CONTENTS

CHAPTER 1

INFINITE COMBINATORICS

Jean A. Larson

1 INTRODUCTION

There is a tension in mathematics between generalization and description in concrete terms. Cantor's extension of number to the transfinite is an example of generalization, and the pull of generalization and the counterbalancing forces are described in the passage that concludes with a famous declaration [Cantor, 1883, 545], "das *Wesen* der *Mathematik* liegt gerade in ihrer *Freiheit*."[1]

> These principles, I think, do not represent a danger for science as some have suggested: first, the conditions under which new numbers can be generated leave little space for arbitrariness; second, every mathematical construct comes with a natural corrective; if it is impractical or uninspiring it will be dropped due to lack of impact. Conversely, any unnecessary constraint on the mathematical research impetus carries the far greater risk—the more so since there is no scientific justification for it; for the *essence* of *mathematics* is its *lack of constraints*.[2]

Concrete structure may be sought through classification schemes, in basis problems, in universal structures, and especially in the search for and expectation of local uniformity, regions of simplicity, and inescapable structure. This aspect is

[1] The quote has been variously translated, e.g. "the essence of mathematics resides in its freedom." This version is the one I first encountered as part of a mural by art students at the University of Florida, and is the one I will be using in the postscript.

[2] (The above translation of the following paragraph was made by Jürg Peters of the University of Florida. When asked why he had translated *Freiheit* as *lack of constraints* rather than *freedom* he said he felt that Cantor's notion of *free* was like that of free groups and that *lack of constraints* better captured that sense.) Es ist, wie ich glaube, nicht nöthig in diesen Grundsätzen irgendeine Gefahr für die Wissenschaft zu befürchten, wie dies von Vielen geschieht; einerseits sind die bezeichneten Bedingungen, unter welchen die Freiheit der Zahlenbildung allein geübt werden kann, derartige, dass sie der Willkür einen äussertst geringen Spielraum lassen; dann aber trägt auch jeder mathematische Begriff das nöthige Correctiv in sich selbst einher; ist er unfruchtbar oder unzweckmässig, so zeigt er es sehr bald durch seine Unbrauchbarkeit und er wird alsdann, wegen mangelnden Erfolgs, fallen gelassen. Dagegen scheint mir aber jede überflüssige Einengung des mathematischen Forschungstriebes eine viel grössere Gefahr mit sich zu bringen und eine um so grössere, als dafür aus dem Wesen der Wissenschaft wirklich keinerlei Rechtfertigung gezogen werden kann; denn das *Wesen* der *Mathematik* liegt gerade in ihrer *Freiheit*.

captured by the following quote, most often used in connection with Ramsey theory: “complete disorder is impossible”. This short sentence is embedded in the following quote from Motzkin [1967]:[3]

> The influence on mathematics of its two neighbors, physics and logic, is sometimes opposite or, at least, complementary. Whereas the entropy theorems of probability theory and mathematical physics imply that, in a large universe, disorder is probable, certain combinatorial theorems show that complete disorder is impossible. Already the “pigeon-hole principle” —out of $t + 1$ objects of t kinds, at least two must be of the same kind—can be interpreted as saying that, while no 2-set (set of two objects) in the given $(t + 1)$-set needs to be composed of objects of different kind (however probable that may be), at least one 2-set of objects of the same kind must occur. A sophisticated generalization of this principle was initiated in [7] by the mathematical logician Ramsey.

The neighbors most relevant to our study are logic and other areas of mathematics. From logic we see that opposite or complementary results may be obtained by extending our basic set-theoretic assumptions in different ways to get consistency results for different aspects of the combinatorial enterprise. Motivated by the Todorcevic paper *Basis problems in combinatorial set theory* [1998a], we shall be especially interested in what he calls *critical members* of classes $\mathcal{S}$ of interest, of which Todorcevic wrote: “Critical objects are almost always some canonical members of $\mathcal{S}$ simple to describe and visualize.”

The Pigeonhole Principle,[4] which was mentioned in the Motzkin quote, is our first example of an assertion of the lack of disorder. It is a fundamental idea that has been around for a long time, and the principle itself is attributed to Dirichlet in 1834, where he used the name *Schubfachprinzip*.[5]

These themes play out in this decade-by-decade review of infinite combinatorics in the 20th Century. The foci are the study of orderings, especially trees, and Ram-

[3] Hans Jürgen Prömel [2005, 3] used the short quote as a theme in the life of Walter Deuber. Theodore S. Motzkin (1908 – 1970) spent the later part of his career at the University of California, Los Angeles.

[4] An early use of the English “pigeonhole principle” occurs in Raphael Robinson’ paper [1941] and the first mention found by searching MathSciNet on December 8, 2010 for the word “pigeon” is the Erdős-Rado paper [1956], where “pigeon-hole principle” is offered as a translation of “the box argument” or “the chest of drawers argument” with “Schubfachprinzip” in parenthesis, and attributed (incorrectly) to Dedekind. Those familiar with pigeon hole desks, sometimes used for sorting letters, recognize the image that almost surely inspired this translation, but those unfamiliar with them may think of the pigeon houses of Iran and Egypt, with their many entrances, as is suggested by one Italian translation, *principio da casa dos pombos*, which appears on Miller’s site (see the next footnote). It would not be surprising if a perusal of British popular articles on mathematics and mathematical puzzles turned up much earlier references. Literary references to the concept go back still further, with an example dating to the 1600s appearing on http://pballew.blogspot.com/2009/06/updating-history-of-pigeon-hole-theorem.html.

[5] It is difficult to ascertain the first usage of any idea, symbol or word. The web page for letter P http://jeff560.tripod.com/p.html of Jeff Miller’s site on *Earliest Known Uses of Some of the Words of Mathematics*, downloaded on December 9, 2010 makes a case for Dirichlet’s priority.

sey theory, especially the partition calculus. Along the way, results about almost disjoint sets, set mappings, transversals, and regressive functions are discussed, and a wide variety of combinatorial tools are mentioned as they come into play. Lists of a variety of results are briefly described to capture the breadth and flavors of different decades. Occasionally conferences and applications to or from infinite combinatorics are described to indicate some of the ways in which the field has benefited by interactions with communities of mathematicians in finite combinatorics, model theory, topology, and other aspects of set theory, including large cardinals and forcing.

My goal in writing this chapter was to be inclusive, that is, to acknowledge multiple discoveries, rediscoveries,[6] and inventions of ideas. I apologize in advance for the omissions and incorrect statements that surely appear in this wide ranging survey. I have not concerned myself greatly about who had the first seed leading to each discovery, nor with assessing the quality of the variety of contributions. Let me state my criteria for attribution of results: (1) they must be directly stated; (2) they must be proved and published in a reputable journal, or in rare cases vetted by someone other than the author or explicitly ascribed by others, usually when the idea of proof has been generalized. This approach is informed by having listened to Paul Erdős describe how he decided at what point and to whom money should be awarded for the solutions to certain of his problems.

1.1 Overview of the history of orderings

A broad overview of the history of orderings starts with foundational work on cardinal and ordinal numbers, including the Schröder-Bernstein Equivalence theorem, fundamentals of cardinal and ordinal arithmetic, and Zermelo's Well-Ordering Theorem. An early treatment in English was Huntington's book on the continuum as a type of order [1905c].

The first systematic study of linear and partial orderings was carried out by Hausdorff, who set the stage at the beginning of century with his first mathematical paper [1901] in which he generalized ordinals. He investigated the family ${}^{\alpha}M$ of functions from an ordinal number α into a linearly ordered set M under various orderings including the lexicographic ordering and eventual domination. Hausdorff identified critical elements, found bases, generating sets, universal linear orders[7] for scattered linear orderings of bounded cardinality, showed every partial order has a maximal well-ordered subset, and set up and studied a classification scheme

[6]At the Scottish Book conference held in Denton, Texas in 1979, Paul Erdős [1981, 36] shared his perspective on rediscovery: "Ulam and I settled many of these questions long ago, but we never got around to publishing our results. These results were rediscovered and published by the Indian mathematician B. V. Rao; when Rao sent me a preprint I urged him to publish. Naturally, I did not tell him that Ulam and I had already done the work. Eventually he found out, however, and asked me why I hadn't said anything about it. I replied that this was the one respect in which I did not want to imitate Gauss, who had a nasty habit of 'putting down' younger mathematicians by telling them he had long ago obtained their supposedly new results."

[7]A linear order $(P, <)$ is *universal for a class* $\mathcal{K}$ of linear orders if every element of $\mathcal{K}$ is order-embeddable into $(P, <)$.

for linear orders that led him to identify critical types, such as what have come to be known as Hausdorff and Rothberger gaps.[8]

In the decade 1910–1920, the Moscow school, founded by Egorov and Lusin, was put on the map by results like Lusin's CH construction of an uncountable set of reals whose intersection with every meager set is countable, Alexandroff's proof that every uncountable Borel set includes a perfect set (independently found by Hausdorff), and Suslin's construction of a non-Borel analytic set. Moore developed postulates for analysis-situ or point-set theory that led to the development of the Texas school of topology. They were the initial source of the axioms he used in his distinctive teaching style: he doled out axioms, a few at a time, and guided the students to deduce consequences from them. Sierpiński published *An Outline of Set Theory* [1912], an early systematic treatment of set theory, and worked on consequences and equivalences of the Continuum Hypothesis. Hausdorff wrote an influential set theory book, *Grundzüge der Mengenlehre* [1914], which included some of his work on linear and partial orders and introduced topology in terms of neighborhoods. Subsequent editions omitted most of the material on orderings in favor of Borel sets, analytic sets and Baire functions, and his pioneering work on ordered sets faded into obscurity for a time.

In the 1920's, mathematics flourished in Poland with schools in Lwów (Banach, Steinhaus, Ulam) and Warsaw, where Janiszewski, Mazurkiewicz, Sierpiński founded *Fundamenta Mathematicae*, an international journal which focused on foundations, point sets and functions and whose first issue appeared in 1920. That issue included a problem (rephrased below in modern language) that was influential in the theory of order and allied areas across the century.

> *Suslin's Problem:* Must every complete, dense linearly ordered set without endpoints for which every pairwise disjoint set of intervals is countable be a copy of the real line?

Also in the 1920's, Dénes Kőnig, working within a graph-theoretic perspective, proved the first result about infinite trees, now known as *König's Lemma* or *König's Infinity Lemma.*[9] Moore constructed his automobile road space essentially using the complete binary tree of sequences of zeros and ones of length $\leq \omega$. Urysohn proved that every ordered set whose increasing chains and decreasing chains are at most countable has power at most the continuum. Sierpiński introduced almost disjoint sets which he and Tarski investigated, and von Neumann introduced his representation of Cantor's ordinal numbers and put transfinite recursion on a firm foundation.

In the 1930's, Marczewski proved that all partial orders can be extended to total (linear) orders; and Mostowski constructed a denumerable partial order universal[10]

[8] See the chapters by Kojman and Steprāns for more on gaps.

[9] König's Theorem refers to the cardinal inequality of Dénes Kőnig's father Gyula, who published under the name Julius Kőnig.

[10] A partial order $(P, <)$ is *universal for a class* $\mathcal{K}$ of partial orderings if every partial order in $\mathcal{K}$ has an order and incomparability preserving embedding into $(P, <)$.

for countable partial orderings. Kurepa made a systematic study of uncountable partial orders, especially trees in the wider framework of ramified sets. He identified the key parameters for a structural analysis of uncountable trees: height, supremum of the sizes of the levels, and cardinality of the set of branches whose length is the height of the tree. In his thesis, he developed the framework within which Aronszajn constructed the first *Aronszajn tree*.[11] Rather than include Aronszajn's construction, Kurepa gave his own construction of an Aronszajn tree as a subtree of $\sigma\mathbb{Q}$, the non-empty bounded well-ordered sets of rationals under end-extension. He extracted the combinatorial content of Suslin's Problem when he showed that a negative answer was equivalent to the existence of a Suslin tree.[12] He constructed the first *special* Aronszajn tree, i.e. one which is a countable union of antichains. The distinction was later recognized as a critical one. Kurepa also showed that Aronszajn trees are special if and only if they have a strictly increasing map into the rational numbers. With his First Miraculous Problem, he started the study of isomorphism types of Aronszajn trees. He studied partial order more generally, in particular monotone functions on them, and proved his Fundamental Relation, an inequality for partial orders bounding the cardinality of the underlying set in terms of the powers of their chains and antichains.

A special Aronszajn tree is implicit in the 1933 construction by Jones of a Moore space which is not metrizable, but this construction was not announced or published until much later. Jones proved his metrization theorem and introduced the Normal Moore Space Conjecture in [1937], which stimulated research related to trees by topologists.

In the 1940's, Dushnik and Miller introduced a notion of dimension for partial orders, investigated endomorphisms of linearly ordered sets, constructed an uncountable rigid set of real numbers, and gave conditions that guarantee that a partial order has a large linearly ordered subset. Kurepa proved that every uncountable subset of $\sigma\mathbb{Q}$ includes an uncountable antichain. Sierpiński constructed a family of $2^{\mathfrak{c}}$ many pairwise non-isomorphic linear orderings each of the cardinality $\mathfrak{c}$ of the continuum. Erdős and Tarski generalized the notion of almost disjointness to partial orders and applied their results to questions related to Kurepa's work. The first steps were taken toward the development of a theory of well-quasi-orders (wqos). Fraïssé's conjecture that there is no infinite strictly decreasing sequence of denumerable order types under the relation of order embeddability became a motivating question in this area. Specker generalized the work of Aronszajn and Kurepa by using GCH to construct κ^+-Aronszajn trees for regular κ. Kurepa focused attention on trees of height ω_1 with more than $\aleph_1$ branches, now known as *Kurepa trees*, and continued the study of subtrees of $\sigma\mathbb{Q}$ and monotone mappings into $\mathbb{Q}$ and $\mathbb{R}$. Knaster introduced what has come to be known as the *Property of Knaster* and proved that for topological spaces this property is equivalent to

[11] An Aronszajn tree is a tree of height ω_1, all of whose levels are countable which has no uncountable branch.

[12] A *Suslin tree* is a tree of height ω_1 in which every branch and every antichain is countable.

being separable,[13] while Marczewski asked if the countable chain condition implies the Property of Knaster. Miller and Sierpiński came up with equivalences in terms of partial orders to the existence of a Suslin line, seemingly unaware of Kurepa's equivalence with the existence of Suslin trees, and Maharam used the existence or non-existence of a Suslin continuum in her investigation of the existence or non-existence of measures on countably complete Boolean algebras, in essence, constructing a Suslin tree from a Suslin continuum as part of one of her arguments.

In the 1950's, interest in Suslin lines and their applications continued: Rudin constructed a Dowker space from a Suslin line; Kurepa showed that the Cartesian square of a Suslin line (level-wise product) is not the countable union of antichains, i.e. does not have the countable chain condition. Sierpiński, Shepherdson, Ginsburg and Johnston, among others, investigated the structure of linear and partial orderings under embeddability. Bing made progress on the Normal Moore Space Conjecture by proving that if there is a Q-set, then there is a separable non-metrizable Moore space.

In the 1960's, the introduction of forcing, Martin's Axiom and the combinatorial principle $\diamondsuit$ and its variants led to an explosion of results, as did the detailed analysis of the fine structure of the constructible universe. By the end of the decade, there were consistency and independence results for Kurepa and Suslin trees, and it was known that, under CH, there are many isomorphism types of Aronszajn trees. The decade featured a catalog of examples of well-quasi-orders (wqos), the invention of better-quasi-orders (bqos) and the study of their closure properties. In particular, the following families of structures were shown to be wqos under structure-preserving embeddings: trees (Kruskal for finite, Nash-Williams for infinite) and σ-scattered linear orders (Laver). Hanf, building on work by Erdős, showed that the generalization to regular uncountable cardinals of König's Lemma, i.e. the tree property, is equivalent to a partition property. Halpern and Läuchli proved a partition theorem for products of finite trees, and an analogous theorem for infinite perfect trees was proved independently (and later) by Laver.

In the 1970's, questions about rigidity, automorphisms and other structural properties of Aronszajn trees and Suslin trees were investigated under various hypotheses. While the concept of special tree dates back to the 1930's, the use of the word "special" for this concept entered the language in the 1970's. Consistency and independence results were proved on the existence or non-existence of special Aronszajn trees and their generalizations to successor cardinals; such results were also proved for trees which have a strictly increasing map into the reals, and for combinations of the various trees under consideration, including Kurepa trees. Milliken proved a Ramsey theorem for finitely branching trees. Basis questions for uncountable linear orders were studied and consistency was shown for the statement that all $\aleph_1$-dense sets of reals are isomorphic. A linear order of size $\aleph_1$ universal for orders of size $\aleph_1$ was shown to be consistent with $2^{\aleph_0} = \aleph_2$. Shelah constructed a Countryman line, an important example of an uncountable linear

[13] A space is separable if it has a countable dense subset.

order.

In the 1980's, Todorcevic surveyed work on trees from combinatorial and set-theoretic perspectives. The elegance of his presentation drew a wide audience for this work. Work continued on exploring consistent possibilities for various types of trees, now looking for results for trees on multiple cardinals, or with required or forbidden types of subtrees. The structure of trees was examined through a variety of lenses: a notion of base, club isomorphism, club embeddability, nearness, existence of a universal Aronszajn tree for club embeddability. Shelah used a complex combinatorial argument to show that adding a Cohen real also adds a Suslin tree; Todorcevic later came up with an elegant construction as part of his introduction of a family of trees constructed from coherent mappings that have become standard examples in the subject. Existence and non-existence of κ^+-Suslin trees were proved for a variety of values of κ in the presence of different values for the continuum. Baumgartner surveyed work on uncountable linear orders, prompting further work in this area. For example, two $\aleph_1$-dense sets of reals were constructed for which provably there is no C^1 isomorphism.

In the 1990's, several consistency results for the existence and non-existence of λ-Aronszajn trees for $\lambda \geq \omega_2$ were found. Cardinality conditions and combinatorial assumptions were found that entail the existence of Suslin trees on successor cardinals larger than $\aleph_1$ which are immune to specialization in cardinal and cofinality preserving extensions. A Suslin tree was constructed from $\clubsuit$ and the assumption that the meager idea $\mathcal{M}$ has a base of size $\aleph_1$. The tree $\sigma\mathbb{Q}$ of all non-empty bounded well-ordered subsets of $\mathbb{Q}$ was shown to be universal (as a partial order) for all Hausdorff trees with strictly increasing embeddings into the reals. Todorcevic and Väänänen studied the quasi-order of all trees of cardinality ω_1 with no uncountable branches under the relation $S \leq T$ if and only if there is a strictly increasing map from S to T (it need not preserve the tree structure). Todorcevic announced a Ramsey-theoretic reformulation, under the Proper Forcing Axiom,[14] for the existence of a two-element basis for the class of Aronszajn lines. Abraham, Rubin and Shelah investigated consistent finite possibilities for the structure of order types of $\aleph_1$-dense homogeneous subsets of the reals under the embeddability relation. Kojman and Shelah investigated the existence problem for universal linear and partial orders in various cardinalities;[15] Todorcevic and Väänänen proved that the class of linear orderings without uncountable well-ordered or conversely well-ordered subsets has no universal element. The possibilities for order types of the form $\mathbb{R} \cap M$ for M a model of set theory were investigated.

[14]The Proper Forcing Axiom (PFA) is a generalization of Martin's Axiom which will be introduced in §8.

[15]A linear (partial) order $(P, <)$ is *universal in power* λ (alternatively *in cardinality* λ) if it has cardinality λ and is universal for the class of linear (partial) orders of cardinality $\leq \lambda$.

1.2 Overview of the history of Ramsey theory

Now let us turn to an overview of the history of Ramsey theory in the 20th Century with a focus on the partition calculus. Conventionally the history is started with Ramsey's famous theorem proved in the late 1920's and published in 1930.[16] In the latter half of the 20th Century, the influence of the habit of thinking "complete disorder is impossible" has spread beyond the communities of combinatorial mathematicians, set theorists, and those studying order whether algebraically, combinatorially or as a special case of binary relations to include set-theoretic topology, semi-groups, dynamical systems, ergodic theory, and Banach space theory. It makes sense to include some of the historical roots of those directions, albeit with only cursory attempts to describe their mathematical interconnections, which can be quite complicated.[17]

Following Hindman in his article [2006] on the mathematics of Rothschild and Graham, we give a short prehistory of Ramsey theory, starting with the result of Hilbert [1892] that for any positive integer n, if the positive integers are partitioned into finitely many cells, there is a natural number a and a finite sequence $\vec{m} = \langle m_1, m_2, \ldots, m_n \rangle$ of n positive integers such that all integers of the form $a + b$ where b is a finite sum of some or all of the integers in $\vec{m}$ are in the same cell of the partition.[18] Next, we state Schur's Theorem [1917] which is Hilbert's Theorem without the a and with $n = 2$: whenever the positive integers are finitely colored, there are positive integers x and y with x, y and $x + y$ all the same color. We conclude the pre-history with van der Waerden's Theorem [1927] that whenever the positive integers are finitely colored, there are arbitrarily long (finite) monochromatic arithmetic progressions.

We start the history in the 1920's with Ramsey's own work by stating the theorem for which he is known by set theorists, and which we call *Ramsey's Theorem*: for finite r and any finite partition of the r-element subsets of an infinite set, there is an infinite subset all of whose r-element subsets belong to the same cell of the partition. He actually used a finite version, which we include as well, naming it *Finite Ramsey's Theorem*. Urysohn and Alexandroff proved that every regressive function on the countable ordinals is constant on an uncountable set, which can be regarded as partition theorem for the singletons of ω_1 for a special type of partition.

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[16]I remember hearing in the 1970's that Ramsey's paper was one of the most cited works in the mathematical literature.

[17]See, for example, the second chapter of *Combinatorics: the Rota way* [Kung *et al.*, 2009] on matching theory, which discusses some of the mathematical connections between Kőnig's Theorem of 1916, Hall's Marriage Theorem, and more of Rado's work on finite systems than is included in this chapter. Kung, Yan and Rota connect matching theory to graph theory, matroids, linear algebra, submodular functions, and transversal theory. It is the latter which is of particular interest in this chapter which includes beautiful work by Milner and Shelah on infinite and uncountable transversal theory and the deep theorem of Galvin and Hajnal on almost disjoint transversals and cardinal arithmetic which was one of the precursors of Shelah's pcf theory.

[18]This overview is the only place where Hilbert's result is mentioned since it predates the 20th Century.

In the 1930's, Dushnik generalized the Alexandroff-Urysohn result to successor cardinals; Skolem rediscovered the Finite Ramsey's Theorem in [1933]; and Erdős and Szekeres rediscovered it on their way to proving a theorem in geometry. Rado introduced the concept of *k-regular* linear systems, and, in what we call Rado's Theorem for Linear Systems with Positive Integer Coefficients, he generalized Schur's Theorem and van der Waerden's Theorem.

In the 1940's, Rado generalized his analysis of k-regularity of linear systems to ones with coefficients from a ring of complex numbers. We discuss the generalizations of Ramsey's Theorem to the uncountable in terms of graphs, the pairs being represented as edges. Dushnik and Miller, with help from Erdős for singular cardinals, proved that for every graph on κ vertices, either there is independent subset of size κ (i.e. no pairs joined) or it has an infinite complete subgraph (i.e. every pair is joined in the subgraph). This theorem grew out of their work on partial orders. Erdős proved that if the edges of the complete graph on more than κ^κ many vertices are colored with κ many colors, then there is a complete subgraph on κ^+ many vertices all of whose edges have the same color. He used GCH to prove that any graph on κ^{++} many vertices either has complete subgraph of size κ^{++} or has an independent set (no edges) of size κ^+. Kurepa proved a related result in 1950 which is implicit in his [1939]: any graph on more than κ^λ vertices either has a complete subgraph on κ^+ vertices or an independent set of λ^+ on vertices. Neither [Erdős, 1942] nor [Kurepa, 1939] received much attention in the 1940's, due to disruptions of World War II and because of the likely related fact that the Erdős paper was published in an obscure Latin American journal. The connection between partition relations and ramification methods (tree methods) was made explicit in a paper of Erdős and Tarski [1943] but proofs did not follow until much later.

In the 1950's, Erdős and Rado initiated Ramsey theory for equivalence relations by identifying families of *canonical partitions* for the n-element subsets of $\mathbb{N}$. Erdős and Rado published a series of papers on what became the partition calculus, summarizing their findings in their joint paper [1956], the first systematic study of the area. This paper delineated the main lines of study, detailed a wide variety of partial results, and provided the important Positive Stepping Up Lemma for transferring results for r-element sets to ones for $(r+1)$-element sets. The modern Erdős-Rado Theorem follows from this lemma. Investigation of variations such as ordinal and polarized partition relations were initiated. Rado's Rectangle Refining Theorem is a canonical polarized partition relation. Specker proved interesting partition relations for ordinals of the form ω^n for finite $n \geq 2$. With multiple individuals contributing, the Regressive Function Theorem reached its standard form. In 1958, Mostowski and Tarski had a seminar that revisited the problems and results for inaccessible cardinals from [Erdős and Tarski, 1943]. By the end of the 1950's the peripatetic Hungarian Erdős and German born Rado, whose academic career was spent in England, were joined by Hungarian Hajnal and by London born Milner who spent most of his career in Calgary, Canada. The group formed the core of a powerful collaborative team that welcomed the participation of a wide group of mathematicians around the world in the study of what was sometimes called "Hungarian" set theory.

