her first move on the kth initial segment and it is relaxed, then she declares the next node label size to be 0 and calls for completion if $(S'_{j,k}, C'_{j,k})$ is ready for completion, and for decision otherwise. Recall that if the architect calls for completion, then the node label of new elements is the empty set. Otherwise, the architect uses the least element of H larger than any used to that point as the size of the next node label, and calls for construction to the next decision node.

The builder always responds conservatively for H, b, and always plays large enough to have the play remain conservative for any possible game that could be constructed using coarsenings of the given trees.

Play stops at the end of the first move by the builder in which he creates a tree $(S_{j,k}, C_{j,k})$ which is relaxed or complete.

In any stage, with any starting initial segment, after finitely many steps of the game, the builder has constructed the required relaxed or complete extension. Since there are only finitely many trees to extend in a given round, eventually each round is finished. Therefore, the construction stops after ω rounds with a set $\overline{X} = \bigcup X_j$ of trees. Let X be the set of complete trees in $\overline{X}$. By construction, X is $\{\omega\}$-free, so by Lemma 10.43, the set $Y := \{ S : (\exists C)((S, C) \in X) \}$ has order type ω^{ω^ω}.

To check that Y is the required set, suppose that $((S^0, C^0), (S^1, C^1))$ is a global pair from the set X with $(S^0, C^0) < (S^1, C^1)$. By the construction, every partition node of $(S^\varepsilon, C^\varepsilon)$ is the maximum of some relaxed segment of $(S^\varepsilon, C^\varepsilon)$, and every splitting node r has $e_\beta(r)$ in $C(\langle m^\varepsilon \rangle)$, where a *splitting node* $r \in S^\varepsilon$ is one of the form $s \cap t$ for distinct partition nodes $s, t \in S^\varepsilon$. Hence there are coarsenings (S^0, D^0) and (S^1, D^1) so that for all $r \in S^\varepsilon$,

$$D^\varepsilon(r) = \begin{cases} \{ \sharp(r, S^\varepsilon(\emptyset, s]) : r \sqsubset s \text{ partition node} \} & e_\omega(r) \text{ successor}, \\ \{ e_\omega(t) : r \sqsubset t \text{ splitting node} \} & e_\omega(r) \text{ limit}, \\ \emptyset & \text{otherwise} \end{cases}$$

Thus $((S^0, D^0), (S^1, D^1))$ satisfies Definition 10.23 and is a global clear pair. If $\max \bigcup S^0 > \max \bigcup S^1$, then the pair is inside, and otherwise it is outside. Use this knowledge in the architect's initial move; use the values of $|D^\varepsilon(r)|$ for the sizes of the node labels in the architect's moves; and orchestrate her moves to create the pair of node labeled trees when the builder is required to use the elements of $\bigcup S^0 \cup \operatorname{ran} D^0 \cup \bigcup S^1 \cup \operatorname{ran} D^1$. Since the architect has no winning strategy, and the builder's plays were large enough for any coarsening, it follows that this run of the game is a win for the builder. Thus $h(S^0, S^1) = 0$ as desired. ⊣

10.7. Completion of the proof

In this subsection, we complete the proof that $\omega^{\omega^\omega} \to (\omega^{\omega^\omega}, 3)^2$ by assembling the appropriate lemmas. We start with $h : [\mathcal{T}(\omega)]^2 \to 2$. We apply the Ramsey Dichotomy 10.32 to h and $N = \omega$ to get $H \subseteq \omega$ infinite, a bounding function b and a favored player.

If the architect has a winning strategy by which she wins $\mathcal{G}(h, N)$ when the builder plays conservatively, then there is a 1-homogeneous triangle by Lemma 10.38.

Otherwise, the builder wins every run of $\mathcal{G}(h, N)$ by playing conservatively, so by Lemma 10.47, there is a set Y of order type ω^{ω^ω} so that all global pairs get color 0. Partition Y into sets Y_n so that $Y_0 < Y_1 < \ldots$, all pairs from Y_n are local, and $\operatorname{ot} Y_n \geq \omega^{\omega^{1+2n}}$. Apply Corollary 9.3 to each Y_n. If for some n, the result is a 1-homogeneous triangle, we are done. Otherwise, we get 0-homogeneous sets $Z_n \subseteq Y_n$ of order type $\omega^{\omega^{1+n}}$, and $Z = \bigcup Z_n$ is the 0-homogeneous set required for completion of the proof of the theorem.

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