In the 1960's, the Canonization Lemma and the Negative Stepping Up Lemma for the partition calculus joined the Positive Stepping Up Lemma in the "Giant Triple Paper" of Erdős, Hajnal and Rado. Years in the making, this classic paper included a nearly complete description, under GCH, of partition relations for cardinals for pairs, and major results for larger r-tuples. It introduced square bracket partition relations for strong counter-examples to ordinary partition relations. The visit by Hajnal to Berkeley in 1964, the appearance of the triple paper in 1965, and the first international meeting after the development of forcing held in 1967 at the University of California, Los Angeles, introduced problems in the partition calculus to an audience with new techniques to apply. The Milner-Rado Paradoxical Decomposition showed that for any uncountable ordinal $\beta < \omega_2$ there is a partition of β into countably many pieces so that no piece has a set of ordinals of order type ${\omega_1}^{\omega}$.

The Halpern-Läuchli Theorem, a beautiful partition theorem about finite products of finitely branching trees, appeared in 1967, making a new connection between two themes, trees and Ramsey theory. The Hales-Jewett Theorem on variable words generalized van der Waerden's Theorem. Nash-Williams, in his quest to prove that the collection of infinite (height at most ω) trees have no infinite antichains and no infinite descending sequences under embeddability, proved a lemma which became known as the Nash-Williams Partition Theorem. Square bracket and polarized partition relations were proved for η, the order type of the rationals. At the end of the 1960's, theorems for Borel and analytic partitions of the infinite subsets of ω were proved, and relative to the existence of an inaccessible cardinal, a Ramsey Theorem for all infinite subsets of ω was shown consistent.

In the 1970's, further extensions, alternate proofs, and applications of theorems on partitions of infinite subsets of ω proliferated, with Ellentuck giving a particularly nice topological approach. Hindman's Finite Sums Theorem published in 1974 was the beginning of the exploration by Hindman of semi-group colorings and another example of a theorem that grew from a result in finite combinatorics, namely the Graham-Rothschild Parameter Sets Theorem. Techniques pioneered in the 1960's and early 1970's were applied to questions about partition relations, with singular cardinals and ordinal partition relations receiving increased attention. The Baumgartner-Hajnal Theorem is a spectacular case: for any coloring of the edges of a graph on ω_1 with red and blue and any countable ordinal α, there is a monochromatic subgraph (all edges red or all edges blue) whose set of vertices has order type α. Chang proved that ω^ω is a partition ordinal, i.e. has a triangle-free graph with no independent set of the full order type, and Hajnal proved that under the Continuum Hypothesis (CH), ${\omega_1}^2$ and $\omega_1 \cdot \omega$ are not. Shelah proved a canonization theorem useful for proving partition theorems for singular cardinals, and Milliken and Blass proved partition theorems for trees.

In the 1980's, Erdős, Hajnal, Máté and Rado published a compendium of results in the partition calculus, the culmination of decades of work. Strengthenings and generalizations of the Baumgartner-Hajnal Theorem for ω_1 were proved consistent, and progress was made on partition relations for triples with a meta-mathematical proof. The Baumgartner-Hajnal Theorem for order types was generalized to trees and partial orders in general. Consistency results were proved for cardinal and ordinal partition relations, including the determination whether certain ordinal products $\kappa \cdot \lambda$ for $\omega \leq \lambda < \kappa$ were or were not partition ordinals under a variety of assumptions. Todorcevic proved a startling square bracket partition result for the uncountable and introduced new technology whose ramifications are still unfolding, and proved a stepping up lemma for negative square bracket partition relations. A brief sketch of work extending the results of Galvin, Prikry, Silver, Mathias, Ellentuck, Hindman and Graham and Rothschild is given. Laver published his proof that the perfect subtree version of the Halpern-Läuchli Theorem holds for the product of infinitely many trees building on work that dated back to 1969.

In the final decade of the 20th Century, Schipperus proved that there is an uncountable family of countable partition ordinals. There were new results for triples in both ordinary and square bracket partition relations. Moreover continuous colorings witnessing negative square bracket partition relations were found. The Erdős-Rado Theorem was strengthened to the Baumgartner-Hajnal-Todorcevic Theorem in both balanced and unbalanced formulations, and consistency results about possible extensions for the Dushnik-Miller Theorem (Komjáth) and the Erdős-Rado Theorem (Foreman and Hajnal) were explored. The inequality $\kappa^* = \aleph_\omega < 2^{\aleph_1}$ for the least cardinal κ^* satisfying a surprisingly simple looking polarized partition relation was show to be equiconsistent with the existence of a measurable cardinal.

The introduction concludes with the remark that the next section covers three decades. This decision has the pleasant effect that the 1930's occur in Section 3, the 1940's in Section 4, and so forth. Moreover, subsections, for example of the 1970's, are numbered 7.1, 7.2. For an introduction to the history of order prior to 1900, see [Alvarez, 1999, 185-188].

2 1900-1930: BEGINNINGS

Fundamental understanding of functions and of "nice" sets (Borel sets, Borel and Baire functions, nowhere dense sets, perfect sets, measurable sets and measurable functions) was developed at the end of the 19th Century and the beginning of the 20th Century as detailed in Kanamori's first chapter.

Georg Cantor had introduced the *order type* of a linearly ordered set A by abstraction to $\overline{A}$, which denotes what remains when the set is considered only with respect to the order of its elements.[19] Cantor used $\overline{\overline{A}}$ to denote what remains

[19] With this notation, two ordered sets A and B have the same order type, $\overline{A} = \overline{B}$, if and only if there is an order-isomorphism between them. Thus $\overline{A}$ may be regarded as the class of linear orders which are order isomorphic to A. In modern notation, the order type of a set A is usually denoted by one of the following: ot(A), otp(A), tp(A). In the classification of orderings, it is

when even the order of the elements is no longer considered.[20] Cantor's choice of η to symbolize the order type of the rationals and ω to symbolize the order type of the non-negative integers were in place, as was the notion of everywhere dense set.

In the first decade of the 20th Century, David Hilbert [1900] enumerated his famous list of problems, including Cantor's Continuum Problem, which is of particular relevance for infinite combinatorics, and Ernst Zermelo [1904] published his Well-Ordering Theorem, the foundation for myriad recursive constructions in set theory. Henri Lebesgue published his classic work [1905] in the theory of definability, described by Moschovakis [2009, 1] as follows:

> It introduced and studied systematically several natural notions of definable functions and sets and it established the first important hierarchy theorems and structure results for collections of definable objects.

Edward Huntington[21] [1905d] developed postulates for the order type of the continuum and wrote two expository papers [1905a], [1905b] which were reprinted by Harvard University Press as a short text [1905c] with a new title, presumably for use by students and other researchers.[22] Veblen opened his review of this work with the following statement [1906, 302]:

> The *Annals of Mathematics* has for some time followed a plan of printing articles expository of subjects which are little known or not easily accessible in the English language. Reprints of these articles are then placed on sale with the double and laudable purpose of making the circulation of the article wider than it would otherwise be and of helping solve the difficult problem of financing a mathematical journal.

Veblen praised the clear style and balance between generality and detail, enumerated the main contents (ordinal theory of integers, rational numbers, and the continuum, with an appendix on the transfinite numbers of Cantor), and mentioned the price, fifty cents. He noted that the text was intended for non-mathematical

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useful to work up to equivalence under the relation of being order-isomorphic, i.e. having the same order type.

[20]With this notation, for two sets A and B, the relation $\overline{\overline{A}} = \overline{\overline{B}}$ holds if and only if there is a bijection between A and B. In modern notation we write $|A| = |B|$.

[21]Edward Vermilye Huntington (April 26, 1874 – November 25, 1952) earned an A.B. in 1895 and an A.M. in 1897, both from Harvard University. He was an instructor in mathematics at Williams College 1897–1899, then studied in Europe, receiving his doctorate in 1901 from the University of Strasbourg (at the time it was part of Germany). He spent the remainder of his academic career at Harvard University, returning in 1901 as an instructor of mathematics. He was promoted to assistant professor in 1905, to associate professor in 1915, and became a full professor of mechanics in 1919. He retired in 1941. He was elected to the American Academy of Arts and Sciences in 1913. To learn more about his life and his work in the foundations of mathematics, see Michael Scanlan's biography [1999].

[22]There was sufficient demand that Huntington updated his brief book on order in [1917].

readers as well as for mathematicians, and pointed out that while it assumes little background knowledge, "it requires for complete comprehension a considerable maturity in abstract reasoning."

Strides were made in working out cardinal arithmetic. Felix Bernstein,[23] as a gymnasium student, participated in Cantor's Halle seminar, and his interaction with Cantor led to his proof of what we shall call the *Schröder-Bernstein Equivalence Theorem*:[24] If there is an injection from a set A into a set B and an injection from B into A, then there is a bijection between them. In his dissertation [1901][25] at Göttingen under Hilbert, Bernstein gave two proofs that the set of countable order types has power the continuum and gave what Plotkin [2005, 8] speculates is the first proof that the order type of the rationals is universal in the sense that all countable order types embed in it. Bernstein [1908] also proved that any Euclidean space may be decomposed into two disjoint totally imperfect sets (i.e. uncountable sets which have no perfect subsets).

An early result in the theory of partial orders introduced the study of chains in partial orders. Gerhard Hessenberg[26] [1906] considered subfamilies of power sets ordered by inclusion and studied general ways to construct well-ordered subsets of them. He stated and proved a fixed point theorem in this connection.[27]

Hessenberg defined the *natural sum* and *natural product* of ordinals which are sometimes called the Hessenberg sum and Hessenberg product. These operations are commutative and associative and the product distributes over the sum. So for example, the natural sum of 2 and ω is $\omega + 2$ which is larger than $2 + \omega = \omega$, and

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[23]Felix Bernstein (February 24, 1878 – December 3, 1956) became an extraordinary professor at the University of Göttingen in 1911, received a medical exemption for military service during World War I, and was made head of the statistical branch of the Office of Rationing in Berlin. After the war he continued in government service, and in 1921 became both Commissioner of Finance and ordinary (full) professor at Göttingen, where he founded the Institute of Mathematical Statistics. He visited Harvard University in 1928, and after he lost his position at Göttingen in 1934 during the Nazi regime, he emigrated to the United States, where he taught at a number of different institutions until his retirement in 1948, when he returned to Göttingen where he was appointed professor emeritus. In the United States his research input was reduced, his own employment irregular, and he sought to help other scientists who had fled Europe in the face of Nazi policies find jobs. While his early work was in set theory, his later work was in applied mathematics, especially statistics, and included work in the mathematical theory of genetics. For more information, see the biography at http://www-history.mcs.st-andrews.ac.uk/Biographies/Bernstein_Felix.html by J. J. O'Connor and E. F. Robertson and the brief article by M. Frewer [1981] which includes a bibliography of Bernstein's work.

[24]The Schröder-Bernstein Equivalence Theorem is sometimes called the Cantor-Bernstein Theorem (e.g. see [Sierpiński, 1965, 34], [Jech, 2003, 28]). See [Kanamori, 2004, Section 4] to learn more about this theorem, and see Dedekind [1968, 447–449] for an early unpublished proof.

[25]A memorable mistake in Bernstein's thesis led to trouble for Julius Kőnig at the 1904 International Congress of Mathematicians as detailed in Kanamori's first chapter and Kojman's chapter.

[26]Gerhard Hessenberg (August 16, 1874 – November 16, 1925) received his 1899 doctorate from the University of Berlin where he studied under Hermann Schwarz and Lazarus Fuchs. He is known for a proof in projective geometry that Desargues' Theorem follows from Pascal's Theorem, and, in linear algebra, Hessenberg matrices are named for him.

[27]This fixed point theorem was rediscovered by Bourbaki [1939, 37] for ordered sets in general with essentially the same proof. For details see [Felscher, 1962].

the natural product of 2 and ω is $\omega \cdot 2 > 2 \cdot \omega = \omega$. Hessenberg proved a division algorithm for arbitrary ordinals and proved that every ordinal can be expressed uniquely in Cantor normal form. The *Cantor normal form* of an ordinal α is its unique representation as $\alpha = \omega^{\beta_0} c_0 + \omega^{\beta_1} c_1 + \dots \omega^{\beta_{k-1}} c_{k-1}$ where $\beta_0 > \beta_1 > \cdots > \beta_0 \geq 0$ and $k, c_0, \dots, c_{k-1}$ are positive integers.

2.1 Hausdorff

Felix Hausdorff[28] analyzed order types in the early part of his mathematical career[29] and touched on many themes that will recur: generalizations of well-orderings, bases for order types, order types universal for nicely defined families, and the study of well-ordered sequences of real numbers.

Hausdorff's first paper [1901] on order types examines a generalization of well-orderings called *graded order types*, where a linear order is graded if no two of its initial segments are *similar*[30] to each other. He analyzes types through decompositions of ordered sets into initial, middle and final pieces, a theme he returns to in later papers. In particular, he generalizes the result he calls the Cantor-Bernstein Theorem, namely that the collection of countable order types has the cardinality of the continuum,[31] to show that the collection of all graded types of cardinality $\mathfrak{m}$ where $\mathfrak{m}^2 = \mathfrak{m}$ has cardinality $2^{\mathfrak{m}}$. On his way to the proof, Hausdorff notes that given a set M of cardinality $\mathfrak{m}$, the power $2^{\mathfrak{m}}$ is the cardinality of the set of all two-valued functions (Cantor's "coverings") with domain M (Hausdorff uses $+1$ and -1) and is also the cardinality of the set of all subsets of M, via the correspondence matching a function f with the subset of points where f takes value $+1$.

In a five-part series of articles sharing the title *Investigations into order types* [1906], [1907], Hausdorff begins a systematic study of ordered sets, moving beyond the study of ordinal numbers, cardinal numbers, their powers, and the order types of subsets of the reals. In the first part, *The powers of order types*, he develops

[28]Felix Hausdorff (November 8, 1868 – January 26, 1942) completed his 1891 doctorate in applications of mathematics to astronomy at Leipzig University, and his Habilitation thesis was in the same area. He was also a writer with interests and a wide range of friends in art, literature, and philosophy, and had plays performed and published under the pseudonym Paul Mongré. Hausdorff began working in pure mathematics around the turn of the century, became an extraordinary professor at Leipzig in 1902, and moved to Bonn in 1910 where the mathematical environment was richer and he became friends with Eduard Study. Hausdorff became an ordinary professor in 1913 at Greifswald where Study had formerly been an ordinary professor. His final professional move was back to Bonn in 1921 where he worked until 1935 when he was forced to retire by the Nazi regime. With his suicide in 1942 in the face of a forced move to Endenich, the world lost a remarkable man.

[29]Hausdorff's beautiful work on order types has been made more accessible by a careful translation by Jacob Plotkin [2005].

[30]Two ordered sets are *similar* if there is an order-isomorphism from one to the other, i.e. they have the same order type.

[31]Cantor provided a sufficiently large collection of examples, and Bernstein, in his thesis, showed there are at most continuum many, which by the Schröder-Bernstein Equivalence Theorem gives the desired equality.

a formal notion of power formation with a base μ and exponent α via sequences indexed by α of elements of μ ("coverings"), where α and μ are arbitrary order types. He studies subsets of these powers under various orderings, especially in the case that the exponent is a well-ordered set. One special case is ordering by first difference, i.e. lexicographic ordering. To ensure a linear order, Hausdorff fixes a principal element m in the base M (of type μ) and looks at the collection of sequences from the representative set A of type α that differ only finitely from m to get powers of the first class. He observes that if the exponent set A has no last element, the resulting type is everywhere dense, and that if the base M and exponent A are ordinal numbers and the principal element m is the least element of the base, then the result is the same as Cantor's power. He generalizes to higher order powers, by replacing the condition that the sequences have all but finitely many places taking value the principal element m to the requirement that the sequences have all but a well-ordered subset of power $\leq \aleph_v$ for some fixed ν that specifies how much higher the power is; ordering by first differences is a linear order on such collections. He matches the collection of subsets of a well-ordered set A with their characteristic functions in a suitably high power with base 2 and exponent A which is linearly ordered by first differences. He concludes the first part of the series with transfinite sums and products where not all the factors are the same.

In the second part, *The higher continua* [1906], he focuses on ordered sets of fundamental sequences, i.e. sequences with domain an initial ordinal number, ordered by first difference. He analyzes these powers by looking at properties of the base, e.g. being dense, and identifies a variety of types. He introduces the critical notions of *cofinal*[32] and *coinitial*, characteristics of segments that became fundamental to the classification scheme he eventually developed.[33] Hausdorff proved that a type cannot be cofinal with two different initial ordinals. Another concept introduced was of a type being *homogeneous* if all its segments are similar.

In the third part of the series, *Homogeneous types of the second infinite cardinality* [1906], Hausdorff sets out a classification scheme for homogeneous types of cardinality $\aleph_1$ based on the cofinality and coinitiality of the segments, types of gaps, and occurrence or non-occurrence of uncountable well-ordered or converse-well-ordered subsequences.

In the fourth part of the series, *Homogeneous types of the cardinality of the continuum* [1907], Hausdorff continues his construction of examples fitting his classification scheme with the constraint that they have cardinality at most the continuum, but in the final part, *Pantachie types*, this constraint is lifted. In Section 1 of this last part, he looks at monotonically increasing functions on ω which go to infinity, and shows that the cardinality of this collection is the cardinal-

[32] Hausdorff used *confinal* for what in English is *cofinal*.

[33] In a note at the end of his introduction to Hausdorff's paper [1906], Plotkin (cf. [Hausdorff, 2005, 42]) pointed out that Hessenberg [1906] had introduced a related notion when he called a set M a *Kern* of an ordinal μ if the set of ordinals less than μ was the same as the set of ordinals for which there is an ordinal in M at least as large. Every Kern of μ is cofinal in μ.

ity of the continuum ([Hausdorff, 1907, 111] or [Hausdorff, 2005, 133]). Hausdorff investigates *final rank ordering* ([Hausdorff, 2005, 137]) of increasing sequences, i.e. ordering under eventual domination. He introduces generalizations of the order type of the rationals, denoted η, and uses the corresponding upper case Greek letter H for them. In Section 3 [Hausdorff, 2005, 145], he focuses on H-types, specifying that an η_1-type is unbounded, everywhere dense, neither cofinal with ω nor coinitial with ω^*; no fundamental sequence in it has a limit; and it contains no (ω, ω^*)-gaps[34] ([Hausdorff, 1909, 323]). He proves the universality of an η_1-type for linearly ordered types of cardinality at most $\aleph_1$ and the uniqueness of an η_1-type of cardinality $\aleph_1$, if it exists. He generalizes the notion to η_ν-types for finite ν.[35] In Section 6, he introduces the scale problem and uses the Continuum Hypothesis (CH) to construct a scale [Hausdorff, 2005, 167]: a sequence of ω_1 increasing functions in ${}^\omega\omega$ which is increasing in the final rank order and has the property that each function in ${}^\omega\omega$ is less than some member of the sequence.

Hausdorff in his *Fundamentals of a theory of ordered sets* [1908] streamlined and extended his earlier work on order. He worked out the basic properties of cofinality in [1908, 440–444], introduced the distinction between regular and singular cardinals, and included a proof in Theorem IV that each set without a last element is cofinal with one and only one regular initial number. He described dense and scattered linear orderings,[36] gave $\{\omega, \omega^*\}$ as a basis[37] for the denumerable linear orders, $\{1, \omega, \omega^*, \eta\}$ as a generating set for all countable linear orders, and discussed type rings and the preservation of properties from the base of such a ring to all its elements (see [Hausdorff, 1908, 454–456], [Hausdorff, 2005, 213–215]). Here is a quote in which Hausdorff describes linearly ordered sets of cardinality less than a fixed $\aleph_\alpha$ [Hausdorff, 1908, 458]:

> Satz XII. Jede Menge ist entweder selbst zerstreut order eine Summe zerstreuter Mengen mit dichtem Erzeuger. Die zerstreuten Mengen der Mätigkeit $< \aleph_\alpha$) ($\aleph_\alpha$ regulär) bilden einen Ring, dessen Basis aus allen regulären Anfangszahlen $< \omega_\alpha$) (inklusive der Zahl 1) and deren Inversen besteht.[38]

[34] A (κ, λ^*)*-gap* in an ordered set $(P, <)$ is a pair $A, B \subseteq P$ such that A is linearly ordered with order type κ, B is linearly order with order type λ^*, for all $a \in A$ and $b \in B$, $a < b$; and there is no $x \in P$ with $a < x < b$ for all $a \in A$, $b \in B$. [Scheepers, 1993] is a very readable survey on gaps in ${}^\omega\omega$ through the early 1990's which is described as "an invitation to study gaps" (see page 441) and has an extensive bibliography. See also the memoir by Farah [2000].

[35] Chang and Keisler [1973, 522], in notes for Chapter 5, date the notions of α-saturated and saturated models back to the η_α-sets of Hausdorff, citing the 1914 text. Plotkin (cf. [Hausdorff, 2005, 109]) quotes from an email of Michael Morley in which he indicated that 1954 was about the time when he set out to generalize the "eta set construction" to other algebraic structures.

[36] A *scattered linear order* is one that does not embed the order type of the rationals.

[37] A basis B for a class $\mathcal{C}$ of linear orders is a set of linear orders in the class which has the property that for any element $(L, \lessdot)$ of $\mathcal{C}$ there is some $(K, <)$ in B which is order-embeddable into $(L, \lessdot)$.

[38] Theorem XII. Every set is either scattered or a sum of scattered sets with a dense generator. The scattered sets of cardinality $< \aleph_\alpha$ ($\aleph_\alpha$ regular) form a ring whose basis consists of all regular initial numbers $< \omega_\alpha$ (including the number 1) and their inverses (translation by Plotkin (cf. [Hausdorff, 2005, 216])).

The "dichtem Erzeuger" [dense generator] may be thought of as an *index set.* The inverse of an ordinal number in the above quote refers to the linear order obtained by reversing the order, also known as the converse ordering. The ring in the above quote is a *type ring*, which is a system of types closed under sum of two types and by sums of types in the ring indexed by a type in the ring. On page 456, Hausdorff lists several collections that form a ring, including unbounded types, types with a last element, and types that are cofinal with a fixed type. Using transfinite recursion, one can transform the type ring of scattered sets of cardinality $< \aleph_\alpha$ into a hierarchy of sets that has become known as the *Hausdorff hierarchy.*

In Theorem XIV [1908, 472], Hausdorff proved that any sequence of length ω_ν under the lexicographic order on the ω_ν-sequences of elements of σ for an ordered set σ will have a cofinal subsequence of one of two types: one with initial segments that form a strictly increasing chain or one where all share a common initial segment and different on the first place not in this common initial segment.

In a substantial part of the paper, Hausdorff detailed his classification of dense types from [1907] and provided existence proofs. He used collections of functions defined on well-ordered domains in interesting ways. Specifically, in Theorem XVIII ([Hausdorff, 1908, 488], [Hausdorff, 2005, 243]), he identified a type universal for ordered sets of cardinality at most $\aleph_\pi$ and in Theorem XIX proved that there is at most one such type of cardinality $\aleph_\pi$. He also showed that there are at least $\aleph_1$ distinct types of power at most the continuum ([1908, 505, footnote]).

Hausdorff [1909] continued his exploration of powers of the alephs, i.e. collections of functions defined on a well-ordered set partially ordered by final rank order, and maximal well-ordered subsets of this partial order. He constructs an (ω_1, ω_1^*)-gap inside ${}^\omega\omega$ ordered by eventual difference using the Axiom of Choice but not the Continuum Hypothesis. He also shows that under eventual domination, any subset of ${}^\omega\omega$ has a maximal linearly ordered set.[39]

In the summer of 1910, Hausdorff moved to the University of Bonn, where he gave a course on set theory. He gave a revised and expanded course in the summer of 1912 and at the same time began work on his set theory book *Grundzüge der Mengenlehre* [1914], [2002] which was completed in 1913, and appeared in April 1914, shortly before the outbreak of World War I.[40] In 1920, Henry Blumberg [1920, 116] gave the book a glowing review in the *Bulletin of the American Mathematical Society*:

> Its most striking feature is that it is the work of art of a master. No one thoroughly acquainted with its contents could fail to withhold admiration for the happy choice and arrangement of subject matter, the careful diction, the smooth, vigorous and concise literary style, and the adaptable notation; above all things, however, for the highly

[39]See Sections 2 and 5 of Steprāns' chapter on the continuum for more on Hausdorff's work on gaps in maximal linear orderings of sequences of integers under eventual domination.

[40]Purkert's *An introduction to Hausdorff* (downloaded from http://www.hausdorff-edition.de/wp/?page_id=20 on February 5, 2011) for the Hausdorff Edition included these details on the timing of Hausdorff's work and a broad overview of his life and mathematical work.

> pleasing unifications and generalizations and the harmonious weaving of numerous original results into the texture of the whole.

In *Grundzüge der Mengenlehre*, there are chapters on basic set operations (sum or union, section or intersection, difference used to get complements), functions and their usage in products and powers, cardinal numbers (Zermelo's Well-Ordering Theorem, non-denumerability of the continuum, existence of a one-to-one correspondence between the real line and the Euclidean plane), ordered sets and their types, well-ordered sets and the ordinal numbers (including a proof Blumberg calls elegant of the fact that $\aleph_\alpha \aleph_\alpha = \aleph_\alpha$), Hausdorff's own work on partially ordered sets, topological spaces (axiomatically described in terms of neighborhoods,[41] including the requirement that the space be what is now called Hausdorff), special topological spaces, metric spaces, and Lebesgue measure.

This text and its subsequent editions were influential in shaping the direction of set theory in the coming decades. Plotkin (cf. [Hausdorff, 2005, 111]) asserts that the "1914 text made *back-and-forth* part of the mathematical mainstream."[42] Hausdorff included his η_α sets, patterned on the rationals. In Chapter VI, §1, Hausdorff considered partially ordered sets and proved the existence of maximal linearly ordered subsets in them. Many of the results on order were omitted from this important book, including the work on gaps and the type ring with its sums over arbitrary elements of the class. As a consequence, a number of Hausdorff's pioneering results on order were rediscovered later in the century.

2.2 1910–1920: Emerging schools

In the decade 1910–1920, Poland and Russia developed schools active in set theory and function theory; R. L. Moore initiated the study of analysis-situ or point-set topology that led to the development of the Texas school of topology; and Schur had an early result in what is now known as Ramsey Theory. The work of Baire became more widely known through the book [1916] by de la Vallée Poussin.

Wacław Sierpiński[43] published *Zarys teorii mnogości* (*An Outline of Set Theory*) [1912], which Kuratowski [1972, 2] described as "one of the first synthetic

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[41]Gregory H. Moore [2008a] observed that Hausdorff's neighborhoods are what today would be called a neighborhood base for the topology. Moore's article on the emergence of open and closed sets situates Hausdorff's "neighborhood spaces" in historical context.

[42]See [Plotkin, 1993] for more on the history of the back-and-forth construction, and see [1907] for Hausdorff's earlier usage.

[43]Wacław Franciszek Sierpiński (March 14, 1882 – October 21, 1969) graduated from the University of Warsaw in 1903, and became a school teacher in Warsaw. He earned his doctorate in Kraków with Stanisław Zaremba at the Jagiellonian University in 1908 and was appointed to the University of Lwów (now Lviv, Ukraine) in 1910 and at that point his interests turned to set theory according to the entry on him in [Gowers *et al.*, 2008]. After a period in Russia during World War I, he returned in 1918 to Lwów, and in 1919 became a professor at the University of Warsaw, where he spent the remainder of his professional life. For further background, see http://www.hpm-americas.org/wp-content/uploads/2010/04/PSMatHPMcorrected_2.pdf, an extended version of the paper by Roman Sznajder entitled *90th anniversary of emergence of the Polish School of Mathematics; Polish mathematics between the world wars* presented at the 2010 East Coast meeting of the History and Pedagogy of Mathematics: Americas Section.

formulations of this theory in the world," and was based on systematic lectures on set theory by Sierpiński in 1909 at University of Lwów.

Investigations of the Axiom of Choice and the Continuum Problem continued. An especially elegant application of the Continuum Hypothesis from this period is the construction from a well-ordering $<_\Omega$ of the real numbers of the subset $S \subseteq \mathbb{R}^2$ of the plane defined by $S = \{(r, s) : r <_\Omega s\}$; it has the property that every vertical section is co-countable and every horizontal section is countable. Sierpiński[44] [1919] published a proof that the existence of a set with these properties is equivalent to the Continuum Hypothesis.[45]

Results of Alexandroff, Lusin and Suslin put the emerging Moscow school[46] founded by Dimitri Egorov[47] and Nikolai Nikolaevich Lusin[48] on the map. Egorov [1911] proved that if a sequence of measurable functions converges pointwise in an interval except for a set of measure zero, then it converges uniformly on the interval except for a set of arbitrarily small measure. This result became known as Egorov's Theorem and then as the Severini-Egorov Theorem, when it was realized that Carlo Severini had independently and earlier published a proof [1910] of this result in a paper on sequences of orthogonal functions.

Lusin [1914] used the Continuum Hypothesis to construct what is now known as a *Lusin set*: an uncountable set of reals whose intersection with each meager set is countable, or equivalently an uncountable set such that every uncountable subset is non-meager.

[44]Kuratowski [1972, 2], in his obituary of Sierpiński, wrote "the outbreak of the war in 1914, interrupted Professor Sierpiński's didactic work. He was interned by the Tsarist authorities, first in Vyatka, and later in Moscow. There, the excellent Russian mathematicians Egorov and Luzin extended a most cordial reception to him and created very convenient conditions for his scientific work. This period brought important joint works by professors Sierpiński and Luzin which laid the foundations for their long co-operation in the domain of analytic and projective sets as well as the theory of real functions."

[45]Erdős [1981, 41] called this result a "very beautiful theorem" and commented that it was "very startling" when he first saw it. In Section 2 of the chapter on the continuum, Steprāns briefly discusses this result.

[46]See [Shields, 1987] and [Paul, 1997] for more information about the Moscow school.

[47]Dimitri Fedorovich Egorov (December 22, 1869 – September 10, 1931) became a Privat Docent in 1893, spent 1902-1903 abroad on paid leave visiting Berlin, Paris and Göttingen, and became a professor at Moscow University in 1903. Alan Shields [1987, 1] speculated that Egorov was one of the first foreign mathematicians to embrace the theory of Lebesgue. Egorov was director of the Institute of Mathematics and Mechanics at Moscow University from 1921 to 1930.

[48]Nikolai Nikolaevich Lusin (December 9, 1883 – January 1950) was a student of Egorov, entering Moscow University in 1901 when his family moved to Moscow from Tomsk. He started study under Bugaev, but it was Egorov, with whom he shared a religious outlook, who invited him to his home and gave him challenging problems that stimulated his creativity. Another influence on Lusin was P. A. Florensky, a fellow student in mathematics at Moscow University, who shifted his studies to theology and became a priest and spiritual advisor to Lusin. Lusin completed his undergraduate studies in the fall of 1905 and went to Paris for further education. He underwent a spiritual crisis in 1905–1906, studied medicine and theology as well as mathematics, and eventually focused on mathematics. He worked on a master's thesis with Egorov, but civil war disrupted his studies, and he spent 1910 to 1914 in Göttingen, continuing to correspond with Egorov and talking with Edmund Landau. He returned to Moscow and received his doctorate in 1915.

Lusin also worked on the Continuum Problem and on ordinals.[49] In a review of the book *Naming Infinity* [2009][50] Cooke [2010, 63] gave the following description of some Lusin's work:

> His notes, which are now in the archives of the Russian Academy of Sciences, are full of attempts to get a complete description of all the ordinals at once. In one valiant attempt, for example, he tried to imitate the method Dedekind had used to define real numbers as equivalence classes of cuts in the rational numbers. He took the class of subsets of the rational numbers between 0 and 1 that are well-ordered in their natural ordering, and introduced the natural order-equivalence relation on this class. Could one then *define* a countable ordinal number to be one of these equivalence classes? Certainly any such equivalence class will be order-isomorphic to a countable ordinal. But how could it be known that every countable ordinal is order-isomorphic to one of these equivalence classes? Luzin struggled with this problem of *naming* the countable ordinals for decades. One can almost hear him sigh as he wrote in one plaintive note that he left behind, "How many times must I write out the set of all ordinals numbers of types I and II?"

Pavel Sergeyevich Alexandroff[51] recalled reading Cantor in his first year (1913) at Moscow University [1979, 281]:

> In the comfortable round reading room of the university library I found Cantor's memoirs on set theory and I began to read them with delight. One of the last mathematics books Eiges had given to me was Kowalewsky's course in analysis, and it had acquainted me, to some extent, with the rudiments of set-theoretical thinking. But when I began to read Cantor in the original and learned what transfinite numbers are, a new world opened up before me, just as had once happened when I first learned about non-Euclidean geometry, and I was in a state of excitement. I was in the same state of excitement when in Baire's book, which was given to me by V. V. Stepanov, I became acquainted

[49] See Roger L. Cooke's survey [1993] of unpublished work and miscellaneous items in the Lusin archive.

[50] Graham and Kantor [2006], [2009] propose that Lusin's religious practice of name-worshiping inspired his mathematics.

[51] Pavel Sergeyevich Alexandroff (November 1896 – May 1982) who grew up in a cultured family was a student of both Egorov and Lusin at Moscow University. He traveled to Europe in the summers from 1923 through 1932, visiting Göttingen, Bonn, Paris and Holland. He obtained his doctorate in 1927 and went on to a successful career in topology at Moscow University, had several students, including Tychonoff, and was elected to membership in a number of academies, including the Russian Academy of Sciences, the Göttingen Academy of Sciences, the National Academy of Sciences of the United States, the London Mathematical Society, and the Polish Academy of Sciences. A. N. Arhangel′kii and A. N. Dranishnikov [1997]: "Alexandroff placed a value on a result not according to technical difficulty of its proof, but, first of all, according to its position in a mathematical theory, according to the new harmony and beauty which it brings into the theory."

> with the Cantor perfect set, which I immediately saw and still see to this day as one of the greatest wonders (and I mean a wonder, nothing less) discovered by the human mind.[52]

Both Hausdorff [1916] and Alexandroff [1916] proved that every uncountable Borel set has a perfect subset. Thereafter, Lusin encouraged Alexandroff to work on proving that the collection of sets which included the Borel sets and whose uncountable members he showed all included perfect sets were exactly the Borel sets and set Mikhail Yakovlevich Suslin[53] to study the work of Henri Lebesgue [1905]. This assignment by Lusin is described by Alexandroff in his chatty autobiography [1979, 375].

In summer 1916, Suslin found an error in Lebesgue's work, and constructed a counter-example. He went on to construct an A-set which is not a B-set and the following year, he [1917] published his work on the these sets, which have variously been called A-sets, analytic sets, and Suslin sets. In this work he introduced operation $\mathcal{A}$, also known as the *Suslin operation*,[54] extracting a key combinatorial aspect of Alexandroff's proof. The collection of analytic sets is the closure of the closed sets under the Suslin operation.

Lusin introduced an alternate approach to analytic sets called the method of sieves. In a modern formulation by Moschovakis [2009, 203], a *sieve* is a map $r \mapsto F_r$ which assigns to each rational number r a subset F_r of a space X. The set *sifted* by the sieve is defined by

$$x \in \text{Sieve}_r\, F_r \iff \{r : x \in F_r \text{ is not well ordered}\},$$

where the standard ordering on the rationals is used. The Suslin operation $\mathcal{A}$ and the method of sieves appear in the study of partitions later in the century. We close this brief aside on analytic sets by noting that the paper by Suslin [1917] discussed above, one by Lusin [1917] which includes work of Suslin, and a pair of papers by Lusin and Sierpiński [1918] and [1923] were important early papers in the theory of analytic sets.

[52] Kowalewsky possibly refers to the German mathematician Gerhard Kowalewski rather than Sonia Kovalevskaya.

[53] In his short biography of Suslin, Igoshin [1996] wrote that Suslin (November 1894 – October 1919), the only child of landowning peasants in the Saratov province of Russia, was recognized as special by his primary school teacher who persuaded some of the more wealthy people in the village to send him to a nearby city for grammar school. By tutoring children of the wealthy, Suslin eventually was able to pay for his studies himself, and finished in Spring 1913. He then went to Moscow University where he attended Egorov's seminar, and in 1914, Lusin's seminar, and was awarded a Diploma of the first degree in 1917. He was given permission to stay an additional two years to prepare for a professorship. During the Russian Civil War, after a period of failing health, he returned to his home village, where he died in 1919 of typhus less than a month prior to his twenty-fifth birthday. His brief mathematical career included a stint of teaching at the Ivanovo-Voznesensk Polytechnic Institute as an Extraordinary Professor starting in 1918. His students appreciated him enough to petition for his return, when he asked for a leave of absence because of poor health.

[54] A modern description of a *Suslin scheme* is a family $\mathcal{P} = \{P_s : s \in {}^{\omega>}\omega\}$ of subsets of a set X indexed by finite sequences of non-negative integers. The *Suslin operation* applied to such a scheme produces the set $\mathcal{AP} = \bigcup_{x \in {}^{\omega}\omega} \bigcap_{n \in \omega} P_{x \restriction n}$.

Egorov and Lusin gathered an increasingly active group of mathematicians around them, and Lusin became a Professor of Pure Mathematics at Moscow University in 1917 prior to the Russian Civil War and the ensuing economic disruption. Lusin and Suslin, among others, moved to Ivanovo-Voznesensk Polytechnic Institute. Lusin returned to Moscow in 1920 and worked together with Egorov to revitalize the Moscow school, and the group of students and collaborators around Lusin came to be called the *Lusitania*.

In [1916] R. L. Moore[55] developed the postulates for what he [1916, 132] called "plane analysis situs or what may be roughly termed the non-metrical part of plane point-set theory". It marked his entry in the emerging field of point set topology. He used *point* and *region* as his basic (undefined) concepts; the concept of region evolved into open sets. He wrote the paper while at the University of Pennsylvania and only in 1920 moved back to the University of Texas at Austin, where he did his undergraduate work. David Zitarelli [2004, 471] has asserted that this paper "firmly solidified his reputation as a first-rate researcher and later formed the basis for the Moore Method." Sometimes called the father of the school of Texas topologists, he had 50 graduate students, the first completing his degree in 1916 and the last in 1969; on September 10, 2010, the Mathematics Genealogy Project site listed 2,661 descendants. In his Math Review MR2103548 (2006a:01005) of Parker's biography [2005] of Moore, Ivan Reilly highlights an interesting measure of the success for graduate students of Moore's teaching methods in which he gave his students axioms (starting with those from his [1916]) and asked them to prove theorems from them in an orchestrated way:

> One measure of this success is described in the Albert Report "A survey of research potential and training in the mathematical sciences", produced in 1957 for the National Science Foundation (page 238). Data was gathered from a very large number of people who received their Ph.D.s in mathematics between 1915 and 1954. This survey collated the work of Ph.D. students after graduation and categorized them by the amount of published mathematical research. Of the top 15 per cent, 25 per cent were graduates from UT (made up almost entirely of Moore students), 20 per cent from Princeton, 16 per cent from Harvard, and 8 per cent from Chicago.

Issai Schur[56] [1917] proved an early result in Ramsey theory.

[55] Robert Lee Moore (November 14, 1882 – October 4, 1974), a student of Oswald Veblen and E. H. Moore, earned his degree at the University of Chicago in 1905. He was a major figure in the mathematical life of the USA in the middle of the 20th Century, noted for his mathematics, his teaching style, and his service to the professional community (he was president of the American Mathematical Society in 1938). From the many resources available about his work and life, one learns that he was a tall Texan, opinionated, racist (see http://www.math.buffalo.edu/mad/special/RLMoore-racist-math.html, accessed September 10, 2010, especially quotes by Vivienne Malone), and a charismatic and inspiring teacher. One of many resources on him is a book by Parker [2005] published by the Mathematical Association of America, an organization that Moore once called the "Salvation Army of Mathematics."

[56] Issai Schur (January 10, 1875 – January 10, 1941) was a student of Frobenius. He worked

> *Schur's Theorem:* When one divides the numbers $1, 2, \ldots, S(k)$ into k (color) classes for sufficiently large $S(k)$, there is always in one of the classes a triple of the form $a, b, a + b$, i.e. a monochromatic solution to the homogeneous linear system $x + y = z$.

Note that the triple $a, b, a+b$ can be thought of as a rather short monochromatic arithmetic progression as well as a monochromatic solution to a linear system. Schur used this result to prove that, for all m, if p is a sufficiently large prime, then the equation $x^n + y^n = z^n$ has a non-zero solution in the integers if you compute all the values modulo p.

2.3 1920–1930: Early structural results

The 1920's saw the flourishing of the Warsaw and Lwów schools and the founding of *Fundamenta Mathematicae*, whose first volume included Suslin's Problem. It saw the publication of Sierpiński's book on transfinite numbers and the second edition of Hausdorff's seminal text. New concepts for the decade included von Neumann ordinals and almost disjoint families. Two results related to infinite trees appeared, König's Lemma (it was expressed graph theoretically) and Moore's road space, a topological example with a tree implicit in its construction. Alexandroff and Urysohn proved their Regressive Function Theorem and van der Waerden proved his theorem on arithmetic progressions.

The Warsaw group in set theory, topology, and their applications prospered under the direction of Zygmunt Janiszewski, Stefan Mazurkiewicz and Wacław Sierpiński, founders of *Fundamenta Mathematicae*[57] (see [Kuratowski, 1980]).

A concept introduced by Sierpiński [1928b], that of an *almost disjoint* family[58] would become important in the study of partitions of infinite subsets of ω. Sierpiński proved that for any infinite set of cardinality κ, there is an almost disjoint family of its subsets of cardinality 2^λ where λ is the least cardinal with $2^\lambda > \kappa$. Alfred Tarski [1928] refined Sierpiński's notion by defining the *degree of disjunction* $\delta(\mathcal{F})$ of an almost disjoint family $\mathcal{F}$ to be the minimum cardinal δ such that for all distinct $M, N \in \mathcal{F}$, $|M \cap N| < \delta$. Tarski [1928], [1929] used the GCH to prove various theorems on the existence of almost disjoint families of large size with specified cardinality of the subsets involved and specified maximal degree of disjointness.

The first volume of *Fundamenta Mathematicae* included the posthumous publication of Suslin's Problem [Suslin, 1920] on linear orders which, like Cantor's Continuum Problem, became a motivating force for a significant body of work in the 20th Century (Problème 3, page 223):

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at the University of Berlin until 1935, and a few years later moved to Palestine. For more details on his life and mathematics see the biography by J. J. O'Connor and E. F. Robertson at http://www-history.mcs.st-andrews.ac.uk/Biographies/Schur.html.

[57] See [Kuzawa, 1970] on the founding and significance of *Fundamenta Mathematicae*.

[58] Sierpiński called a family $\mathcal{F}$ *almost disjoint* if for every two distinct sets $M, N \in \mathcal{F}$, the cardinality of the intersection $M \cap N$ is less than $|M|$ and less than $|N|$.

> Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel que tout ensemble de ses intervalles (contenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinare)?[59]

Here is a modern take on the problem. A linear order satisfies the *countable chain condition* (alternatively, say it has countable *cellularity*) if every collection of pairwise disjoint non-empty open intervals is countable.[60] Suslin asked if every linearly ordered set without endpoints whose order is (1) dense, (2) *complete* in the sense that every non-empty bounded subset has a greatest lower bound and a least upper bound, and (3) has the countable chain condition (i.e. countable cellularity) is isomorphic to the real line. If the requirement that the linear order have the countable chain condition is replaced by the requirement of having a countable dense set (being *separable*), then such a linear order is isomorphic to the real line. Call a linearly ordered set without endpoints satisfying the three conditions but not isomorphic to the real line a *Suslin continuum.* Call a dense linear order without endpoints which satisfies the countable chain condition but is not separable a *Suslin line.* Note that the existence of a Suslin line implies the existence of a Suslin continuum by passage to the completion of the line. *Suslin's Hypothesis* (SH) asserts the non-existence of Suslin lines.

Sierpiński published his book *Leçons sur les Nombres Transfinite* [1928a]. Émile Borel wrote the preface, lauding the work of the author, welcoming the inclusion of Lusin's work, and calling attention to the modern material featured in *Fundamenta Mathematicae.* Borel remarked (cf. [Sierpiński, 1928a, VI]): "Ce qui le distingue surtout, c'est le fait que M. Sierpinski croit effectivement à la réalité de *tous* les nombres transfinis, et admet sans restriction les raisonnements tels que celui par lequel M. Zermelo a ≪démontré≫ que le continu peut être bien ordonné."[61] In Chapter VII Sierpiński discussed order types and in the very last section, §64, he listed three properties (1^o no end points, 2^o continuous,[62] 3^o has a countable dense set) that characterize the order type λ of the continuum. At the end of the section, he noted that ordered sets which have a countable dense subset also have the property 4^o that all sets of non-empty pairwise disjoint intervals are at most denumerable, and concluded [1928a, 153]:

> Or, on ne sait pas si tout ensemble ordonné, jouissant des propriétés 1^o, 2^o and 4^o, est nécessairement du type λ, et ce problème [dû à M.

[59] Must a (linearly) ordered set which has neither jumps nor gaps and in which every set of disjoint [non-encroaching] intervals (all containing more than one element) is at most denumerable, must it, of necessity, be a (ordinary) linear continuum [copy of the real line]?

[60] Recall that the countable chain condition for a partial order P is the condition that every set of pairwise incompatible elements is countable. The countable chain condition in the restatement of Suslin's Problem is applied to the partial order of open intervals of order by inclusion.

[61] What most distinguishes it [the book] is the fact that Mr. Sierpinski actually believes in the reality of all transfinite numbers, and allows, without restriction, reasoning such as that by which Mr. Zermelo has shown that the continuum can be well ordered.

[62] We would say complete where Sierpiński said continuous.

Souslin (^{2})] semble très difficile.[63]

The Lwów school of mathematics grew around Hugo Steinhaus[64] and his brilliant student, Stefan Banach.[65] At the end of the decade (in 1929 to be precise), Stefan Banach and Hugo Steinhaus founded the journal *Studia Mathematicae* in Lwów accepting papers in international languages, playing a role for the Lwów school comparable to that of *Fundamenta Mathematicae* for the Warsaw school. Among those publishing in the first year were Banach, Stanisław Mazur, Władysław Orlicz, Juliusz Schauder and Steinhaus. Other notable members of the Lwów school were Stanisław Ulam, Mark Kac, Antoni Łomnicki, Herman Auerbach and Stanisław Ruziewicz.

Banach and Kazimierz Kuratowski[66] looked at generalizations of Lebesgue's Measure Problem, and it was a question of Banach in their [1928] that led Ulam[67]

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[63]However, we do not know if any ordered set, having the properties 1^o, 2^o and 4^o, is necessarily of type λ, and this problem [due to Mr. Souslin (^{2})] seems very difficult. The reference (^{2}) is to the statement of Suslin's problem [1920].

[64]Steinhaus was a student of Hilbert who earned his degree in 1911 in Göttingen.

[65]The book [Kuratowski, 1980, 43] includes a quote from Stanisław Mazur to the effect that Banach's thesis, presented in June 1922 for the Ph.D. at University of Lwów and published in *Fundamenta Mathematicae* [Banach, 1922], was the beginning of functional analysis as a field. Kuratowski's book gives delightful descriptions of the mathematicians of the day and some of their ways of working. To learn more about the intellectual atmosphere of the time, see the wide variety of problems recorded in a notebook in the Scottish cafe in Lwów and recollections recorded from a conference on the this notebook [Mauldin, 1981].

[66]Kazimierz Kuratowski (February 2, 1896 – June 18, 1980) studied for one year at the University of Glasgow in Scotland before the outbreak of World War I in August 1914. He then moved to Warsaw University in 1915 and received his doctorate in 1921. On the Mathematics Genealogy Project site, Mazurkievic and Janiszewski are listed as his advisors; note that Janiszewski died before the thesis was awarded. Kuratowski taught at Warsaw University 1925–1927 and was then appointed to a professorship at Lwów University. In 1934, after the closure of the mathematics department in Lwów, he became a professor at Warsaw University where he remained until 1952, when he became a member of the Polish Academy of Sciences.

[67]Stanislaw (Stan) Marcin Ulam (April 13, 1909 – May 13, 1984), was born in Lwów, Poland. He received his bachelor's degree in 1931, his master's in 1932 and his doctoral degree in 1933, all from the Lwów Polytechnic Institute. He visited Princeton for three months in 1935 at the invitation of John von Neumann and returned the following year for a longer period. He was a Harvard Junior Fellow and then a lecturer at Harvard during 1936–1940, visiting Poland in the summers until he left Poland in 1939 to return to the United States a mere month before World War II arrived in Poland. He was on the faculty of the University of Wisconsin, Madison 1941–1944, and then went to Los Alamos Laboratories 1944-1967, and remained a consultant for the remainder of his life. He continued his academic career 1965–1977 at the University of Colorado and at the University of Florida 1974–1984. His mathematical interests were broad, including set theory and abstract measures at the beginning of his career, topology, ergodic theory, computational approaches to mathematics, and bio-mathematics. He enjoyed conversations with mathematicians, physicists, and biologists. He held von Neumann in high regard, and valued their close association which lasted many years. He used to joke during visits by Erdős that the two had known each other since before I was born. I was privileged to enjoy the hospitality of Stan and his wife Françoise in Santa Fe where other guests and visitors included Gian-Carlo Rota, Dan Mauldin, Richard Laver, and the Mycielskis. For more information on Ulam, including further references, see his autobiography [1976], selected works [1974], the biography at http://www-history.mcs.st-andrews.ac.uk/Biographies/Ulam.html by J. J. O'Connor and E. F. Robertson and the brief article by Ciesielski and Rassias [2009].

to invent what are now called Ulam matrices in his ground-breaking paper [1930]. The definition of measurable cardinals traces back to this seminal work of Ulam; their properties as cardinals and their utility in combinatorial arguments took awhile to be fully recognized. See Kanamori's two chapters and Mitchell's chapter on inner model theory and large cardinals for more on the impact of this discovery. For more on the history of this accomplishment, see Chapter 1, Section 2 of [Kanamori, 1994].

Hausdorff published a second edition of his text under the new title of *Mengenlehre* in 1927, where he expanded the treatment of Borel sets, Suslin sets, Baire functions, continuous mappings and homeomorphisms. To keep the length reasonable, he had to cut material from the first edition and found it "easiest to sacrifice, of the topics treated in the first edition, most of the theory of ordered sets, a subject that stands somewhat by itself" (translation by John Aumann in [Hausdorff, 1957, 5]). The growing familiarity of the theory of ordered sets, as witnessed by Sierpiński's 1928 book, may have been one of the reasons that Hausdorff omitted this theory from the second and third editions of his textbook.

Next we turn to results related to trees. In 1925 Dénes Kőnig submitted *Sur les correspondences multivoques des ensembles* which includes a version of *König's Lemma*[68] described in terms of sequences and relations [1926, 120]:

> E) Soit $E_1, E_2, E_3, \dots$ une suite dénombrable, d'ensembles finis, non vides, et soit R une relation telle qu'à chaque élément x_{n+1} de chaque ensemble E_{n+1} corresponde au moins un élément x_n de E_n, lié á x_{n+1} par la relation R, c'est ce que nous écrivons sous la forme $x_n \, R \, x_{n+1}$ $(n = 1, 2, 3, \dots)$. Alors on peut choisir dans, chaque ensemble E_n un élément a_n de sorte que, pour la suite infinie $a_1, a_2, a_3, \dots$, on ait $a_n \, R \, a_{n+1}$ $(n = 1, 2, 3, \dots)$.[69]

Kőnig remarks in a footnote to his paper that the proof is based on an idea he owes to M. St. Valkó.[70]

The modern formulation of König's Lemma is for trees considered as partial orders: $(T, <)$ is a *tree* if $<$ is a partial order such that for each element $t \in T$, the set of predecessors of t is well-ordered by $<$. By a common abuse of notation the tree is often referred to by the underlying set T alone. For an ordinal α, the αth *level* of the tree is the set of all elements $t \in T$ whose set of predecessors is order-isomorphic to α. The *height* of a tree T is the least α for which the αth level is empty. A *branch* is a maximal linearly ordered subset. With this terminology

[68] See [Franchella, 1997] for information of the history of this important result.

[69] E) Let $E_1, E_2, E_3, \dots$ be a countable sequence of finite non-empty sets, and let R be a relation such that to each element x_{n+1} of each set E_{n+1} there corresponds at least one element x_n of E_n which is linked by the relation to x_{n+1}, which we will write as $x_n \, R \, x_{n+1}$ $(n = 1, 2, 3, \dots)$. Then one can choose in each set E_n an element a_n such that, for $a_1, a_2, a_3, \dots$, one has $a_n \, R \, a_{n+1}$ $(n = 1, 2, 3, \dots)$.

[70] Stephan Valkó who was his co-author in [Kőnig and Valkó, 1926], a paper which was received by the journal in December 1924.

in hand, we can give the now usual formulation of König's Lemma: If T is a tree of height ω such that all levels of T are finite, then T has an infinite branch.

In the late 1920's, Moore started to revise his axioms and in the process constructed his famous road space. Moore presented a paper at the annual meeting of the American Mathematical Society in December, 1926 and his abstract [1927] gave a brief commentary on Axioms 1, 2, and 4 and Theorem 4 of his early work [1916], including a list of types of spaces in which either Axiom 1 or Axiom 4 failed to hold, together with a proposed Axiom 1'. He gave a series of four colloquium lectures on point set theory (we would call it point set topology) for the Summer Meeting of the American Mathematical Society in August, 1929 in Boulder, Colorado. The synopsis he passed out to the participants gave a variation of Axiom 1' which he further modified in the course of his lectures; he eventually ended up with a version that allowed him to prove what is known as the Moore Metrization Theorem [1935]. On his way home from the meeting, Moore discovered his automobile road space [1942].[71] Mary Ellen Rudin [1975b], in her personal survey on the metrizability of normal Moore spaces, described road spaces in terms of trees where a connecting road or copy of an interval of the real line is added between a node and each of its immediate successors, and any terminal nodes are also extended. The Moore automobile road space is then the space obtained from the complete binary tree of sequences of length $\leq \omega$.

Important tools for the understanding of the infinite were developed. John von Neumann[72] developed what are now known as the *von Neumann ordinals* and put transfinite recursion on a firm foundation in the 1920's as detailed in Kanamori's first chapter. With concrete representatives of Cantor's ordinal numbers, the order

[71] See [Jones, 1997, 99–1000] for the timing, a description of the road space, and a graphical illustration. See [Fitzpatrick, 1997] for a carefully documented history of these lectures, their part in the development of the axioms for a Moore space, and their connections to the Moore metrization theorem and road spaces.

[72] John (János) von Neumann (December 28, 1903 – February 8, 1957), while at the Lutheran Gymnasium, was tutored by professors at the University of Budapest at the behest of his mathematics teacher László Rátz. In 1920, as a teenager, von Neumann submitted his first mathematical paper for publication. He studied chemistry at the University of Berlin 1921–1923 and then went to Zurich where he received his diploma in chemical engineering in 1926. He had an ongoing interest in mathematics, had been admitted to the University of Budapest in 1921, and did well on mathematics examinations there without taking any courses there. In Zurich he talked with Weyl and Pólya about mathematics. In the same year he received his chemical engineering degree he was awarded his doctorate with a thesis in set theory from the University of Budapest. The Mathematics Genealogy Project site lists the harmonic analyst Lipót Fejér (he is particularly well known for *Fejér's Theorem* on convergence of Fourier series) as his doctoral advisor. von Neumann held a Rockefeller Fellowship 1926–1927 which enabled him to study with Hilbert in Göttingen. He lectured at the University of Berlin 1926–1929, and in Hamburg from 1929–1930. He became a visiting lecturer at Princeton University in 1930, was appointed professor there in 1931, and in 1933 became one of the first six mathematics professors in the newly founded Institute for Advanced Study in Princeton. This remarkable man contributed to many fields of pure and applied mathematics, physics, computer science and received wide recognition for his work. Since the work of von Neumann related to the topics of this chapter occurred when he was young, we have restricted this brief sketch of his academic career to the early years.

types of well-orderings now had canonical representatives, and Hausdorff's initial ordinals received the names $\omega = \omega, \omega_1, \ldots$. Specifically, the order type of the non-negative integers, which Cantor denoted by ω, was now the order type of the von Neumann ordinal ω, which minimized the distinction between the order type of an ordinal number and its representative as a von Neumann ordinal.

In response to a question of Sierpiński about the size of an ordered set whose increasing chains and decreasing chains are known to be at most denumerable, Pavel Urysohn[73] [1924] built on work of Hausdorff [1914] to prove that the cardinality of such an ordered set is at most that of the continuum. The proof generalizes to the inequality $|S| \leq 2^{U(S)}$ where $U(S)$ is the supremum over the cardinalities of increasing and decreasing well-ordered chains, which we will call the *Hausdorff-Urysohn Inequality.*

Alexandroff and Urysohn, in the supplemental material of [1929],[74] made a simple but fundamental observation about *regressive functions*[75] which we will call the *Regressive Function Theorem* (its generalizations will be indexed under this name):[76]

> *Théorème:* Supposons qu'on ait défini, pour tout nombre ordinal a de la seconde classe, un nombre ordinal $\mu(\alpha)$ sous la seule condition $\mu(\alpha) < \alpha$.
>
> Il existe alors un ensemble indénombrable de nombres ordinaux
>
> $$\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots, \alpha_\omega, \ldots, \alpha_\lambda, \ldots \quad |\Omega$$
>
> tels que les $\mu(\alpha)$ correspondant sont tous identiques:
>
> $$\mu(\alpha_1) = \mu(\alpha_2) = \cdots = \mu(\alpha_n) = \cdots = \mu(\alpha_\omega) = \cdots = \mu(\alpha_\lambda) = \ldots \quad |\Omega$$[77]

[73] Pavel Samuilovich Urysohn (February 3, 1898 – August 17, 1924) started studying physics at the age of fifteen under the direction of P. Lasareft, a member of the Academy of St. Petersburg, and Urysohn already had a publication in 1915. Influenced by courses taken with Egorov and Lusin, he switched to mathematics and finished his undergraduate degree at Moscow University where his focus was analysis. Lusin persuaded him to continue with graduate studies, and Urysohn was awarded a doctorate in 1921 under Lusin's direction, after which he became an assistant professor there. He is perhaps best known for his work in topology, specifically for Urysohn's Lemma and the metrization theorem bearing his name. He died after swimming in stormy seas off the coast of Brittany near Batz-sur-Mer, and is buried in the cemetery there. For more information, see the brief biography at http://www-history.mcs.st-andrews.ac.uk/Biographies/Urysohn.html by J. J. O'Connor and E. F. Robertson, and the obituary by Alexandroff [1925]. Another resource of possible interest to those reading Russian is a memoir entitled *Joy of Discovery* written by his sister, Lina Neiman, which discusses his life, and includes articles by mathematicians on his mathematics.

[74] See Kojman's chapter for more on the Alexandroff and Urysohn paper.

[75] A function f on ordinals into ordinals is *regressive* if $f(\alpha) < \alpha$ for all non-zero α.

[76] In the preface to [Alexandroff and Urysohn, 1929], Alexandroff noted that the research was done in the summer of 1922, written up in winter 1922-1923, and accepted in 1923 by the editors of *Fundamenta Mathematicae*, but did not appear there for technical reasons. Math Review MR0370492 (51 #6719) of a reissue in Russian of the monograph noted that its first appearance was delayed for six years "mainly because its length posed problems for the editors of *Fundamenta Mathematicae*, and it was the intervention of L. E. J. Brouwer that led to its publication, some years after the appearance of the six articles in which the results first appeared in print."

[77]

In modern language, every regressive function on $\omega_1 \setminus \omega$ is constant on an uncountable set. They show that there cannot be a function ν with the property that for all α, $\nu(\alpha)$ is the least ordinal number greater than α such that $\mu(\beta) > \alpha$ when $\beta > \nu(\alpha)$. Thus there is some α such that for unboundedly many β, $\mu(\beta) \leq \alpha$ and, by cardinality considerations, uncountably many of these β have the same value for $\mu(\beta)$. They used this theorem to prove that ω_1 with the order topology is a non-metrizable topological space.

The Regressive Function Theorem can be regarded as a generalized pigeonhole principle in which one is required to distributes items in a list into indexed pigeonholes so that each item is placed in a pigeonhole with a smaller index.

2.4 Ramsey and van der Waerden

Bartel Leendert van der Waerden[78] was an algebraist, whose textbook *Modern Algebra* [1930][79] has been translated into English and Russian, was reissued in

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Theorem Suppose one has defined for every ordinal number α of the second number class, an ordinal number $\mu(\alpha)$ satisfying the sole condition $\mu(\alpha) < \alpha$.

Then there exists an uncountable set of ordinal numbers of the second class

$$\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots, \alpha_\omega, \ldots, \alpha_\lambda, \ldots \quad |\Omega$$

such that the corresponding $\mu(\alpha)$[s] are all identical:

$$\mu(\alpha_1) = \mu(\alpha_2) = \cdots = \alpha_\lambda = \ldots \quad |\Omega$$

Todorcevic, after noting that for limit ordinals $\alpha < \varepsilon_0$, there is a natural recursive way to choose an increasing sequence of ordinals with limit α, made the following comment about this result of Alexandroff and Urysohn in a footnote [2007b, 19]: “one is tempted to believe that the recursion can be stretched all the way up to ω_1 and this is probably the way P. S. Alexandroff discovered the phenomenon that regressive mappings on ω_1 must be constant on uncountable sets.”

[78] Bartel Leendert van der Waerden (February 2, 1903 – January 12, 1996) started studying mathematics at the University of Amsterdam in 1919 when he was sixteen, where Brouwer was the teacher he most respected. Most of his graduate study was at the University of Amsterdam where, in 1925, he defended his doctoral thesis written with Hendrik de Vries, and he also studied in Göttingen where he shifted to algebra after contact with Hellmuth Kneser and Emmy Noether. After a postdoctoral visit to Göttingen, he went to Hamburg for a semester with Hecke, Artin and Schreier and started work on the first volume of *Moderne Algebra* [1930] (in German) based on lectures by Artin and Noether. Van der Waerden held positions in Groningen and Göttingen before becoming a professor at the University of Leipzig in 1931, where he continued working during the war until his house was bombed in 1943. He and his family stayed in various places with friends and relatives and in 1945 returned to the Netherlands. Van der Waerden visited Johns Hopkins University in 1947, returned in 1948 to a chair in mathematics at the University of Amsterdam, and in 1951 moved to the University of Zurich where he spent the remainder of his academic career.

[79] *Moderne Algebra. Teil 1* was published as volume 33 in the first Springer series of higher mathematics texts, started by Richard Courant in 1920. In the preface to the translation of the second edition, he shared his attitude toward set theory in [1937], translated to English in [1953, v–vi]: “I have tried to avoid as much as possible any questionable set-theoretical reasoning in algebra. . . . I completely omitted those parts of the theory of fields which rest on the axiom of choice and the well-ordering theorem. Other reasons for this omission were the fact that, by the well-ordering principle, an extraneous element is introduced into algebra, and furthermore, the

1991, and is cited to this day. Van der Waerden [1927] (cf. [Graham *et al.*, 1990]) used a double induction argument to prove that for every partition of the positive integers into finitely many pieces there are arbitrarily long arithmetic progressions $(a+b, a+2b, a+3b, ..., a+\ell b)$ in one of the cells, and thereby solved a conjecture he attributed to a Dutch mathematician Baudet.[80] Van der Waerden wrote an account[81] of how he came to prove the theorem. It started with a lunch time conversation with Emil Artin and Otto Schreier, continued in Artin's office, where the trio exchanged questions and ideas, with lots of diagrams on the board, and ended with van der Waerden having an insight into a method for finding the arithmetic progressions based on an example about which he said [1927, 259]: "This final idea was accompanied by a feeling of complete certainty. I felt quite sure that this method of proof would work for arbitrary k and ℓ. I cannot explain this feeling; I can only say that mathematicians often have such conviction. When a decisive idea comes to our mind, we feel that we have the whole proof we are looking for: we have only to work it out in detail."

A critical theorem in infinite combinatorics is Ramsey's Theorem. In 1928 Frank P. Ramsey[82] submitted a paper [1930] addressing a problem in formal logic which included a proof of his oft-cited theorem as a lemma.[83] For a photograph of Ramsey and a discussion of the beginnings of finite Ramsey theory, see Harary [1983] and Spencer [1983]. Below is his statement of what he labeled *Theorem A* [1930, 264].

> *Ramsey's Theorem:* Let Γ be an infinite class, and μ and r positive integers; and let all those sub-classes of Γ which have exactly r members, or, as we may say, let all r-combinations of the members of Γ be divided in any manner into μ mutually exclusive classes C_i $(i = 1, 2, \dots, \mu)$, so that every r-combination is a member of one and only one C_i; then, assuming the Axiom of Selections, Γ must contain an infinite sub-class Δ such that all the r-combinations of the members of Δ belong to the same C_i.[84]

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consideration that for virtually all applications the special case of countable fields, in which the counting replaces the well-ordering, is wholly sufficient."

[80]Schur independently made the conjecture and shared it with his students (cf. Soifer [2009]).

[81]For van der Waerden's account, see [1954], reprinted in [1998] and, in English, see [1971].

[82]Frank Plumpton Ramsey (February 22, 1903 – January 19, 1930) was the son of Arthur Ramsey, President of Magdalene College, Cambridge and a mathematics tutor. He completed his undergraduate work at Trinity College in 1923 and was elected a fellow of King's College in 1924. The Mathematical Genealogy Project site lists him as having a University of Cambridge doctorate with no mention of date or advisor. With his first major work (*The Foundations of Mathematics*) published in 1925, it is likely that his doctorate was awarded after he became a fellow of King's College and prior to being appointed a university lecturer in mathematics at King's College in 1926. His interests ranged widely and included foundations, philosophy and economics.

[83]Ramsey's paper was published posthumously.

[84]The Axiom of Selections is presumably another name for the Axiom of Choice. Bertrand Russell discussed the "theory of selections" in Chapter XII of his book *Introduction to Mathematical Philosophy* [1920, 117ff], including Zermelo's Theorem.

Ramsey's Theorem is an example of a *balanced* partition theorem, since the size (infinite) of the set all of whose r-combinations belong to the same cell of the partition does not depend on which cell it is.

Ramsey's proof uses induction on the number of cells μ in the partition, with the critical case being $\mu = 2$, and induction on the size r of the combinations. He observed that the theorem is clearly true for $r = 1$ and for $\mu = 1$. Fixing $\mu = 2$, he assumes the statement is true for $r = \rho - 1$, and proceeds to the induction step with $r = \rho$. Having fixed a partition C_1 and C_2 of the ρ-combinations of Γ, he defines as long as possible sequences $x_1 < x_2 < \dots$ and $(\Gamma \supseteq)\Gamma_1 \supseteq \Gamma_2 \supseteq \dots$ so that $x_1 \in \Gamma$, $x_i \in \Gamma_{i-1}$ for $i > 1$ and every ρ-combination obtained by adding x_i to $\rho - 1$ members of Γ_i is in C_1. If the process goes on infinitely, he concludes that $\Delta = \{x_1, x_2, \dots\}$ is the desired set. If the process stops after finitely many steps, say with the definition of Γ_n, then he initiates a new process of choosing $y_1 < y_2 < \dots$ and $\Gamma_{n-1} \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \dots$ with $y_1 \in \Gamma_{n-1}$, $y_i \in \Delta_{i-1}$ for $i > 1$, and every ρ-combination obtained by adding y_i to $\rho - 1$ members of Δ_i is in the same C_i, which can be arranged by applying the induction hypothesis for $\rho - 1$. Since the first process stopped after finitely many steps, it must be the case that $C_i = C_2$ at every stage in the second process, so that $\Delta = \{y_1, y_2, \dots\}$ is the desired set. He then completes the induction on r. He reduces the case for $\mu > 2$ many cells to at most $\mu - 1$ cells, by combining the last two cells together to get a new partition and applying the induction hypothesis to it to get an infinite set Δ and a specified cell containing all r-combinations from it. If the specified cell is not the new cell obtained by joining two old ones, Δ is the desired set, and otherwise he applies the theorem again with $\mu = 2$ to get the required infinite $\Delta' \subseteq \Delta$.

We will call a sequence $\{z_1, z_2, \dots\}$ where the combination of any element z_i with any $\rho - 1$ later elements lands in the same cell C_i *begin-homogeneous*.[85] With this language, Ramsey's basic induction step argument starts with an to attempt to construct a sequence begin-homogeneous for which all the ρ-combinations end in the first cell, and if it does not work, repeat starting from the point of failure to get a sequence begin-homogeneous and verify that all the ρ-combinations end in the second cell.

Corresponding to the theorem discussed above, sometimes also called the *Infinite Ramsey's Theorem*, Ramsey proved a finite version quoted below [1930].

> *Finite Ramsey's Theorem:* Given any r, n and μ we can find an m_0 such that, if $m \geq m_0$ and the r-combinations of any Γ_m are divided in any manner into μ mutually exclusive classes C_i $(i = 1, 2, \dots, \mu)$, then Γ_m must contain a sub-class Δ_n such that all the r-combinations of the members of Δ_n belong to the same C_i.[86]

Ramsey followed the statement of the theorem with the following remarks:

[85] The use of *begin-homogeneous* here is designed to parallel *end-homogeneous* used in later constructions.

[86] Here Γ_m denotes a class with m members.

"This is the theorem which we require in our logical investigations, and we should at the same time like to have information as to how large m_0 must be taken for any given r, n, and μ. This problem I do not know how to solve, and I have little doubt that the values for m_0 obtained below are far larger than is necessary." The study of this question and its generalizations is the core of finite Ramsey theory.

3 1930-1940: EARLY RAMIFICATIONS

In the 1930's Erdős and Rado, two pioneers in the partition calculus, forged their initial results; Sierpiński constructed a key example which shows that there is no direct uncountable analog of Ramsey's Theorem; Jones proved his metrization theorem and constructed a special Aronszajn tree[87] but did not publish the work until later; and Kurepa made the first systematic study of uncountable trees, giving an equivalent formulation of Suslin's Problem in terms of trees. Partial orders were shown to be extendible to linear orders, and a denumerable partial order universal for countable partial orders was constructed. The Regressive Function Theorem was generalized and large families of almost disjoint sets were constructed. Equivalence relations became part of the tool kit for analysts;[88] graph theory became a subject in its own right[89] with the publication of *Theorie der endlichen und unendlichen Graphen* [Kőnig, 1936], [1990], the first book to present graph theory as a subject.

There were ups and downs in mathematical publishing. *Zentralblatt für Mathematik* was founded in 1931 by Springer Verlag in cooperation with the Prussian Academy of Science.[90] *The Journal of Symbolic Logic* was founded in 1936 (cf. [Ducasse and Curry, 1962]), and Alonzo Church included a bibliography of symbolic logic in the first issue. However, on September 1, 1939 Germany invaded Poland and volume 33 of *Fundamenta Mathematicae* was not published.[91]

Hausdorff published the third edition of his set theory text; Moore [1932] published his monograph *Foundations of Point Set Theory* with the revised version of his axioms that he had been working toward in the 1920's.[92]

Sierpiński published a monograph [1934] on the Continuum Hypothesis. It naturally included his own work, and also included that of Lusin, Tarski, Kuratowski and others of the Polish school. It started with equivalences of the statement and

[87] Recall that a *special Aronszajn tree* is a tree of height ω_1 all of whose levels are countable and which is the union of countably many antichains.

[88] Equivalence relations were one of the topics in [Bourbaki, 1939], a 51-page overview of set theory designed to prepare individuals for the study of analysis.

[89] See the AMS Math Review by Robin Wilson [1991].

[90] See the *History of Zentralblatt* page whose url follows:
http://www.zentralblatt-math.org/zmath/en/about/history/ downloaded September 7, 2010.

[91] See [Kuzawa, 1970, 489–490] for an overview of the loss of life and property of the mathematical community of Poland during the war period 1939–1945.

[92] Jones [1997, 97], in his article *The beginning of topology in the United States and the Moore school*, assessed this work of Moore and *Topology* by Solomon Lefschetz [1930] as follows: "While definitely limited, these books were of considerable help in formulating the knowledge and tools necessary for research."

continued with discussions, all the while assuming CH, of Lusin sets, the duality between sets of measure zero and sets of first category, the proof that there is no measure on all subsets of the real numbers which is (1) not identically zero; (2) zero on singletons; (3) countably additive.

The founders of the Moscow school, Egorov and Lusin, fell on hard times in the 1930's. In September 1930 Egorov was arrested for his religious beliefs, Name Worshipping, and exiled to a camp near Kazan on the Volga River, and he died after a hunger strike initiated because the prison guards would not let him practice his religious beliefs.[93] The Moscow Mathematical Society had elected Kolman, a loyal Bolshevik, as president. Lusin was alarmed by what happened to Egorov, so he left Moscow University for the Steklov Institute of the Central Aero-Hydrodynamics Institute in Leningrad, but kept his position as the head of the mathematical group of the Russian Academy of Sciences. Kolman's denunciation of Lusin in print prevented Lusin from attending the International Congress of 1932. The Russian Academy of Sciences and the Steklov Institute were moved to Moscow in 1934. In July 1936, articles in Pravda appeared accusing Lusin, in particular, of claiming results of his students as his own and of publishing his best papers in the West. These articles led to a series of meetings of the Moscow Mathematical Society which drafted a resolution against Lusin that led to the loss of his university position and loss of good will from some of his colleagues, but not his position as an academician. Lusin made a statement in response to the resolution promising, in part, to publish primarily in the Soviet Union. What became known as the *Lusin Affair* had a chilling effect: mathematicians in the Soviet Union basically stopped publishing their results abroad.[94]

The study of transversal theory was launched when Philip Hall [1935] proved his celebrated theorem on matching:

> *Hall's Marriage Theorem:* Given n boys, if, for each positive $k \leq n$, any k of the boys between them know at least k girls, then it is possible to marry each boy to a girl that he knows.

The result can also be formulated as an existence theorem for a system of distinct representatives: For each boy b, there is a set G_b of girls he knows, and a selection of distinct girls from each of these sets yields the desired outcome. D. Kőnig [1931] had an equivalent but initially lesser known result published in Hungarian. Transversals are regarded in a variety of ways: as injective choice functions for a sequence of sets; as sets whose intersection with every set in a family is a singleton; as systems of unique representatives.

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[93] See [Graham and Kantor, 2007] for details.

[94] See [Demidov and Levshin, 1999] and the Math Review MR1790419 (2001k:01066) by F. Smithies of it for more information.

3.1 Extensions

This section includes extensions of results from earlier decades for linear and partial orderings, almost disjoint families and regressive functions. Also included is the following result which discusses an extension of a different kind.

Edward Marczewski[95] [1930] proved a theorem that is known in the literature by his birth name:

> *Szpilrajn's Theorem:* Every partial order has an extension to a total (linear) order.

For Marczewski, a linear order was a transitive, irreflexive, trichotomous binary relation. Marczewski first proved a lemma than any partial order with an incomparable pair can be extended to a partial order in which the pair has a specified order. He next observed that the union of an increasing well-ordered chain of partial orders under extension is a partial order. He concluded that there must be a maximal extension, hence one which is a linear order, citing [Kuratowski, 1922] and [Hausdorff, 1914]. In his paper, he remarks that the theorem is already known, citing the third edition of Sierpiński's book, *Zarys teorji mnogości*, but that the proofs of Banach, Kuratowski and Tarski had not yet been published.

Andrzej Mostowski[96] [1938] constructed a denumerable partial order universal for denumerable partial orders,[97] extending the Bernstein result for countable linear orders which Hausdorff had extended under CH to larger size linear orders.

Hausdorff [1936] revisited ordered sets and constructed an (ω_1, ω_1^*)-gap inside the "dyadic number sequences." He commented on its relationship to his earlier work in a footnote [Hausdorff, 1936, 244, footnote 1]:

> Für Folgen reeler order rationaler Zahlen habe ich dies bereits 1909 gemacht (*Graduierung nach dem Endverlauf*, Abh. Sächs. Ges. d. Wiss. 31). Da dieser Arbeit whol wenig bekannt is und die Konstruktion für dyadische Folgen, die zum vollen Beweis des Satzes I er-

[95] Marczewski had the surname Szpilrajn until 1940. Kuratowski directed his thesis at the University of Lwów. Other influential colleagues there included Mazurkiewicz and Sierpiński.

[96] Andrzej Mostowski (November 1, 1913 – August 22, 1975) wrote up his doctoral thesis, *On the Independence of the Definition of Finiteness in the System of Logic* in 1938 and defended it at Warsaw University in February 1939 (recall that Poland was invaded in September 1939). Kuratowski was the official supervisor, but Mostowski counted himself a student of Tarski. Lindenbaum had suggested that he work on the method of independence proofs sketched by Fraenkel (cf. [Crossley, 1975, 44]). This conversation was the starting point of the main content of Mostowski's dissertation, namely, what came to be known as Fraenkel-Mostowski permutation models. In conversation with Crossley [1975], Mostowski shared some memories of his three escapes during World War II, and his decision to take some bread instead of his "nice, very big, wonderful notebook with all these discoveries" which subsequently was burned. For more on his life and his interactions with mathematicians inside and outside of Poland, see [Ehrenfeucht *et al.*, 2008].

[97] Recall a partial order $(P, <)$ is *universal for a class* $\mathcal{K}$ of partial orders if every partial order in $\mathcal{K}$ has an order and incomparability preserving embedding into $(P, <)$. I only know of this result from the reference to it in [Johnston, 1956].

> forderlich ist, doch einige Modifikationen velangt, möchte ich die Sache vollständig darstellen.[98]

Sierpiński [1931] investigated the collection of all subsets of a denumerable set D partially ordered by the subset relation. He proved that an order type φ is realizable as a family of subsets of D linearly ordered by the subset relation if and only if it is a denumerable real type, where a *real type* is the order type of some set of real numbers with their usual order. At the end of his paper he asked the following question.

> *Question:* What are necessary and sufficient conditions for an order type φ to be realizable as a family of denumerable sets linearly ordered by the subset relation?

He concluded with the comment that all such types are necessarily realizable as subsets of $(1+\lambda)\,\omega_1$, where λ is the order type of the set of real numbers.

Sierpiński [1937], who had introduced the notion of almost disjoint families in [1928b], published his proof that for any cardinal κ there is a family of cardinality at least κ^+ of increasing κ-sequences which, pairwise, have disjoint tails, i.e. there is an almost disjoint family of size 2^λ of subsets of κ where λ is the least cardinal μ with $2^\mu > \kappa$.

Ben Dushnik[99] [1931] extended the Regressive Function Theorem of Alexandroff and Urysohn to non-limit [successor] cardinals κ using the well-foundedness of the ordinals and basic cardinal arithmetic to find a subset of size κ on which the function is constant. He gave the example of the function $F : \aleph_\omega \to \omega$ defined by $F(\alpha) = 0$ if α is finite and otherwise $F(\alpha) = n$ if and only if $|\alpha| = \aleph_n$ to show some limitation was necessary.

3.2 Erdős and Rado

Now we turn to Ramsey-theoretic results of Erdős, Rado and Sierpiński.

Richard Rado[100] was a student of Schur in Berlin. Rado in his thesis [1931] (see also his [1933]) studied partitions of positive integers into finitely many classes

[98] I already did this in 1909 for sequences of real or rational numbers (*Graduierung nach dem Endverlauf*, Abh. Sächs. Ges. d. Wiss. 31). Since this work is undoubtedly little known and since the construction for dyadic sequences, which is necessary for the full proof of Theorem 1, needs some modifications, I would like to present the matter completely. (Translation by Plotkin cf. [Hausdorff, 2005, 307])

[99] In the middle 1920's, Ben Dushnik joined the faculty of the University of Michigan (see [Kaplan, 1971, 184-185]). He received his Ph.D. from the University of Michigan under Theophil H. Hildebrandt in 1931.

[100] Richard Rado (April 28, 1906 – December 23, 1989), son of Leopold Rado, a Hungarian from Budapest, was born in Germany, and met and married Luise Zadek there. Since opportunities for those of Jewish descent were slim under Hitler, they went to Cambridge, England to study further. After earning a second doctorate under Hardy in 1935 (see [Richards, 1971] for details), he was a lecturer at Cambridge and then at the University of Sheffield. He was appointed Reader in Mathematics at King's College, London, before becoming a Professor of Mathematics at Reading in 1954. He was elected a Fellow of the Royal Society in 1978. He worked in a variety of

and homogeneous systems of linear equations with positive integer coefficients. He called such a system of linear equations *k-fold regular* (now shortened to k-regular) if for every partition of the positive integers into k classes, there is one class which has a solution to the system. Homogeneous systems which are k-regular for all k are called *partition regular*, and Rado classified partition regular systems, generalizing van der Waerden's Theorem [1927] and Schur's Theorem [1917]. We will call this *Rado's Theorem for Linear Systems with Positive Integer Coefficients.*

Here is a modern take on the theorem (see Hindman [2006, 4] for example). Say that an $n \times n$ matrix A with rational entries and columns $A = [\vec{c}_0, \vec{c}_1, \ldots, \vec{c}_n]$ satisfies the *columns condition* if and only if there is a partition $\mathcal{I} = \{I_0, I_1, \ldots, I_{m-1}\}$ of the indices such that

1. $\sum_{k \in I_0} \vec{c}_k = \vec{0}$; and
2. for all $j < m$, if j is positive, then the sum $\sum_{k \in I_j} \vec{c}_k$ is a linear combination of the vectors in $\bigcup_{\ell < j} \{\vec{c}_m : m \in I_\ell\}$.

Call a matrix A with rational entries *partition regular* over $\mathbb{N}$ if and only if for any finite coloring of $\mathbb{N}$, there is a solution $\vec{e}$ to $A\vec{x} = \vec{0}$ such that all entries of $\vec{e}$ receive the same color. Here is Rado's Theorem reformulated this terminology.

> *Rado's Theorem for Linear Systems with Positive Integer Coefficients:* A is partition regular if and only if A satisfies the columns condition.

At a celebration of his 65th birthday, Richard Rado spoke of his beginning steps in mathematics ([Richards, 1971]):

> Having been registered as a research student at Berlin University for some time, I was in the habit of staring gloomily at a blank sheet of paper wondering how I could ever cover it with any worthwhile mathematics. One day I attended a seminar conducted by a group of distinguished mathematicians. A fellow research student gave a lecture on van der Waerden's theorem on arithmetical progressions. To me the theorem sounded quite incredible and the proof a string of fallacies. I went away determined to shatter whatever belief there could exist in the truth of such a theorem. But on studying the matter more closely I had to admit that the theorem was true and the proof sound. This gave me my start in mathematics and I have never looked back.

In 1931, Rado's future collaborator, Paul Erdős[101] was a first-year undergradu-

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fields and was particularly known for his work in combinatorics, including abstract independence structures, transversal theory, and Ramsey theory, especially the partition calculus. For more on his life and work, see [Rogers, 1998].

[101] Paul Erdős (March 26, 1913 - November 9, 1996) moved out of Hungary because of World War II and eventually became an itinerant mathematician whose travels knit together a world-wide community of mathematicians. See [Bollobás, 1997] for a short biography; see [Graham and Nešetřil, 1997] for personal memories of many close to him, and [Hajnal, 1997] for a discussion of his work in the partition calculus by a close collaborator.

ate taking a graph theory course from Dénes Kőnig, when he proved a generalization of Menger's Theorem[102] for infinite graphs, which only appeared in 1936 at the end of [Kőnig, 1936]. While at the university, Erdős interacted with a dozen or so young mathematicians, including György Szekeres and Eszter Klein. Sós [2002, 88] speculated:

> It might be that this intensive collaboration, which was quite unusual at the time, has been influential in shaping the future character of the mathematical life in Hungary.

Erdős received his Ph.D. from Péter Pázmaány University[103] in 1934. About his university professors, Erdős said in an interview:[104] "I learned a lot from Lipót Fejér, but very probably, I learned the most from László Kalmár." However, Erdős had learned the basics of set theory from his father, who was a high school teacher, and Erdős "soon became fascinated with 'Cantor's paradise' " according to András Hajnal [1997, 352]. Erdős and Szekeres [1935][105] (see also [Spencer, 1973]) rediscovered the Finite Ramsey's Theorem.[106] They comment that Thoralf Skolem published a proof of the Finite Ramsey's Theorem [1933]. Erdős and Szekeres developed what Graham and Nešetřil [2002] called an *ordered pigeonhole principle.*

> *Ordered Pigeonhole Principle:* Let m, n be positive integers. Then every sequence of $(m-1)(n-1)+1$ distinct integers contains either a monotone increasing m-set or monotone decreasing n-set.

They then used their version of the Finite Ramsey's Theorem for quadruples to prove a Euclidean partition property.

After his doctoral degree, Erdős went to Manchester, England at the invitation of Mordell, where he met Rado and Ulam, among others. Erdős' interest in generalizing Ramsey's Theorem began early, as is illustrated in the obituary of Rado by Rogers which included a sentence from a letter to Rado from early in 1934 [1998, 190]: "Erdős asked: 'Is it true that when S is a set of infinite cardinal[ity], and the countable subsets of S are split into two classes, then there is always an infinite subset S^* of S all of whose countable subsets are in the same class?' Rado replied with a counterexample almost immediately." In 1938 Erdős was offered a fellowship at the Institute for Advanced Study in Princeton, where he spent a year and a half, and after that he became an itinerant mathematician, fostering

[102] Menger's Theorem [Menger, 1927] says that for any vertices x and y in a graph, the minimum number of vertices that need to be removed from a graph to end up with x and y not connected by a path is the same as the maximum size of a family of paths from x to y with no two paths sharing an edge. For a strong version of Menger's Theorem that answers a question of Erdős, see [Aharoni and Berger, 2009].

[103] The university was renamed Loránd Eötvös University in 1948.

[104] The interview was published in 1979 in Hungarian [Staar, 1979] in the monthly *Természet Világa* and quoted by Sós [2002] in an English translation.

[105] The paper was submitted December 7, 1934.

[106] In [Erdős, 1942, 363] and [Erdős, 1996, 113] the rediscovery of the Finite Ramsey Theorem is attributed to Szekeres.

collaborations everywhere he went. For more on his life, including passport and visa troubles with Hungary and the United States and the story of how he became a dual citizen of Israel, I recommend the article by Bollobás [1997].

Ramsey's Theorem is called a *positive* result, since it guarantees the existence of an infinite set whose n-element subsets are indistinguishable with respect to a given partition, coloring, or n-ary symmetric relation. An important early *negative* result (published in 1933) is the *Sierpiński partition* which shows that there is no direct uncountable analog of Ramsey's Theorem. Bronisław Knaster[107] had asked the following question (cf. [Sierpiński, 1933, 285]):

> Existe-t-il une relation symétrique R, dont le champ E est non dénombrable, telle que dans tout sous-ensemble non dénombrable de E existent deux éléments différents α et β, tels que $\alpha R \beta$, et deux éléments différents γ et δ, tels que γ non $R\delta$.[108]

For the domain E, Sierpiński selected the set of finite and denumerable ordinals.[109] He applied the Axiom of Choice to obtain a one-to-one function $r : E \to \mathbb{R}$, and defined $\alpha R \beta$ if and only if $r(\min(\alpha, \beta)) < r(\max(\alpha, \beta))$. The symmetric binary relation R may be cast as a graph (without loops) by letting E be the set of vertices and letting the edges be the unordered pairs $\{\alpha, \beta\}$ for which $\alpha R \beta$ (and $\beta R \alpha$)). Thus this example shows that Ramsey's Theorem cannot be directly extended to the uncountable.

3.3 Jones

F. Burton Jones[110] in a one-page abstract for a paper presented at the October 28, 1933 meeting of the American Mathematical Society at Columbia University included the statement of a metrization theorem [Jones, 1933]: "a separable normal space satisfying the first three parts of Axiom 1 of R. L. Moore's *Foundations of Point Set Theory* is completely separable and therefore metric, provided that in this space every uncountable point set contains a subset of power $\mathfrak{c}$." The reference is to Moore's book [1932]. Spaces which satisfy Axiom 0 and the first three parts

[107] Knaster (May 22, 1893 – November 3, 1990) received his Ph.D. degree from the University of Paris in 1925 under the supervision of Stefan Mazurkiewicz. He was a professor in Lwów starting in 1939, and in Wrocław from 1945.

[108] Is there a symmetric relation R, of which the domain E is uncountable, such that in every uncountable subset of E there are two different elements α and β, such that $\alpha R \beta$ and two different elements γ and δ such that not $\gamma R \delta$ (translation by the author).

[109] Formally, he let E be the set of all ordinal numbers of the first and second class of Cantor.

[110] Floyd Burton Jones (November 22, 1910 – April 15, 1999) was a student of R. L. Moore who received his degree at the University of Texas at Austin in 1935, where he taught until 1950 with the exception of a couple years during the war at the Harvard Underwater Sound Laboratory. For information on Burton Jones, see the dedication by Jones' first doctoral student [Transue, 1969] and the overview of Jones' work in the conference proceedings [McAuley and Rao, 1981]. Both were recommended by M. E. Rudin [1997]. Other resources include [Rogers, 1999] and the brief biography at http://www-history.mcs.st-andrews.ac.uk/Biographies/Jones_Burton.html by J. J. O'Connor and E. F. Robertson.

of Axiom 1 were named *Moore spaces* by Jones [1937, 675]. A space is *normal* if disjoint closed sets P and Q can always be separated by disjoint open supersets $U \supseteq P$ and $V \supseteq Q$.

After a four-year gap, Jones published the proof of his Theorem 5 [1937, 676]:

> *Jones Metrization Theorem:* If $2^{\aleph_1} > 2^{\aleph_2}$, then every separable metric space M is completely separable and metric.[111]

Below this theorem and on the same page, Jones commented: "The author has tried for some time without success to prove that $2^{\aleph_1} > 2^{\aleph_0}$." He then stated what has come to be known as the Normal Moore Space Conjecture: "Is every normal Moore space M metric?"

Jerry Vaughan [1980] asked Jones why he thought that $2^{\aleph_0} < 2^{\aleph_1}$ was true, and here is the response Vaughan recorded from their October 12-13, 1979 conversation [1980, 40]: "Think of a countable sequence, say ω, as being an initial segment of an uncountable set, say ω_1. Well, it is perfectly obvious to even the most stupid person, and I guess maybe it helps to be a little stupid, if you take all the sequences which run through the first infinity of ordinals, which gives a set of cardinality $2^{\aleph_0}$, look how many more you can get by running the rest of the way through ω_1 with uncountable sequences. It is clear that there ought to be a whole lot more of them. I thought surely that I could prove that overnight, maybe even in an hour, but I could not get anywhere on it. I remember one week I worked on that problem pretty hard and could not see it. I happened to run into Moore walking across campus and told him that I was losing my mind: I could not prove $2^{\aleph_0} < 2^{\aleph_1}$ even though it ought to be true. Well, he just laughed and said 'Neither could Sierpinski.' It had not occurred to me to look in Sierpinski's book; so I looked in there and found some things equivalent to this. He did not do much with it, but he did enough to convince me that I was wasting my time." Peter Nyikos [1980a, 28] completes the story: "The real cause of the delay, as Jones has informed us, is that he was hoping to settle the entire normal Moore space problem $\cdots$."

Jones [1980][112] shared reactions of his thesis advisor Moore to this work:

> When I was a student at Texas, Moore apparently didn't consider work in abstract spaces very highly. $\cdots$ I think he just felt there was more substance and beauty in other less abstract kinds of problems -- continua, decompositions, etc. So when I proved that if $2^{\aleph_0} < 2^{\aleph_1}$ then every separable, normal Moore space was metrizable [1], I wasn't asked to present the proof in class but rather to the Mathematics Club (as it was called). And when I wanted to use it together with other things for a thesis, Moore didn't like the idea at all but chose instead

[111] Peter Nyikos [1980a] pointed out that Jones and Moore used *completely separable* for *second countable*, i.e. there is a countable family of open sets such that every open set can be represented as a union of some of them.

[112] The paper [Jones, 1980] appeared in a volume dedicated to his mathematical work and influence and containing some of the papers presented at a conference held in Greensboro, North Carolina, in October, 1979.

> a couple of embedding theorems I had gotten. With hindsight he was certainly right. Since I thought that one ought to be able to prove $2^{\aleph_0} < 2^{\aleph_1}$, I might still be there!

Jones made a presentation entitled *A false proposition of logic* to the Mathematics Club of the University of Texas in the spring of 1933 (see footnote 6 of [1953]) in which he presented the construction/example of an Aronszajn tree as he much later recounted for the American Mathematical Society on December 29, 1952 [1953].[113] A modern description of his example follows [1953, 32–33]:

> *Jones' Example 1:* There exists a well-ordered family F of collections $H_1, H_2, \ldots, H_z, \ldots (z < \omega_1)$ such that (1) for each ordinal x, $x < \omega_1$, H_x is a countable collection of mutually exclusive sets, (2) if $x < y < \omega_1$, each set of H_y is a proper subset of H_x, (3) if $x < y < \omega_1$, and h_x is an element of H_x, there exists a sequence $h_x, h_{x+1}, h_{x+2}, \ldots, h_y$ such that $h_x \supset h_{x+1} \supset h_{x+2} \supset \cdots \supset h_y$, and for each z, $x \leq z \leq y$, H_z contains exactly one of the sets $h_x, h_{x+1}, h_{x+2}, \ldots, h_y$, but (4) there exists no sequence $h_1, h_2, h_3, \ldots$ such that $h_1 \supset h_2 \supset h_2 \supset \ldots$ and for each x, $x < \omega_1$, H_x contains exactly one of the sets $h_1, h_2, h_3, \ldots$.

Jones used his example to construct a non-metrizable Moore space [1954, 31–33]. He compared his example with the one in Kurepa's thesis, and described it as follows [1954, 34]: "Aronszajn's example is similar to the one I have constructed. It yields Example 1 but is not as suitable for the application to Moore spaces."

3.4 Kurepa

Đuro Kurepa[114] made a systematic study of uncountable partial orders, especially trees in the wider framework of ramified sets, starting in his thesis, *Ensembles et*

[113]While Jones' example was published two decades after its construction (the paper included suitable references to related work that had been published in the meantime), it is included here because of its influence on students in the Texas school of topology, who heard him talk about it and the related road spaces constructed from it.

[114]Đuro Kurepa (August 16, 1907 – November 2, 1993) was a proud, tall, energetic individual from a region known for tall people. He spent a substantial portion of his career at the University of Zagreb, an ethnic Serb in Croatia. Specifically, he got his diploma there in 1931 and became an assistant in mathematics there. He spent 1932-1935 in Paris at the Faculté des Sciences and the Collège de France. He obtained his doctoral degree at the Sorbonne in 1935 where the president of his committee was Paul Montel, his supervisor (we would say advisor) was Maurice Fréchet, and whose third member was Denjoy. Knaster was one of the reviewers of his thesis (cf. Kurepa's Math Review MR0423284 (54 #11264)). He spent some postdoctoral time in Paris and Warsaw. Then he returned to the University of Zagreb, where he became an assistant professor in 1937, was promoted to associate professor 1938, and full professor in 1948. According to Hrovje Šikić, a Professor at the University of Zagreb and currently editor-in-chief of *Glasnik Matematički*, Kurepa was interested not only in mathematics, but also in languages, and would sometimes include in his lectures comments about idiosyncrasies in Serbian and Croatian word formation. An example Šikić recalled was the words for curve, which is *krivulja* in Croatian and *kriva* in Serbian. They came up a semi-popular seminar, about which Kurepa said neither should be used since they failed to follow the rule of names ending in "un", and he recommended that *krivun* be used instead to great laughter in his audience. Kurepa moved to the University of

Tableaux Ramifiées. This section starts with a detailed review of his thesis which includes the definition of a Suslin tree;[115] the equivalence of Suslin's Problem with the non-existence of a Suslin tree; the construction of an Aronszajn tree as a subtree of the tree of all non-empty bounded well-ordered sequences of rationals under end-extension; and the definition of the key parameters for the structural analysis of uncountable trees: height, supremum of the sizes of the levels, and cardinality of the set of branches whose length is the height of the tree. The review of his thesis is followed by an overview of results from the 1930's beyond the thesis: his construction of a special Aronszajn tree; his initial results on $\mathbb{Q}$-embeddable and $\mathbb{R}$-embeddable[116] Aronszajn trees; his Fundamental Relation bounding the size of a partial order in terms of the sizes of its well-ordered and conversely well-ordered subsets and the sizes of its antichains; and his use of a super-position of two linear orders to define a witness to the sharpness of the bound in his Fundamental Relation.

In the opening paragraph of the published version of his thesis, Kurepa [1935, 1] listed two logical problems that the theory of ordered sets had run up against: (1) the existence of a well-ordering of an arbitrary set,[117] and (2) the existence of a cardinal number between any cardinal κ and its power 2^κ.[118] They shaped much of the research in set theory in the 20th Century, and provided a context for Suslin's problem [1920], the motivating force in Kurepa's thesis.

While working on his thesis, Kurepa shared his progress in a series of four announcements in *Comptes Rendus*: [1934a], [1934b], [1934d], [1934c].[119] Kurepa [1934a] announced a positive solution to Suslin's Problem in February with a theorem asserting the equivalence of seven statements about a linearly ordered set E, the first of which posits that E is a continuum isomorphic to each of its intervals and not the product of two sub-continua and the last of which asserts that the set E is dense, it has neither a first nor a last element, is metrizable and is complete. This note introduced the notion of the development or atomization of a linearly ordered set. In modern language this is the process of repeatedly partitioning into subintervals to end up with a family of intervals which are pairwise either disjoint or have one a subset of the other, and so that the family forms a tree under the subset relation. Kurepa discussed developments in §12 of his thesis, and

Belgrade in 1965 and remained there until his retirement in 1977. Kurepa was instrumental in the introduction in 1966 of *Glasnik Matematički* Series 3 as the mathematical successor to *Glasnik matematicčko-fizički i astronomski* Series 2. He died tragically in the time of Milošević and rampant inflation after being beaten by thugs and hidden from view under stairs.

[115] Recall that a *Suslin tree* is a tree of height ω_1 in which every branch and every antichain is countable.

[116] A partially ordered set is $\mathbb{Q}$-*embeddable*, respectively $\mathbb{R}$-*embeddable*, if it has a strictly increasing map into $\mathbb{Q}$, $\mathbb{R}$ respectively.

[117] This problem was addressed by Zermelo and Fraenkel as discussed in Kanamori's first chapter.

[118] On this problem, Kurepa wrote [1935, 1, footnote 2], [1996, 11, footnote 2] "La résponse présumée negative à cette problème est appelée l'Hypothèse de Cantor."

[119] Carlos Alvarez [1999] has written on the history Suslin's Problem, and his discussion of these announcements is particularly interesting and detailed.

the development of a Suslin line led to the equivalence of the existence of a Suslin line and the existence of a Suslin tree.

The second announcement, [1934b] in March, also claimed a positive solution to Suslin's problem, and as a step toward the proof introduced the two cardinals:

- $p_1E = \inf\{|F| : F \subseteq E$ is a dense subset of $E\}$, and
- $p_2E = \inf\{|\mathcal{F}| : F$ is a family of disjoint and non-empty open intervals of $E\}$.

Now p_1E is written $d(E)$ and called the *density*, and p_2E is written $c(E)$ and called the *cellularity*. These cardinals represent the beginning of Kurepa's generalization of the Suslin problem to a wider context, and they figure in Kurepa's Fundamental Relation discussed at the end of this section. In §7 [1996, 58–64] of his thesis, Kurepa defined and investigated p_1E and p_2E for a linearly ordered set E. He focused on the relationship between these cardinals and proved in Lemma 10 that to prove $p_1E = p_2E$ it sufficed to prove it for linearly ordered continua. He also explored when the supremum of p_2E is attained, and proved in Lemma 10′ that if the supremum p_2E is attained for every linearly ordered continua, then it is attained for every linearly ordered set. In §12.C.3, Theorem 2, Kurepa proved that $p_2E \leq (p_1E)^+$.[120] In §12.C.4, Lemma 5, he proved that if the density p_1E is singular, then the cellularity $p_2E = p_1E$ is attained. Todorcevic (cf. [Kurepa, 1996, 299]) combined Lemma 4 of §12.C.3, Lemma 5 of §12.C.4 and Theorem 3 of §11.4 [Kurepa, 1935, 110], [Kurepa, 1996, 100] to point out that Kurepa showed that if the cellularity $c(E) = p_2E$ is singular then it is attained. In footnote 11 [1935, 131][121] (the degree of) cellularity is extended to abstract spaces as the supremum of the sizes of disjoint families of open sets.

Now we turn to Kurepa's systematic study of set-theoretic trees[122] (ramified tables) within the context of ramified sets and ramified tables. Since the definition of a ramified table essentially filled six pages of the published version of the thesis as printed in [1996], we look at an important family of examples.

Suppose that $(P, <)$ is a strict partial ordering and that the relation $>$ is defined from it in the usual way. Let $\parallel$ be defined on P by $a \parallel b$ if and only $a \not< b$, $b \not< a$ and $a \neq b$. Let $\equiv$ be the relation of identity. Then $* = (\equiv, <, >, \parallel)$ is a *ramification relation* if for all points $a, b, c \in P$, if $a < c$ and $b < c$, then a and b are comparable, i.e. the set of predecessors of any point forms a chain.[123] Since all pairs are related

[120] Todorcevic [1996, 299] called this the first important result on the relationship between density $d(E) = p_1E$ and cellularity $c(E) = p_2E$.

[121] Unfortunately, the supplementary material in pages 127–135 of [Kurepa, 1935] entitled *Complément des résultats précédents* were omitted from the version of Kurepa's thesis in [Kurepa, 1996].

[122] Arthur Cayley studied finite trees, especially rooted finite trees, coming up with the generating function to list the number of rooted trees with n nodes in [1857], made connections with isomers in chemistry in [1875], and counted the number of distinct rooted trees with n nodes in [1881].

[123] The notation and splitting of all pairs recalls the formal splitting by Hausdorff [1909, 300], [2005, 273] of real-valued functions with a common ordered domain by final rank ordering into $f < g$, $f = g$, $f > g$, $f \parallel g$.

by one of $\equiv$, $<$, $>$ and $\|$, the set P is a ramified set with respect to this ramification relation.

Kurepa gave, as his first example of a ramified set, a genealogical tree [1996, 67], and his first example of a ramified table was the genealogical tree again [1996, 70], so it is clear that he intended to generalize the notion of tree.

For Kurepa, a ramified table was the result of the process of *ramification* applied to a partial order P, i.e. the stratification of the elements of the partial order into levels, by identifying a first level, removing the first level, letting the second level be the first level of the reduced set, and continuing in this fashion. Some partial orders fail to be suitable for ramification. A set $E \subseteq P$ *has a first level* [première rangée] if for every point $a \in E$, the set of predecessors of a in E has a least element; in such a case, the *first level of* E is the set of the least elements of the sets of predecessors of elements of E. Kurepa attributed this version of the definition to Fréchet; his original definition of first level was that it was the unique set F, if it existed, such that every element of $E \setminus F$ was preceded by one and only one element of F.

A ramified set is a *ramified table* if each of its non-empty subsets has a first level. In the lemma immediately following the definition, he showed that it is equivalent to require that each linearly ordered subset be well-ordered and commented in footnote 10 (see [1996, 70]) that Fréchet used this statement as the definition and that part of the proof of the lemma of equivalence of these definitions is due to Fréchet. Thus in the example we are considering, the definition is equivalent to and, in the alternate form, remarkably close to, the current standard definition of a set-theoretic tree as a partially ordered set in which the predecessors of each point are well-ordered.

Immediately after the discussion of the definition of ramified table T, Kurepa recursively defined the αth level [rangée], which he denoted $R_\alpha T$, and used the levels to define the *height* [rang] as the order type of the set of α for which $R_\alpha T$ is non-empty, the *length* [longeur] as the cardinality of the height, the *cofinality* [longeur réduite] (the cofinality of the height), and the *width* [largeur] as the upper bound of the cardinalities of the $R_\alpha T$s.

The name ramified table suggests that Kurepa envisioned a distribution of the elements of T into levels, with boxes to contain the elements at each level and immediate successor boxes all stacked above the box that is their common immediate successor, as might be drawn to illustrate the atomization of a linear order by intervals into a tree. Indeed, in the opening paragraph of the second chapter of his thesis Kurepa cites the idea of subdivision as motivation for his generalization of the relation of order to the relation of ramification. He used this approach to the construction of ramified tables and thus was able to show that if all ramified tables of cardinality ω_1 have either a chain or an antichain of cardinality ω_1, then there is no Suslin line.

Many graduate students have been challenged by their dissertation advisors to give examples that show their theoretical approaches are substantive, and Kurepa was not the only one who received help from a fellow graduate student in producing a desired example. Nachman Aronszajn,[124] also a student of Maurice Fréchet, constructed the first tree of uncountable height with all levels countable. The Aronszajn construction was announced in an abstract of Kurepa [1934c] and is briefly described in a footnote to Theorem 6 of §9 of Kurepa's thesis [1935, 96], [1996, 88]. Essentially, he built a suitable subtree of the set of one-to-one sequences of countable ordinal length of rationals in $\mathbb{Q} \cap (0,1)$ whose sums are finite and partially ordered them by end-extension. Aronszajn may have constructed such a tree prior to Kurepa, but Kurepa came up with his own construction of what is now known as an Aronszajn tree inside the tree $\sigma\mathbb{Q}$ of bounded well-ordered subsets of the rationals ordered by end-extension. He started with an initial level consisting of all rational singletons. At successor levels of the construction, he extended every node on the immediately preceding level in all possible ways. At limit levels α, he took any denumerable subset F of the collection of well-ordered subsets of $\mathbb{Q}$ all of whose initial segments were in levels already constructed which had the property that it and the tree up to that point were dense in each other. More generally, one can build trees inside the ramified table σE, which consists of the collection of non-empty bounded well-ordered subsets of a ordered set E ordered by end-extension.

In §10.1, Kurepa [1935, 98], [1996, 89] introduced the notion of *similarity* for ramified tables and called the similarity classes types of ramification. He showed that if E and F are similar linearly ordered sets, then so are the ramified tables σE and σF of the well-ordered bounded non-empty subsets of E, respectively F, ordered by end-extension. In §10.2, Kurepa sketched the proof that $\sigma\mathbb{Q}$ is *homogeneous*, i.e. that for every element $s \in \sigma\mathbb{Q}$, the set of $t \in \sigma\mathbb{Q}$ properly extending s is similar to $\sigma\mathbb{Q}$.

In §10.3, Kurepa introduced *distinguished ramified suites* [suites ramifiées distinguées]. A ramified table T is a *ramified suite* if for every $a \in T$, the set of $t \in T$ for which either $t <_T a$ or $a <_T t$ has cardinality $|T|$ (see §8.12 [1935, 76], [1996, 82]). A *distinguished ramified suite* is a ramified table S whose height κ is a regular initial ordinal and satisfies additional conditions, which, in the case $\kappa = \lambda^+$ is a successor cardinal may be phrased as follows: S has width λ [ambigu, i.e. ambiguous, neither narrow, nor wide]; S does not have a cofinal branch [descente monotone] and for every element $s \in S$, the set of all $x \in S$ with the same

[124] Nachman Aronszajn was born July 26, 1907 in Warsaw and died February 1980 in Corvallis, Oregon (cf. [Aronszajn, 2011], [Mardešić, 1999]). He received a University of Warsaw doctorate in 1930 under the direction of Stefan Mazurkiewicz, and a second doctorate [1935] from Université de Paris in 1935 where Fréchet was his advisor. In his Math Review MR0746781 (85m:01086a), Gerlach describes Aronszajn's working life as follows: 1930–1940 in France, 1940–1945 in England, 1945–1948 in France again, and the United States after 1948. He spent time at Oklahoma Agricultural and Mechanical University, leaving after the university fired Ainsley Diamond for failing to sign a loyalty oath (cf. [Magill, 1996]), and moved to the University of Kansas where he was a professor from 1951 until his retirement in 1977. He then moved to Corvallis where he was appointed an adjunct professor. Aronszajn was an analyst known for the Aronszajn-Smith Theorem that every compact operator on any Banach space of dimension at least two has an invariant subspace [1954] and for his work in reproducing kernel Hilbert spaces [1950]. For more on his life see [Szeptycki, 1983] (in Polish). For a list of his publications, see [Szeptycki, 1980].

set of predecessors as s has cardinality λ. Let us call S *uniformly branching* if the last condition together with the condition that every element s has successors on every larger level both hold. Then in modern language, the distinguished ramified suites of height λ^+ are the uniformly branching λ^+-Aronszajn trees.[125] At the end of §10.3, he asserted the existence of uniformly branching $\omega_{\beta+1}$-Aronszajn trees without stating the hypotheses (GCH and ω_β regular) that would make his proof generalize. He also raised the question of the existence of a κ^+-Aronszajn tree for an inaccessible cardinal.[126]

Kurepa opened §10.4 with the statement that we do not know if every uniformly branching Aronszajn tree is necessarily similar to a subset of $\sigma_0 = \sigma\mathbb{Q}$, and then stated the following questions:

> *Kurepa's First Miraculous Problem:* Must every two uniformly branching Aronszajn trees necessarily be isomorphic?[127]

He pointed out that the problem is equivalent to showing that every uniformly branching Aronszajn tree is homogeneous, and in [1935, 100, footnote 4], [1996, 91, footnote 26] posed the question of the existence of a homogeneous uniformly branching Aronszajn tree.

Next we examine results of Kurepa beyond his thesis. Recall that a tree is *special* if it is the union of countably many antichains. Kurepa constructed a special Aronszajn tree inside $\sigma\mathbb{Q}$ not long after his thesis in [1937a, 147–156], [1996, 130–138] introducing a class of trees that has resonated through the century.[128] Note that any Suslin tree is an Aronszajn tree, but since any countable collection of antichains whose union is uncountable must include an uncountable antichain, no Suslin tree can be special. To construct his special Aronszajn tree, Kurepa started with a function $g : \omega_1 \to \mathbb{R}$ with the property that for all $\beta < \omega_1$, the restriction of g to the interval of ordinals $[\omega_2\beta, \omega^2(\beta+1))$ is one-to-one and has range dense in $\mathbb{R}$. He used recursion to define $\varphi : \omega_1 \to \sigma\mathbb{Q}$ so that for all $\alpha < \omega_1$, $g(\alpha) = \sup \varphi(\alpha)$, and so that for each $\beta < \omega_1$, the image under φ of $[\omega^2\beta, \omega^2(\beta+1))$ is the βth level of the tree he is building, requiring essentially the same properties he required in his thesis, with the additional constraint imposed by g, and the requirement that for all $\xi < \zeta$, if $\varphi(\xi)$ is an initial segment of $\varphi(\zeta)$, then $g(\xi) < g(\zeta)$.[129] Most of the work in the proof is to ensure that the values assigned by g to limit levels are attained and are assigned such that the last mentioned inequality is always satisfied. At the end of the proof he considers two alternative restrictions on the

[125] For regular κ, a *κ-tree* is a tree of height κ all of whose levels have cardinality $< \kappa$, and a *κ-Aronszajn tree* is a κ-tree with no branch of length κ.

[126] Kurepa's definition of inaccessible is regular limit cardinal [1996, 22]. He does not mention GCH, under these and the (strongly) inaccessible cardinals coincide.

[127] For the original statement of the "problème miraculeux 1," see [1935, 100] or [1996, 91].

[128] Kurepa only noted in passing (see [Kurepa, 1996, 138]) that he was able to construct an Aronszajn tree which is a denumerable union of antichains; that is, this fact is not one of the numbered theorems, corollaries and lemmas in the paper.

[129] It is known today that care must be taken at the limit levels to ensure that such a tree is special. See [Devlin, 1972] discussed in §7.7.

functions: (a) for all β, the range of g restricted to $[\omega_2\beta, \omega^2(\beta+1))$ is the same as the range restricted to $[0, \omega^2)$; and (b) g is one-to-one. In case (a) with the range being the set of rational numbers, his proof gives a strictly increasing map into the rationals, i.e. it is $\mathbb{Q}$-embeddable, but he does not mention this fact explicitly. Instead he observes that for any x the set of elements of his tree whose sup is x is an antichain, so in this case the tree is a countable union of antichains. He also proved (see the discussion of [1996, 118] and Théorème 3^{bis} of Section 29 [1996, 140]) that there is an $\mathbb{R}$-*embeddable* Aronszajn tree, that is, one with a strictly increasing mapping into the set of real numbers. The paper also includes a proof by Aronszajn of the existence of an Aronszajn tree whose elements are closed subsets of a compact subset of a metric space ordered by the superset relation (see Théorème 2 [Kurepa, 1937a, 135],[130] [1996, 121]).

Kurepa [1940a, 837] proved that a partially ordered set admits a strictly increasing mapping into the rationals if and only if it is the union of countably many antichains. Let us expand the scope of the concept of *special* to apply to partially order sets in general. Then Kurepa's result may be restate as follows: a partial order is $\mathbb{Q}$-embeddable if and only if it is special.

Kurepa announced in December [1937b] an upper bound on the cardinality of a partially ordered set E. He used pE to denote the cardinality of E (p for puissance, power). For a non-empty E, he denoted the supremums of the cardinalities of several families of subsets of E as follows: p_cE is the sup of sizes of well-ordered subsets (c for croissant, increasing); p_dE is the sup of sizes of conversely well-ordered subsets (d for decroissant, decreasing); and p_sE is the sup of sizes of subsets of pairwise incomparable elements. He then set $p_0E = \max(p_cE, p_dE)$ and $bE = \max(p_0E, p_sE)$, prior to introducing his *fundamental relation* [1937b, 1196], [1996, 387]:

$$\textit{Kurepa's Fundamental Relation:} \quad bE \leq pE \leq (2p_sE)^{p_0E}.$$

Kurepa also showed that any well partial order in which all well-ordered subsets and all subsets of pairwise incomparable elements are countable must itself be countable [Kurepa, 1937b, Section V].

The proof of Kurepa's Fundmental Relation [1939] was published just before World War II, overlooked by reviewing journals, and reprinted in [1959b] in order to distribute it more widely. Note that the Fundamental Relation for partial orders generalizes the Hausdorff-Urysohn Inequality [Urysohn, 1924] for linearly ordered sets (i.e. those with $p_sE = 1$): $pE \leq 2^{p_0}$, using the notation given above. The proof is divided into two cases. In case A, Kurepa proved the theorem for partial orders in which all linearly ordered subsets are well-ordered, and in case B, Kurepa considered an arbitrary partial order, superimposed a well-order of the underlying set on it to get a new partial order in which all linearly ordered subsets are well-ordered, and applied case A to conclude the inequality held in general.

[130] Unfortunately, pages 135-138 are omitted from the copy of this paper posted on the site http://elib.mi.sanu.ac.rs/ of the eLibrary of the Mathematical Institute of the Serbian Academy of Sciences and Arts.

To show the sharpness of the bound in his Fundamental Relation, Kurepa gave the example of the lexicographic ordering on the set E of sequences of 0's and 1's of length $|n|$ since for any cardinal n (which we may assume is an initial ordinal), one has $p_0E = n$, $p_sE = 1$ and $pE = 2^n$. Thus the partial order considered in case B to prove the Fundamental Relation is the superposition of a lexicographic order and a well-order, which is now known as the Sierpiński poset.[131]

In his Math Review MR0058687 of [Erdős and Rado, 1953], Kurepa asserted that "the idea of using superposition of orderings of a same set was, as far as we know, first introduced in 1937 by the reviewer." He referred the interested reader to the paper discussed above and to the announcement in [1937c] of a two-part paper [1940a], [1941], specifically pointing to page 487 in the latter (see also footnote 28 on page 179 of [1996]), where he had given the superposition of a well-order on another partial order.

Kurepa was a pioneer in the study of uncountable partial orders and what have come to be called trees, and, as with many pioneers, his ground-breaking work was not immediately recognized.

4 1940-1950: PIONEERING PARTITION RESULTS

The 1940's featured pioneering results in the Ramsey theory of the uncountable and the development and refinement of some basic tools of the trade. Progress was made on understanding the structure of partial orders and trees, and Maharam used a Suslin line in her exploration of the theory of measure algebras. Specific combinatorial results for this decade include the Δ-system lemma, the first unbalanced generalization of Ramsey's Theorem to the uncountable, and foundational results in partitions of pairs from a set. König's Lemma was generalized to measurable cardinals, and generalizations of Aronszajn trees to cardinals above ω_1 were made. Questions were raised that stimulated interest in quasi-orders and in ω_1-trees with many branches.

In this decade, Gödel [1940] published his monograph on the constructible universe, establishing the consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the axioms of set theory. Garrett Birkhoff's *Lattice Theory* [1940] became an oft-cited reference in the theory of orders. Mostowski and Tarski [1949] announced that the first-order theory of well-ordered sets is decidable.[132]

[131] For example, Fraïssé [2000, 47] defined the *Sierpiński poset* to be the superposition of the usual ordering on the countable ordinals with an order induced from the real numbers via an injection of the set of countable ordinals into the set of real numbers. Recall that Sierpiński [1933] used this approach to define a symmetric binary relation, which via translation is now known as the Sierpiński partition, but it is not a partial order.

[132] Slomson in his Math Review MR0519800 (80d:03027a) observed that the decidability of the first-order theory of well-ordered sets was first obtained by Mostowski and Tarski in 1938–1941. It was one of the items in Mostowski's notebook that was burned during World War II. The proof via an axiomatization and quantifier elimination argument appears in a paper joint with John E. Doner, a student of Tarski, in [1978]. Slomson noted that he had an essentially similar

World War II had a devastating effect on mathematical life with many mathematicians killed outright or sent to concentration camps, and others decamping: Erdős left Hungary; Rado emigrated to England; and Tarski emigrated to the United States. After Germany invaded Poland and disrupted intellectual life there, an underground university sprang up in Warsaw, and Kuratowski, Mostowski and Sierpiński were among those who taught in it. After the end of the war, they rebuilt the group in logic and set theory. Publication of journals and books were slowed or disrupted as in the case of *Fundamenta Mathematicae*, and authors shifted to journals they had not used before and did not return to after the war. With the German occupation of France during World War II, the publication of the treatise *L'Énumération transfinite, Livre I* [1946] by Arnaud Denjoy[133] was postponed to 1946.[134] This book included a bibliography compiled by Gustave Choquet[135] (cf. [Denjoy, 1946, xxv–xxxvii]) consisting mainly of references appearing after 1920 and organized by topics, including the following most closely related to the subject of this chapter: (a) linearly ordered sets; the general case; (b) characterization of the linear continuum; Suslin's Problem; (c) well-ordered sets; ordinal numbers; (d) ordinals of the second number class; enumeration; (e) inaccessible ordinal numbers. It included Erdős' paper of [1942] with the foundational results in the partition calculus and papers by Dushnik and Miller prior to their [1941], and listed Kurepa's work.

4.1 Combinatorial tools

Before we get into the specifics for order and Ramsey theory, we review advances in the basic tools of combinatorial set theory in a short subsection for ease of reference. A collection of sets is a Δ-*system* if there is a set K such that any two distinct sets in the collection have K as their intersection. Nikolaĭ Šanin[136] [1946] proved the following lemma, which I have restated in modern terminology.

> *The Δ-System Lemma for Finite Sets:* Any uncountable collection of finite sets has an uncountable subcollection which is a Δ-system.

Another tool in combinatorial set theory is the use of equivalence relations, which were discussed by Stoll [1949] in an *American Mathematical Monthly* article,

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axiomatization in [1972].

[133] Arnaud Denjoy received his doctorate in 1909 from École Normale Supérieure Paris, where Baire was his advisor. His main focus was the theory of functions of a real variable.

[134] In a footnote to the preface dated March 1946, Denjoy [1946, vi] wrote "Les conditions créées par l'occupation allemande en France de 1940 à 1944 ont fait que de ce Traité le Livre I, composé typographiquement en première épreuve dès les premier mois de 1942, n'a pu être édité qu'en 1946.

[135] Choquet earned his doctorate at École Normale Supérieure Paris in 1946 with Arnaud Denjoy as his advisor.

[136] Nikolaĭ Aleksandrovič Šanin (May 25, 1919 –) was a student of A. A. Markov at Leningrad State University. His Ph.D. dissertation *Extensions of Topological Spaces* was defended in 1942 and his D.Sci. dissertation was defended in 1946. (See G. Mints, http://logic.pdmi.ras.ru/šhanin/biogr2.html downloaded on August 11, 2009.)

particularly in connection with the study of algebra. We quote from page 377: "we mention that the set E of all possible equivalence relations definable over a given set S has been made an object of study," and note his examples: [Birkhoff, 1940] and [Ore, 1942].

Rado [1949a] introduced a useful principle, which he called "a combinatorial lemma" in his article extending Whitney's notion of rank for finite sets to rank for sets of arbitrary cardinality. Below is a modern formulation using functions.

> *Rado Selection Principle:* Let $\mathcal{F} = \{F_i : i \in I\}$ be a finite family of sets and for each finite $J \subseteq I$, let $g_J : J \to \bigcup \mathcal{F}$ be such that $g_J(j) \in F_j$ for all $j \in J$. Then there is a function $h : I \to \bigcup \mathcal{F}$ such that for all finite $M \subseteq I$ there is a set N with $M \subseteq N \subseteq I$ so that for all $n \in M$, $h(n) = g_N(n)$.

The Rado Selection Principle was the compactness principle of choice for combinatorialists of his day and played an important role in transversal theory.

4.2 Ordered sets and their structure

We start this section with a result of Rothberger which is pertinent to the Normal Moore Space Conjecture, followed by two questions of Erdős and Tarski.

The *pseudointersection number* $\mathfrak{p}$ is the minimal cardinality of a family of subsets of ω with the strong finite intersection property (all intersections of its finite subfamilies are infinite) which has no infinite pseudointersection, i.e. no single set that is almost contained in every member of the family. Fritz Rothberger [1948] proved that if the pseudointersection number is greater than $\aleph_1$, then there is a *Q-set*, i.e. an uncountable subset of the real line every subset of which is a G_δ[137] in the relative topology.

The last part of the paper [1943] by Erdős and Tarski is subtitled *General remarks on inaccessible numbers* and includes a list of six problems concerning a cardinal $\mathfrak{n}$, of which the last two (see [1943, 327–328]) are most pertinent to this section.

> Problem 5. (*The Ordering Problem.*) Is it true that every ordered set N of power $\mathfrak{n}$ contains a subset X of power $\mathfrak{n}$, which is either well ordered, or becomes well ordered if we invert the ordering relation?
>
> Problem 6. (*The Ramification Problem.*) Let ν be the smallest ordinal number such that the power of all ordinals $\xi < \nu$ is $\mathfrak{n}$. Is it true that every ramification system of the ν^{th} order has power $< \mathfrak{n}$ for every $\xi < \nu$, contains a well-ordered subset of the type ν. (By a ramification system S we understand a partially ordered set which has the property that, for every $x \in S$, the set $S(x)$ of all elements $y \leq x$ is well ordered; If the set $S(x)$ is of the type ξ the element x is said to be of the ξ^{th}

[137] A set is a G_δ if it is the countable intersection of open sets.

> order. The order of the whole set S is the smallest ordinal number greater than the order of all elements of S.)

For the Ordering Problem, Erdős and Tarski note that the answer is obvious for $\mathfrak{n} = \aleph_0$, and for uncountable $\mathfrak{n}$ which are not inaccessible they credit Hausdorff [1914, 145–146] with the solution, noting ([Erdős and Tarski, 1943, 328]): "he does not state the solution explicitly, but it can be deduced easily from his results." For the Ramification Problem, the authors note that a solution for $\mathfrak{n} = \aleph_0$ was given by Dénes Kőnig [1927]. Erdős and Tarski continued: "for numbers $\mathfrak{n} > \aleph_0$ which are not inaccessible it was given by Aronszajn" (sic). It is surprising that the authors do not mention Kurepa, who brought the word *ramification* into mathematics for this type of system. Tarski presumably was acquainted with Kurepa's work after Kurepa's visit to Warsaw in 1937 (see [Kurepa, 1937a]), but Erdős was not. In his prior [1942], Erdős had already used the approach he came to call the *ramification method* without using this name.

Disjointness

In a paper received by the journal in 1946, Rado [1949b] investigated the circumstances under which it is possible to divide a system of intervals $\langle I_\nu \mid \nu \in N\rangle$ into k (k is finite) pairwise disjoint families, where N is an abstract index set. At the beginning of Section 2 on page 510, he sets the scene: "Sets occurring in this note may have any arbitrary cardinality. Zermelo's axiom will be used freely." He proved that a necessary and sufficient condition is that no $k+1$ intervals I_ν belonging to $k+1$ distinct indices have a point in common. He gave a different characterization in the following proposition.

> *Rado's Disjoint Interval Theorem:* If any subsystem of $k+1$ distinct interval $I_{\nu_0}, I_{\nu_1}, \ldots, I_{\nu_k}$ can be subdivided into k groups of pairwise disjoint intervals, then the same kind of subdivision is possible for the whole system.

Erdős and Tarski in [1943] generalized almost disjointness to partial orders and investigated sizes of mutually exclusive subsets of a partial order $(P, \leq)$. They declared y and z from $(P, \leq)$ to be *disjoint* if there is no non-null element x with $x \leq y$ and $x \leq z$, and called a subset of P *mutually exclusive* if its elements are pairwise disjoint in this generalized way. For $F \subseteq P$, they set $\mathfrak{d}(F)$ to be the least cardinal greater than the power of every mutually exclusive subset of F. Define a closely related notion by setting $m(F)$ to be the least upper bound of the set of cardinalities of mutually exclusive subsets of F. Erdős and Tarski showed that for any partial order P, if $m(P)$ is not an uncountable inaccessible cardinal, then P contains a mutually exclusive set of power $m(P)$. Their proof involved an analysis of non-decreasing functions from P to the cardinal numbers. They applied their results to Boolean algebras, rings of sets and fields of sets. In particular, they showed that if F is a field of sets and $m(F)$ is not a regular uncountable inaccessible cardinal, then there is a mutually exclusive subset of power $m(P)$; and if κ is a regular uncountable inaccessible cardinal, then there is a field G with $m(G) = \kappa$ which does not contain a mutually exclusive subset of power κ. Erdős and Tarski also showed in Corollary 3 and its proof that in a topological space X with uncountable cellularity which is not a regular limit cardinal, the supremum $c(X)$ is realized as the size of a pairwise disjoint family of $c(X)$ many open sets of X. This result extends Kurepa's result [1935] for spaces with singular cellularity.

Dushnik and Miller

Dushnik and Edwin Miller[138] [1940] examined linearly ordered sets and order-preserving embeddings. Recall that every infinite well-ordering is order-isomorphic to a proper subset. The first question that Dushnik and Miller asked was if the same is true for every infinite linear order, and they proved that it is true for denumerable linear orders. They observed that for a well-ordering $(A, <)$, every order-preserving embedding $f : A \to A$ has the property that $f(a) \geq a$ for all $a \in A$, and asked if this property characterizes well-orderings among linear orderings. They show that this property does characterize the well-orderings among the denumerable linear orderings.

In the same paper, they go on to construct, using the Axiom of Choice, an uncountable subset E of the reals which is rigid, i.e. the only order-endomorphism is the identity. They appear not to be aware of the type ring for scattered linear order types discussed in [Hausdorff, 1908] or the fact that he showed that every type is either scattered or a sum over a set of dense order type of scattered types.

In a subsequent paper, Dushnik and Miller [1941] introduced a notion of dimension for partial orders, namely the minimal size of a family of linear orderings whose intersection results in the partial order. They constructed examples showing that for every cardinal κ, there is a partial order whose underlying set has cardinality κ and whose dimension is κ. They revisited their rigidity result from [1940] by constructing an example N that has the property that for any uncountable subset $M \subseteq N$, M is not similar to any proper subset of itself which differs from M in more than a denumerable infinity of points.

In Theorems 5.24 and 5.25, Dushnik and Miller showed that in a infinite partial order P, on the one hand, if every subset of power $|P|$ has two incomparable elements, then P has a denumerable linearly ordered subset, and on the other hand, if every denumerable subset has two incomparable elements, then P has a

[138] Edwin Wilkinson Miller and Dushnik were fellow students at the University of Michigan: Miller received his Ph.D. from Raymond Wilder in 1930 (Raymond Wilder's doctorate was from the University of Texas at Austin, where he was student of R. L. Moore). In a letter dated October 15, 1929 (parts of this letter are quoted in [Fitzpatrick, 1997, 47]), Hildebrandt [1929] wrote to Moore about Miller's work on whether two of Moore's axioms from his 1929 Colloquium Lecture for the Summer Meeting of the AMS in Boulder define a metric space. Thus Miller was familiar with families of sets which were nested or disjoint. In 1934, when Hildebrandt, began a term as chair of the department, Miller became a member of the department and continued there until his death of a heart attack in 1942 (see [Kaplan, 1971, 184–185]).

linearly ordered subset of power $|P|$. Note that the above results of Dushnik and Miller for uncountable partial orders follows from the approach Kurepa took to showing that any partial order in which all well-ordered subsets and all subsets of pairwise incomparable elements are countable must itself be countable in [1937b, Section V], and from his proof that any narrow tree has a linearly ordered subset of cardinality the power of the entire tree in Théorème 5 [1935, 80], [1996, 76]. A proof appears in [Kurepa, 1948a, 30, Lemma 5], [1996, 209] that every uncountable well-founded partial order with all finite levels in its stratification has an uncountable chain.

Further work of Miller on order is discussed in §4.3, and additional Ramsey related work of Dushnik and Miller is discussed §4.4.

Order types

Sierpiński [1946] showed there are at least $2^{\mathfrak{c}}$ many real types by defining a family of that many pairwise incomparable order types of sets of real numbers of power $\mathfrak{c} = 2^{\aleph_0}$ (he did not use the Continuum Hypothesis).

Philip Carruth [1942] showed that the maximum order type of the union of sets of ordinals A and B of order type α and β respectively is the Hessenberg sum of α and β.

Noberto Cuestra Dutari [1943, 243] showed in Theorem 15 that the collection of dyadic sequences of length ω_α which are eventually zero, ordered lexicographically, is universal for linearly ordered sets of cardinality $\aleph_\alpha$, reproving a result known to Hausdorff. Recall that in the case of countable linear orders, the set of rationals is a countable linear order universal for all countable linear orders.

Kurepa [1948b] characterized in [1996, 200, Théorem I] the denumerable scattered linear order types as the denumerable order types that do not embed the order type of some countable ordinal. He noted that the supremum of the order types of the well-ordered subsets of η, the order type of the rationals, is $\aleph_1$, citing [Hausdorff, 1914, 99].

We continue the discussion of order types in the next section with Fraïssé's paper which includes results for order types so as not to split the discussion of it between two subsections.

Well-quasi-order theory

First steps toward the development of a theory of well-quasi-orders[139] were taken. Roland Fraïssé,[140] in his paper [1948] on the comparison of order types, wrote

[139]A *quasi-order* is a reflexive transitive binary relation $\leq_Q$ on a set A; it is a well-quasi-order (wqo) if for every infinite sequence $\langle a_i : i < \omega \rangle$ of elements of A, there are $i < j$ such that $a_i <_Q a_j$.

[140]Roland Fraïssé (1920 – March 30, 2008) received his doctorate from Université de Paris in 1953 under René de Possel (cf. http://www-history.mcs.st-andrews.ac.uk/Biographies/Possel.html). Denjoy was de Possel's advisor (cf. [Pouzet, 1992]). In his thesis Fraïssé [1953], [1954] used the back-and-forth method to determine whether two model-theoretic structures are isomorphic. He introduced amalgamation and developed a

$S < T$ to indicate there is a subset of T of type S and $S \ll T$ for $S < T$ and not $T < S$, and called a type T poor (pauvre) if $T \ll \eta$, i.e. T is countable and scattered. He used Sierpiński [1928a] as a reference and announced several results, a few of which are listed below, including an independent discovery of the characterization of countable non-scattered ordinals given by Kurepa above:

1. A necessary and sufficient condition for a type to be non-scattered is that it embeds every denumerable well-ordered set.
2. For every finite or denumerable set of scattered types T_i, there corresponds a scattered type T with $T_i \ll T$ for all i.
3. Every scattered type T is the sum of a finite sequence, an ω-sequence or an ω^*-sequence of scattered types T_i with $T_i \ll T$.

Here are the four *Fraïssé Conjectures* he made in [1948, 1331]:

> *Hypothèses.* – Nous considérons comme vraisemblables les propositions suivantes:

I. Il n'existe pas de suite infinie de types dénombrables $T_1, T_2, \ldots, T_i, \ldots$ avec $T_{i+1} \ll T_i$ pour tout i.

II. Si T est pauvre, le nombre des classes d'équivalence définies par la relations $\approx$ sur l'ensemble des types $< T$ est au plus dénombrable.

III. Soient E une famille, finie ou dénombrable, de types dénombrables t; $T(E)$ un type quelconque $> t$ quel que soit t de E; $\mathcal{T}$ un type inférieur à tout $T(E)$; alors $\mathcal{T}$ est inférieur à l'un des types t.

IV. Tout type indécomposable pauvre T est similaire au type d'une ω-suite ou d'une ω^*-suite de types $T_i \ll T$, ces types T_i étant soit indécomposables, soit de puissance 1.[141]

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method for constructing countable homogeneous models from finite structures via amalgamation (see his influential book [2000]). Fraïssé held a chair of mathematics in Algiers 1941–1959, and then spent most of his career at Université de Provence in Marseille, France. He was one of the founders of Bourbaki (cf. [Delsarte, 2000]).

[141] *Conjectures.* – We consider the following propositions plausible:

I. There is no infinite sequence of denumerable types $T_1, T_2, \ldots, T_i, \ldots$ with $T_{i+1} \ll T_i$ for all i.

II. If T is poor $[T \ll \eta]$, the number of equivalence classes defined by $\approx$ of the set of types $< T$ is at most denumerable.

III. Let E be a family, finite or denumerable, of denumerable types t; let $T(E)$ be an arbitrary type $> t$ for every t in E; let $\mathcal{T}$ be a type strictly below $T(E)$; then $\mathcal{T}$ is below one of the types t.

IV. Every indecomposable poor T is similar to the type of an ω or ω^* sequence of type T_i strictly below T, with the types of the T_i being indecomposable or of power 1.

Erdős [1949] asked the following problem in the *American Mathematical Monthly* that is related to the first Fraïssé Conjecture:

> If a set A of integers has the *divisor property* (i.e. for every sequence $\langle a_i : i < \omega \rangle$ of elements of A, there are $i < j$ such that $a_i | a_j$), then does is follow that the set of all products of n-elements from A has the divisor property.

This question stimulated interest in the notions of well-quasi-order (wqo) and well partial order.[142] Eric C. Milner [1985] attributes the first development of wqo theory to Graham Higman [1952], in his analysis of divisibility in algebras. Higman used the equivalent notion of having the *finite-basis property.*[143]He proved the equivalence of the finite-basis property with the non-existence of both infinite antichains and infinite descending sequences, and referred to parallel work by Erdős and Rado (see [1952b], [Rado, 1954b][144]).

4.3 Around Suslin's Problem

The definitions of Aronszajn and Kurepa trees grew out of Kurepa's investigation of Suslin's Problem.

Aronszajn and Kurepa trees

Kurepa [1940b] revisited the family $\sigma_0 = \sigma\mathbb{Q}$ of all (non-empty bounded) well-ordered subsets of the rationals introduced in his thesis and proved (see Lemma 1 on page 148) that every uncountable subset of this tree under end-extension includes an uncountable antichain. On the same page, in footnote 7, he commented that we do not know if there is a Suslin line which is universal for Suslin lines. He also proved that every $\mathbb{R}$-embeddable uncountable ramified table T has an antichain of cardinality $|T|$.

Note that the result described in §3.4 for partial orders of the equivalence of being the countable union of antichains and being $\mathbb{Q}$-embeddable applies to Aronszajn trees: they are special if and only if they are $\mathbb{Q}$-embeddable.

Kurepa continued his work on the classification of trees with a paper [1942][145] that focused on suites T[146] of height ω_1 in which every level is countable and has the further property that if the set of predecessors of an element is a successor

[142]A *well partial order* is a well-founded antisymmetric partial order $(P, <)$ with no infinite antichains.

[143]The *closure* of a subset A of a quasi-order $(P, <)$ is the set $\mathrm{cl}(A)$ of all $q \in P$ for which there is some $a \in A$ with $a \leq q$. A *closed subset of* P is a set which is its own closure. A quasi-order has the *finite-basis property* if every closed subset is the closure of a finite subset.

[144]Rado [1954b] used the concept of *partial well-order*, a well-founded relation with the finite-basis property but not necessarily transitive.

[145]Todorcevic (cf. [Kurepa, 1996, 8]) referred to [Kurepa, 1942] as "one of Kurepa's best written and most frequently cited papers."

[146]Recall that for Kurepa, a suite is a particular kind of ramified table.

ordinal or empty, then there are denumerably many elements with this set of predecessors; while if its set of predecessors has limit order type, then it is the unique element with this set of predecessors. That is, he focused on those T which are a denumerable forest of uniformly branching ω_1-trees. As Kurepa's earlier work was missed by many, we note that in this paper Kurepa referred to his thesis and the proof of the existence of an Aronszajn tree and the equivalence of a negative answer to Suslin's Problem to the existence of a Suslin tree. He formulated an equivalence of the Continuum Hypothesis in terms of a particular kind of ramified suite.

Kurepa proved that every denumerable forest T of normal ω_1-trees with the property that every element of T is in an uncountable branch must have uncountably many uncountable branches, and pointed out that we do not know how to evaluate the cardinality of the set of uncountable branches. The existence of an ω_1-tree with more than $\aleph_1$ branches is now known as *Kurepa's Hypothesis*, abbreviated KH and such trees are called *Kurepa trees.* Kurepa constructed an example μ of a denumerable forest of ω_1-trees in which every element is in an uncountable branch as follows: the elements of μ are finite sequences of countable ordinals ordered by $\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle < \langle \beta_0, \beta_1, \ldots, \beta_k \rangle$ if and only if $n \leq k$, $\alpha_n < \beta_n$ and for all $i < n$, $\alpha_i = \beta_i$. Note that s and $s^\frown\langle 0 \rangle$ are incomparable and have the same set of predecessors, so each element has infinitely many companions with the same set of predecessors. Also, the sequence of all $s^\frown\langle \xi \rangle$ for $\xi < \omega_1$ is a branch, so that every node is on an uncountable branch. Kurepa proved that every denumerable forest of ω_1-trees contains a copy of μ.

While Ernst Specker[147] was a graduate student, Roman Sikorski visited and asked the following question. For a regular cardinal κ, suppose collections D_α for $\alpha < \kappa$ of sequences of length α of 0's and 1's are given which enjoy the following four properties: (1) D_1 is non-empty; (2) if $\alpha < \beta < \kappa$, then every element of D_β is an extension of an element of D_α; (3) if $\alpha < \beta < \kappa$, then every element of D_α has an extension in D_β; and (4) for all $\alpha < \kappa$, $|D_\alpha| < \kappa$; must there be a sequence of length κ which, for all $\alpha < \kappa$, extends an element of D_α? The condition that the sequences be binary was dropped to make the question more interesting. Henry Helson [1949] had already shown the existence of a family of sets of sequences with the first three properties and no sequence of length κ whose initial segments all appear in the appropriate D_α. Specker set himself the task of constructing a counter-example, that is, the levels of a κ-Aronszajn tree for κ the successor of a regular cardinal. He succeeded by assuming the GCH and using increasing functions (see [Specker, 1949]).

[147] Specker was born February 11, 1920 in Zurich, received his doctorate from Eidgenössische Technische Hochschule (ETH) in 1949, and spent his professional life there. He worked on Quine's set theory New Foundations, and is perhaps best known for the Kochen-Specker Theorem in quantum mechanics. On August 14, 2009, the Mathematics Genealogy Project site listed 42 students for him, including Walter Deuber and Hans Läuchli, and selected works have been collated in [Specker, 1990].

Suslin Problem variations

Suslin's Problem continued to stimulate and influence research. During this decade various topological questions related to Suslin's problem were investigated; Miller and Sierpiński came up with equivalences in terms of partial order; and Maharam used Suslin's Problem in her investigation of measure for Boolean algebras. The lack of references to Kurepa by Miller, Maharam, and Sierpiński show that Kurepa's [1935] formulation in terms of a ramified table (Suslin tree) was not well-known at the time.

We note in passing that Y. S. Ochan[148] [1941] investigated two strengthenings of a topological variation of Suslin's problem and showed that they were equivalent.

Say a topological space T has the *Property of Knaster* if every uncountable family of open sets of T contains an uncountable subfamily of sets which pairwise have non-empty intersection. In April 1941,[149] Knaster [1945] proved that for complete linearly ordered sets, the Property of Knaster is equivalent to separability, i.e. having a countable dense set. Thus Suslin's Problem is equivalent to the question of whether, every Suslin continuum has the Property of Knaster. In May 1941, Knaster and Marczewski (under the name of Szpilrajn) asked in problem 192 of the *Scottish Book* [Mauldin, 1981, 265–266] if there is a topological space with the countable chain condition which does not have the Property of Knaster. Marczewski went on to ask whether the product of two partial orders with the countable chain condition necessarily has the countable chain condition. Both questions turned out to be independent of set theory; for details see [Mauldin, 1981] and Marczewski [1945], [1947].

Edwin Miller [1943][150] described a partial order whose existence is equivalent to the existence of a Suslin line: P has power $\aleph_1$; every subset of power $\aleph_1$ contains two comparable elements and two non-comparable elements; and if x and y are non-comparable elements, then there exists no z such that $x < z$ and $y < z$.

To prove the existence of such a partial order from a Suslin line, Miller recursively defined a sequence of non-overlapping (closed) intervals of length ω_1 and ordered them by reverse inclusion.[151]

To prove the existence of a Suslin line from such a partial order, he used two results that follow from the Dushnik-Miller Theorem (see below) to build a tree where each node has denumerably many successors; he linearly ordered the successors in type $\omega^* + \omega$; he passed to the lexicographic order, noted that there are no jumps, and filled any gaps by adding new elements to get a complete linear order.

[148] Yuri Semyonovich Ochan is perhaps best known for his book of problems in mathematical physics [1967]. His name has been variously transliterated as Ju. S. Očan, Y. S. Ochan, G. Otchan.

[149] The date for Knaster's proof is in the *Scottish Book* [Mauldin, 1981].

[150] A footnote in Miller's paper noting that the paper was received July 10, 1942 also notes that the author died on July 23, 1942.

[151] The idea of using a sequence of non-overlapping intervals to study a space is natural to those familiar with Moore spaces.

Sierpiński [1948, 165] also gave an equivalence to the existence of a Suslin line in terms of families of infinite subsets partially ordered by the subset relation, which is, in essence, another description of a Suslin tree:

> *Problème P.* Soit F une famille infinie d'ensembles jouissant de trois propriétés suivantes:
>
> 1. X et Y étant deux ensembles de la famille F, on a ou bien $X \subset Y$, ou bien $Y \subset X$, ou bien $XY = 0$.
> 2. Chaque sous-famille de F formée d'ensembles disjoints est au plus dénombrable.
> 3. Chaque sous-famille F_1 de F telle que, dans chaque couple de ses éléments, il existe un élément qui est contenu dans l'autre, est au plus dénombrable et a un ensemble maximal (c. à d. contenant tous les autres ensembles de F_1).
>
> La famille F est-elle nécessairement dénombrable?[152]

On the one hand, Sierpiński appeared unaware of Kurepa's work on Suslin's Problem, while on the other, he referenced [Denjoy, 1946] for literature on the problem, and this reference includes a bibliography complied by Choquet, as mentioned at the beginning of Section 4, which listed Kurepa's work. Sierpiński, in a note added in proof, cited work of Miller [1943] and Maharam [1947] on Suslin's Problem.

Dorothy Maharam[153] [1947] provided a set of necessary and sufficient conditions

[152] *Problem P.* Let F be an infinite family of sets enjoying the following three properties:

1. If X and Y are two sets of the family F, then one has pairwise $X \subset Y$, or $Y \subset X$, or $X \cap Y = \emptyset$.
2. Each subfamily of F of pairwise disjoint sets is at most denumerable.
3. Each subfamily F_1 of F such that, in each pair of its elements, there is one element which is contained in the other, is at most denumerable and has a maximal set (that is to say, containing all the other sets in F_1).

Is the family F necessarily denumerable?

[153] Dorothy Maharam was born in West Virginia in 1917. She did her undergraduate work at Cornell University and obtained her doctorate at Bryn Mawr College under the supervision of Anna Pell Wheeler in 1940 with a dissertation titled *On Measure in Abstract Sets.* Next she held a postdoc at the Institute for Advanced Study at Princeton, where she met Arthur H. Stone, whom she married in April 1942. Stone was at Princeton University during 1941–1942 and held a series of short appointments (Purdue 1942–1944, Geophysical Laboratory of the Carnegie Institute in Washington, D.C., Trinity College) before becoming a lecturer at Manchester University 1948–1957, where Maharam joined the staff in 1952. They both obtained professorships at the University of Rochester in 1961 where they stayed until retirement (cf. [Cohn, 2000]). For a brief biography of Maharam, see [Oxtoby, 1989] and for an overview of her work, especially on measure theory, see [Choksi *et al.*, 1989]. David Fremlin [1989] declared her "the chief architect of the present theory of measure algebras" in his masterful exposition of measure algebras in the *Handbook of Boolean Algebras.* A recent paper of Boban Veličković includes a survey of recent work on Maharam algebras, i.e. complete Boolean algebras that admit probability measures. Arthur Stone described working with her in an interview with W. Wistar Comfort [1997].

for a countably complete Boolean algebra to have a measure.[154] In a discussion of whether one of her conditions for the existence of a continuous outer measure can be replaced by the countable chain condition, she indicated the thinking of the times about Suslin's hypothesis [1947, 164]: "We now show that an affirmative answer is unlikely—or at all events can hardly have an easy proof—by showing that it would imply the truth of a well-known hypothesis of Souslin." She called a family of sets A a *Souslin system* if

1. every pair is either comparable under $\subseteq$ or disjoint,
2. every subfamily whose sets are pairwise disjoint is countable,[155] and
3. every subfamily whose sets are pairwise comparable is countable.

Maharam stated that "it was conjectured by Souslin that every Souslin system is countable." By suitable thinning and selection of maximal disjoint families a level at a time, Maharam transformed an uncountable Souslin system into a Suslin tree (or forest of trees) under the subset relation. She used this tree to construct a Boolean σ-algebra with no non-trivial outer measure (see Theorem 5).

The following year, Maharam [1948] proved that all Souslin subsystems of Boolean algebras are countable if and only if on each non-atomic Boolean algebra satisfying the countable chain condition there exists a real-valued function f such that (1) f is monotone non-decreasing, (2) f vanishes only at 0 and (3) every non-zero element contains non-zero subelements on which f is arbitrarily small. She further showed that for a fixed non-atomic Boolean algebra E with the countable chain condition, to show that all its Souslin subsystems are countable it is sufficient to have such functions satisfying the three conditions for every such subsystem, but it is not sufficient to have a single function for E since under the assumption that Suslin's Hypothesis is false, she gave a counter-example.

In setting the context for the paper, Maharam recalled her result that Suslin's Hypothesis is implied by the existence of a non-trivial outer measure satisfying certain conditions and in a footnote on the first page attributed the implication from the existence of such a measure to Gödel. She asserted that the equivalence of Suslin's Hypothesis to the statement that all Souslin systems are countable was well-known, citing [Miller, 1943] together with some results from [Dushnik and Miller, 1941].

4.4 Ramsey theory

Marshall Hall [1948] extended Philip Hall's Marriage Theorem to an infinite system of finite subsets of a given set. Independently, C. J. Everett and G. Whaples [1949]

[154] She thanked Arthur H. Stone in a footnote at the bottom of the first page, where she noted that the paper is based in part on work done during 1941-1942 when she held the Alpha Delta Xi Fellowship of the American Association of University Women and that she had presented a preliminary version of the paper to the American Mathematical Society in November 1942.

[155] Maharam [1948] called this constraint the *countable chain condition* where Kurepa [1935] used *countable degree of cellularity.*

extended Hall's Marriage Theorem to infinite sets.[156] Other versions include one due to Paul Halmos and Herbert Vaughan [1950] using Tychonoff's Theorem, and another due to Walter Helbig Gottschalk [1951], who derived it as a corollary to the Rado Selection Principle (see Subsection 4.1).

Rado [1943] extended his classification of which systems of linear equations with positive integer coefficients are k-regular to systems of homogeneous linear equations whose coefficients are in a ring of complex numbers.

We now turn to the first steps in extending Ramsey's Theorem for partitions of pairs to the uncountable, and we will look at three papers: the Dushnik and Miller paper [1941] discussed above for its results on order types, a less well-known paper [1942] by Erdős published in a Latin American journal, and a related paper [1953] by Kurepa who obtained the results in 1950, extending previous work from 1939. We discuss the mathematics of Kurepa's paper here and postpone comment on its publication date in 1953 to the section on the 1950's.

Dushnik and Miller [1941] formulated a positive generalization of Ramsey's Theorem for pairs from an uncountable set of power κ partitioned into two color sets, K_0 and K_1, by weakening the conclusion to ask for either a set of size κ all of whose pairs are in K_0 or a countable set all of whose pairs are in K_1. They discussed graphs and partitions of sets of pairs in terms of the structure theory for order, having observed that any partial order gives rise to a graph (binary, symmetric relation) under the relation of comparability. Here is the statement of their Theorem 5.22 [1941, 606] under the name by which it is now known.[157]

> *Dushnik-Miller Theorem:* If G is a graph of power $\mathfrak{m}$, where $\mathfrak{m}$ is a transfinite cardinal, and if every subset of G of power $\mathfrak{m}$ contains two connected elements, then G contains a complete subgraph of power $\aleph_0$.

Dushnik and Miller acknowledge help from Erdős [1941, 606, footnote 6]:

> We are indebted to P. Erdős for suggestions in connection with Theorems 5.22 and 5.23. In particular, Erdős suggested the proof of 5.22 for the case in which $\mathfrak{m}$ is a singular cardinal.[158]

The above result is the first *unbalanced* generalization of Ramsey's Theorem. Theorem 5.23 is dual of 5.22; it showed that if every subset of G of power $\aleph_0$ contains two connected elements, then G contains a complete graph of power $\mathfrak{m}$. These two theorems are translations to the language of graphs of Theorems 5.24 and 5.25 discussed in §4.3.

Dushnik and Miller [1941, 608] exhibit a counter-example to a possible extension of these theorems in the case when $\mathfrak{m} = \aleph_1$, namely the comparability graph of the superposition, on a set of cardinality $\aleph_1$, of a linear order obtained by mapping it into the reals and a linear order obtained by mapping it into "the well-ordered series

[156] In a footnote, Everett and Whaples credit the basic idea of the proof to Erdős.

[157] This theorem is sometimes called the Erdős-Dushnik-Miller Theorem.

[158] In [Hajnal, 1997] the proof by Erdős is described as requiring a good technical knowledge of the set theory of the time.

consisting of all the ordinals of the first and second class" [finite and countable ordinals]. That is, they used the comparability graph of what is now known as the Sierpiński poset.

Soon after his work for [Dushnik and Miller, 1941],[159] Erdős [1942] stated and proved some of the foundational results of what has become known as the partition calculus. Here he used, as was common at the time, "sum" for "union" [1942, 364]:

> *Theorem I:* Let a and b be infinite cardinals such that $b > a^a$. If we split the complete graph of power b into a sum of a subgraphs at least one of them contains a complete graph of power $> a$.
>
> In particular: If $b > c$ (the power of the continuum) and we split the complete graph of power b into a countable sum of subgraphs; at least one subgraph contains a non-denumerable complete graph.
>
> Theorem I is best possible. As a matter of fact, if $b = a^a = 2^a$ we can split the complete graph of power b into the sum of a subgraphs, such that no one of them contains a triangle. For the sake of simplicity we show this only in the case $b = c = 2^{\aleph_0}$.

Erdős described the example[160] as the complete graph G on points of the interval $(0, 1)$ and for each positive integer k, the edges of G_k are those pairs $\{x, y\}_<$ such that

$$\frac{1}{2^{k-1}} \geq y - x = \frac{1}{2^k}.$$

Erdős, in his Theorem II, assumed the Generalized Continuum Hypothesis and supposed that the complete graph on $m = \aleph_{\alpha+2}$ many points is the sum $G = G_1 + G_2$, and proved that if G_1 does not contain a complete graph of power m, then G_2 contains a complete graph of power $\aleph_{\alpha+1}$.

Prior to his proofs of the two theorems, he wrote "Tukey and I have shown by a result of Sierpiński that the complete graph of power $\aleph_1$ can be decomposed into the countable sum of trees. Without assuming the continuum hypothesis we can not decide whether this also holds for the complete graph of power $\aleph_2$." (See [1942, 365]; the result that Tukey and Erdős used is in [Sierpiński, 1924].)

Both proofs used a construction process that used a graph G to build a tree or *ramification system*, as it was called in his paper [1943] with Tarski; we describe only the proof for Theorem I. If G is a graph on $b > a^a$ points and the edges of a G are partitioned into a subgraphs enumerated as $\langle G_\alpha : \alpha < |a| \rangle$, then the process starts by choosing an arbitrary p, splitting the remaining points into sets Q_α where $q \in Q_\alpha$ if α is the least index such that the pair $\{p, q\}$ is in G_α. At successor stages $\xi = \eta + 1$, this approach is repeated; if $s : \eta \to |a|$ is the index of a non-empty set Q_s, then p_s is chosen as an arbitrary element of Q_s and the remaining points are split into sets $Q_{s^\frown\langle\alpha\rangle}$ where $q \in Q_{s^\frown\langle\alpha\rangle}$ if α is the least index

[159] Baumgartner [1997, 71] also used "soon after" to indicate when Erdős proved the results of [1942].

[160] Hajnal [1997, 354] reported that Erdős mentioned that the example was pointed out to him by Gödel.

such that the pair $\{p_s, q\}$ is in G_α. At limit stages, intersections are taken. The construction stops when all the points have taken on the role of being the chosen point. By cardinality considerations, the number of chosen points whose index is a sequence of length less than $|a|^+$ is at most $|a| \cdot |a|^+ = |a|^+$, so there must be a point r not chosen at any level of the construction of ordinal height $< |a|^+$. Since at the ηth level, every point of G has either been chosen or is in one of the sets with an indexing sequence of that length, it follows that r is in the intersection $\overline{G}$ of the level sets Q_s to which it belongs. For some α^* there must be $|a|^+$ many sequences s with $\overline{G} \subseteq Q_{s^\frown\langle\alpha^*\rangle}$ and the set of points p_s for which this is true is a complete subgraph of G_{α^*} of order type $|a|^+$. Erdős did not point out that r may be added to get a sequence of length $|a|^+ + 1$.

Erdős [1942] attributed the result that there is a graph on $2^{\aleph_\alpha}$ which does not have a complete subgraph of power $\aleph_{\alpha+1}$ nor an independent[161] subset of that power to Sierpiński, who proved it for graphs of size $\aleph_1$.

The last part of Erdős-Tarski [1943], subtitled *General remarks on inaccessible numbers*, includes a list of six problems, of which problems 5 and 6 were discussed in §4.3. The following is a problem for an inaccessible cardinal $\mathfrak{n}$.

> Problem 4. (*The Graph Problem.*) Is it true that if a complete graph G of power $\mathfrak{n}$ is split into two graphs G_1 and G_2, at least one of the contains a [complete] subgraph of power $\mathfrak{n}$? (A graph is to be defined as an arbitrary set of non-ordered couples (x, y) with $x \neq y$. By a complete graph of power $\mathfrak{n}$ we mean the set of all such couples formed from the elements of a set N of power $\mathfrak{n}$.)

In the footnote (page 328) with information on the Graph Problem, the authors cite Ramsey's Theorem for $\mathfrak{n} = \aleph_0$ (where the answer is positive), and a paper by Erdős to appear in *Revista de Tucumán*[162] for uncountable $\mathfrak{n}$ which are not inaccessible (where the answer is negative).

The final paragraph of the footnote lists implications of the form positive solution to problem a implies positive solution to problem b. Of particular note is the fact that a positive solution of the Ramification Problem discussed in §4.3 leads to a positive solution of the Graph Problem. Proofs of these implications were postponed.

Some connections between graphs and partial orders were investigated. Marczewski [1945, 307] had shown that every graph can be represented by the disjunction relation[163] on some collection of sets. Kazimierz Zarankiewicz [1947] connected Marczewski's representation of graphs via the disjunction relation to

[161] An *independent* subset of a graph is one in which no pair of points is joined by an edge of the graph.

[162] The paper by Erdős mentioned here is his [1942] discussed above.

[163] Given a graph represented as $G = (V, E)$, Marczewski observed that the collection $\mathcal{D}$ of non-empty sets $D_x = \{\{x, y\} \notin V \mid y \in V \setminus \{x\}\}$ has the property that $\{a, b\} \in E$ if and only if $D_a \cap D_b = \emptyset$.

symmetric relations.[164]

At some point in the late 1940's and early 1950's, Kurepa expanded his focus beyond ramified tables, partial orders and the study of monotone functions of them, to include graphs. Since a disjunction relation was one of the constituent parts of Kurepa's relation $* = (<, >, \equiv, \|)$ of ramification,[165] as explicated in his thesis [1935], the paper [Marczewski, 1945], cited in the paper of Kurepa discussed below, may have been the one that led Kurepa to think about applying his Fundamental Relation (see the end of §3) to graphs.

In May 1950 Kurepa proved the following theorem for a binary symmetric relation[166] $G = (G; \rho)$, where kG is the cardinality of the vertex set, k_cG is the supremum of the sizes of connected subsets of G,[167] and k_sG is the supremum of the sizes of disconnected subsets of G.[168] We will call this theorem *Kurepa's Graph Relation* ([1953], [1996, 399]):

> *Theorem 0.1* For any graph $G = (G; \rho)$ one has
>
> (0.4) $\quad kG \leq (2k_sG)^{k_cG}$.
>
> For any $\aleph_\alpha$ and any cardinal number $n \leq 2^{\aleph_\alpha}$ there is a graph $g = g(\aleph_\alpha, n)$ such that
>
> (0.5) $\quad k_cg \leq \aleph_\alpha,\ k_sg \leq \aleph_\alpha,\ kg = n$;
>
> in particular (for $n = 2^{\aleph_\alpha}$) there is a graph M_α such that
>
> (0.6) $\quad k_cM_\alpha = \aleph_\alpha k_sM_\alpha,\ kM_\alpha = 2^{\aleph_\alpha}$.

With his orientation toward the study of partial order and the fact that he was updating his proof of the Fundamental Relation from [1939], Kurepa referred to the totally disconnected sets as antichains and the pairwise connected sets as chains.

Toward an overview of Kurepa's proof, for $\alpha < (k_cG)^+$, let $T_\alpha = {}^\alpha G$ be the set of functions (sequences) from α into G. Note that $T = \bigcup_\alpha T_\alpha$ is a tree under end-extension. Let $\delta(X)$ denote a maximal disconnected subset of X. Kurepa used δ to define sets $D_\alpha \subseteq T_\alpha$ (see below) by recursion on α such that $D = \bigcup\{D_\alpha : \alpha < (k_cG)^+ \wedge D_\alpha \neq \emptyset\}$ is a subtree of T closed under initial segments that has the properties that (1) each element is an enumeration of a chain (complete subgraph) in G, and (2) each element with domain a limit ordinal is the union of

[164] Kurepa cited Zarankiewicz' paper in [Kurepa, 1959a]. Erdős, in Math Review MR0023047 (9,297c), translated the Zarankeiwicz results for symmetric relations into the language of finite graph theory and noted that the author's results can be deduced from [Turán, 1941].

[165] Kurepa used $\|$ as the incomparability relation; $\equiv$ was an equivalence relation, i.e. equality in the case of trees; and $>$ was the converse of $<$.

[166] Kurepa followed Sierpiński [1933] in his usage of binary symmetric relations.

[167] The *connected subsets* may be regarded as pairwise related subsets, complete subgraphs, or cliques.

[168] The disconnected subsets may be regarded as pairwise unrelated subsets, subgraphs in which no pairs are joined by an edge, or independent subsets.

its initial segments. It follows that each branch through D may be thought of as an enumeration of a chain of G, and hence has cardinality at most k_cG. Moreover, for each non-terminal element $s \in D$, the set of $a \in G$ such that $s^\frown\langle a\rangle \in D$ is an antichain (disconnected subset, independent subset) of G. Thus for all ξ with $D_\xi \neq \emptyset$, the subtree $\bigcup_{\eta\leq\xi} D_\eta$ is injectively embeddable into the tree ${}^{\xi\geq}(k_sG)$. Consequently, for positive $\xi < (k_sG)^+$, $|\bigcup_{\eta\leq\xi} D_\eta| \leq (k_sG)^{|\xi|} \leq (2\cdot(k_sG))^{(k_cG)}$. Since $D \subseteq T$ has no branches of length $(k_cG)^+$ and $(k_cG)^+ \leq 2^{(k_cG)}$, it follows that $|D| \leq (2\cdot(k_sG))^{(k_cG)}$. Kurepa ensured that for each $a \in G$ there are $\alpha < (k_cG)^+$ and a element $s \in D_{\alpha+1}$ with $s(\alpha) = a$. Therefore $|G| \leq |D|$ and the theorem follows.

For the interested reader, we describe the construction of the sets D_α from $G = (G;\rho)$ where ρ is a reflexive symmetric relation. It uses a function $E : \wp(G) \to \wp(G)$ defined by $E(X) = \{y \in G \mid \forall x \in X\rho(x,y)\}$. Note that since ρ is symmetric, $X \subseteq E(X)$. By an abuse of notation, for $s \in T$, write $E(s)$ for $E(\mathrm{ran}(s))$ and $E(s)\setminus s$ for $E(\mathrm{ran}(s))\setminus \mathrm{ran}(s)$. To start the recursion, $D_0 = \{\emptyset\}$. For $\xi = \eta+1$ successor, D_ξ is the set of all $s^\frown\langle a\rangle$ for $s \in D_\eta$ and $a \in \delta(E(s)\setminus s)$. For ξ limit, D_ξ is the set of all $s \in T_\xi$ such that $E(s)\setminus s \neq \emptyset$ and all proper initial segments of s are in $\bigcup\{D_\eta \mid \eta < \xi\}$.

Kurepa showed that his graph inequality is sharp for graphs of size $2^{\aleph_\alpha}$ as follows. He let M_α be the collection of all ω_α-sequences of rational numbers, and defined a partial order $\leq$ on M_α as the superposition of a well-ordering of M_α and the lexicographic ordering. His example is the comparability graph of this partial order, $(M_\alpha, \leq)$.[169]

Kurepa used Hausdorff's language for the ordering "by first difference" to describe the lexicographic order, and used Hausdorff's Theorem XIV [1908, 472] in the proof of the lemma that states that the subsets of the collection of ω_α-sequences of rationals which are well-ordered or converse well-ordered under the first difference order have power at most ω_α, and that there is a subset S of such sequences which either has order type ω_α or has order type ω_α^*.

Kurepa derived two corollaries and applied his results to a question of Sierpiński [1933]. Kurepa's Corollary 0.1 [1953, 400] is the assertion that if a graph has cardinality at least $\left(2^{\aleph_0}\right)^+$, then either it contains an uncountable connected (complete) induced subgraph or it contains an uncountable totally disconnected induced subgraph (independent set), which is an independent rediscovery of part of Theorem I by Erdős [1942].[170]

The corollary may be obtained by expressing the graph relation as an implication and taking the contrapositive. That is, the graph relation is equivalent to

[169] Ginsburg, in his Math Review MR0071485 (17,135b) noted that $(M_\alpha;\leq)$ given on page 91 of [Kurepa, 1953] is a generalization of an example in [Dushnik and Miller, 1941]. Recall that at the end of §3, we discussed the fact that in the late 1930's Kurepa used a partial order obtained as the superposition of a well-order and the lexicographic order on ${}^\omega 2$ to show his fundamental inequality for partial orders was sharp.

[170] Apply the Erdős result to the partition of the complete graph, i.e. the set of all pairs of vertices, into K_0, the set of connected ones, and K_1, the set of disconnected ones.

the following conditional statement: for any graph $G = (V, E)$, if the supremum of the cardinalities of its complete subgraphs is κ and the supremum of the cardinalities of its independent subsets is λ, then $|V| \leq (2\kappa)^\lambda$. While Kurepa did not do so, the modern reader, taking the contrapositive of the above conditional statement in the case where κ and λ are infinite cardinals and using basic cardinal arithmetic, derives the statement that if a graph (G, ρ) has at least $\left(\kappa^\lambda\right)^+$ points, then either there is a complete subgraph of size κ^+ or there is an independent set of size λ^+. For the situation of Erdős' Theorem II [1942] (and others like it with different size goals), this modern interpretation does not require the GCH which Erdős used, instead the guarantee that any graph on at least $({\aleph_{\alpha+1}}^{\aleph_\alpha})^+$ vertices will have either a complete subgraph on $\aleph_{\alpha+2}$ vertices or an independent set of size $\aleph_{\alpha+1}$. On the other hand, for the situation with equal size outcomes of size κ^+, neither Kurepa nor Erdős used GCH; both start with the same size set but Erdős' Theorem I [1942] works for κ many colors.

Kurepa applied his theorem to a question of Sierpiński on a problem of Knaster. Kurepa's description of Knaster's problem appears below, translated by the author:

> Does there exist a non-denumerable infinite graph G so that each infinite nondenumerable subgraph of G contains both: two distinct connected points and two distinct disconnected points.

Sierpiński came up with an example of a graph on $\aleph_1$ points meeting Knaster's requirement, and asked if one could have an example with $\aleph_2$ points. Kurepa pointed out that the graph relation means that the cardinality of any graph satisfying Knaster's condition has power at most the continuum, giving a negative answer to Sierpiński's question if the Continuum Hypothesis holds. Since his counterexample M_0 (essentially the generalization of Sierpiński's example to the continuum) is a graph of cardinality the continuum in which every uncountable subset contains both connected and disconnected pairs, a positive answer to Sierpiński's question is equivalent to the negation of the Continuum Hypothesis.[171]

Kurepa's Corollary 0.2 is the inequality for partial orders corresponding to the graph relation, i.e. equation (0.4) in Theorem 0.1. It is an updated version of the Fundamental Relation for partial orders announced by Kurepa in [1937b] and published in [1939], obtained by using the supremum of the cardinalities of *all* chains rather than restricting attention to increasing and decreasing well-ordered chains.

To place Kurepa's paper [1953] in context, note that the results on graphs are closely related to those of Dushnik and Miller [1941] and Erdős [1942], discussed above. Moreover, the proof of the Fundamental Relation that Kurepa gave in [1939] can be immediately adapted to prove his graph relation (0.4). Kurepa had the tools in hand early, but missed the opportunity to be the first to state and prove these fundamental results in Ramsey theory.

[171]Ginsburg, in his Math Review MR0071485 (17,135b) noted that Erdős [1942] already answered the question of Sierpiński.

5 1950-1960: FOUNDATION OF THE PARTITION CALCULUS

In the 1950's, a systematic study of partitions of finite subsets of an ordered set was initiated, the main lines of study delineated, and the useful arrow notation for partition relations was introduced. In a classic paper, Erdős and Rado proved a Positive Stepping Up Lemma from which the modern Erdős-Rado Theorem follows. Erdős and Rado generalized partition relations from cardinals to ordinals, from colorings of r-element sets to products of them, and introduced canonical partition relations as a way to treat partitions with potentially many classes. Kurepa gave an extended treatment of his approach to graphs and hypergraphs. Late in the decade, a seminar of Mostowski and Tarski revisited the Erdős and Tarski results for inaccessible cardinals which had been announced in 1943. By the end of the decade, all the members of the synergistic group consisting of Erdős, Hajnal, Milner and Rado had met one another and were working on partition relations.

The Regressive Function Theorem reached its modern form, and general Δ-system lemmas were proved. Interest in Suslin lines and objects that can be derived from them continued. Progress was made on the Normal Moore Space Conjecture: Bing proved that the existence of a Q-set implies the existence of a separable non-metrizable normal Moore space. Investigations into the basic structure of partial orders and their mappings were made and a partial order of size 2^κ universal for partial orders of size κ was constructed.

Sierpiński published *Cardinal and Ordinal Numbers* [1958], an expanded version his *Leçons sur les Nombres Transfinis* [1950]. In his Math Review MR650095787, Gillman commented "Professor Sierpiński writes in his usual kindly, patient style which makes reading a pleasure." Rodolfo A. Ricabarra published *Ordered and Ramified Sets (Contribution to the Study of Suslin's Problem)* [1958]. This book is an extended look at trees and linear order types related to them using Kurepa's ramified sets and tables, and has a chapter on equivalences and implications of Suslin's Problem.

We conclude this introduction with some results about combinatorial tools.

In answer to a question of Tarski, Robert Vaught [1952] showed that the statement "every family of sets has a maximal disjointed family" is equivalent to the Axiom of Choice. The statement can be derived by an application of Zorn's Lemma, and Vaught gave a simple argument that from it, one can find a transversal for any family of pairwise disjoint non-empty sets. Kurepa [1953], [1996, 401] used Vaught's result to show that the statement "every graph contains a maximal disconnected subgraph" is equivalent to Zermelo's axiom.

In 1958, Erdős and Rado submitted the paper [1960] in which they prove the general Δ-system lemma, with an interesting bound in the case of finite collections of finite sets. That year, Ernest Michael [1962] submitted his independently proved result equivalent to the general Δ-system lemma:[172] he generalized the result of

[172] Michael used *quasi-disjoint* for Δ-system. Michael is best known for the Michael Selection Principle [1956]; see also volume 155 of *Topology and its Applications*, a special issue dedicated to the theory of continuous selections of multivalued mappings and Michael's 80th birthday.

Šanin [1946] to higher cardinals. In a footnote added in proof, Michael indicated that Erdős had alerted him to the equivalent Erdős-Rado result and its longer proof. It may be that Michael's paper alerted Erdős to the existence of Šanin's paper as well. It is unsurprising that such a useful result would be discovered multiple times.

András Hajnal[173] met Erdős in 1956 and in [1997, 361–362] recounted the delightful story of how they came to starting working together on set mappings at that very first meeting (it included climbing 300 odd stairs to the top of the local cathedral). The resulting paper is [Erdős and Hajnal, 1958]. Set mappings are an important part of infinite combinatorics and useful for proving partition relations, so I take this opportunity to quote a result by Hajnal under the name by which it is now known [1961, 123]:

> *Hajnal's Set Mapping Theorem:* Assume κ is an infinite cardinal, $\kappa > \lambda$, and let $f : \kappa \to \mathcal{P}(\kappa)$ be a set mapping with the property that $|f(\alpha)| < \lambda$ for every $\alpha < \kappa$. Then there is a set $X \subseteq S$ of cardinality κ free with respect to f.[174]

5.1 The partition calculus

This subsection details chronologically the work of Erdős, Hajnal, Kurepa, Rado, and others as they explored a variety of approaches to finding subsets with uniform behavior in graphs, binary symmetric relations, and in distributions, partitions and colorings of r-tuples. A chronological presentation was chosen to place in historical context the May 1950 results of Kurepa [1953] whose mathematical context was discussed in §4.4.

The Erdős-Rado paper [1950] has been frequently cited as the beginning of the study of *canonical partitions.*[175] Any decomposition or coloring function f defined on r-tuples of a set A induces the "same color" equivalence relation on the r-tuples of A. This equivalence relation is defined even when the number of colors is infinite. Erdős and Rado proved that for each finite r, there is a finite set $\mathfrak{C}_r$ of equivalence relations on the r-element subsets of $\mathbb{N}$ with the property that for every equivalence relation E on the r-tuples of A there is some infinite $B \subseteq \mathbb{N}$ and some $F \in \mathfrak{C}_r$ such that the restrictions of E and F to the r-tuples of B are identical. We

[173] András Hajnal (May 13, 1931 –) was awarded his doctorate in 1957 at Jószef Attila University in Szeged, Hungary where his advisor was László Kalmar, a logician. His thesis was on relative constructibility (see Kanamori's first chapter for more on Hajnal's thesis work). The Mathematical Genealogy Project site lists 1956 for the year of his doctorate, but a curriculum vitae accessed September 9, 2010 from Hajnal's Hungarian website at http://www.renyi.hu/ ahajnal/hajn2006.pdf lists 1957.

[174] A *set mapping* is a function $f : S \to \mathcal{P}(S)$ so that $f(s) \notin s$ for every $s \in S$. A subset $X \subseteq S$ is *free* with respect to a set mapping f on S if for all $x, y \in X$, $x \notin f(y)$. See [Komjáth and Shelah, 2000] for the history of problems on set mappings and for some set mapping results about $\aleph_n$ at the end of the century. The history dates back to the thirties and a question of Paul Túran, and includes results by Sophie Piccard and Stanisław Ruziewicz, one of the founders of the Lwów School.

[175] The seminal paper [Ramsey, 1930] has an earlier classification of r-ary relations.

will call such a collection $\mathfrak{C}_r$ a *canonical basis* for the equivalence relations on the r-tuples of A. For concreteness, we note that for pairs, the canonical equivalence relations are $s \equiv_{\min} t$ iff $\min(s) = \min(t)$; $s \equiv_{\max} t$ iff $\max(s) = \max(t)$ and $s \equiv_{=} t$ iff $s = t$; and the equivalence relation in which all s and t are equivalent. For each finite positive r and each $I \subseteq r$, let E_I be the equivalence relation on the r-element subsets of $\mathbb{N}$ defined by

$$\{x_0, x_1, \dots, x_{r-1}\}_< \; E_I \; \{y_0, y_1, \dots, y_{r-1}\}_< \iff x_i = y_i \text{ for all } i \in I.$$

Here is a precise statement of Erdős and Rado's oft-quoted canonical partition relation for r-tuples of natural numbers:

> *Canonical Partitions for* $\mathbb{N}$: For every positive $r < \omega$ and every equivalence relation E on the r-element subsets of $\mathbb{N}$, there is an infinite set $M \subseteq \omega$ and an index set $I \subseteq \{0, 1, \dots, k-1\}$ such that the restrictions of E and E_I to r-tuples from M are identical.

In [Erdős and Rado, 1952a], the authors looked at partitions of order types and proved (cf. Theorem 4, pp. 428-9) that if the set of pairs of rational numbers is partitioned into two pieces, then either there is an infinite monotonic sequence whose pairs are in the first cell or there is a subset dense in an open interval all of whose pairs are in the second cell.[176] Rado,[177] in his joint work with Erdős [1952a], used the Axiom of Choice to define a partition of the collection T of all infinite subsets of an infinite set S into two classes, $T = K_0 \cup K_1$, so that there is no infinite set all of whose infinite subsets are in the same class as follows. Start with a well-ordering $<$ of T. Let K_0 be the set of all $A \subseteq T$ for which there is an $A' < A$ with $A' \subseteq A$. Set $K_1 = T \setminus K_2$. Observe that if S' is in T and A is its $<$-least infinite subset, then A is in K_2. Since A is infinite, using the Axiom of Choice, we may partition $A = \{b_i \mid i < \omega\} \cup C$, where the b_j are all distinct, none of them is in C, and C is infinite. Let $B_j = \{b_i \mid i < j\} \cup C$. Then for some m_0, B_{m_0} is the $<$-least among the B_j's. In particular, B_{m_0+1} is a proper subset of B_{m_0} with $B_{m_0} < B_{m_0+1}$, so B_{m_0+1} is in K_1.

Erdő and Rado called attention to the following problem [1952a, 418]:

> *All Finite Subsets Problem:* Given an infinite set S, is it possible to divide all finite subsets of S into two classes in such a way that every infinite subset of S contains two finite subsets of the same number of elements but belonging to different classes?

They indicate that such partitions exist if $|S| \leq 2^{\aleph_0}$,[178] but indicate that nothing is known for sets of larger cardinality. We conclude with the comment that Erdős

[176] Erdős mentioned this result in his paper [1996, 120] on his favorite theorems.

[177] The attribution of this result to Rado was made in [Hajnal, 1997, 354], and it dates back to 1934, as discussed in §3.2.

[178] In Example 2 [1952a, 435], Erdős and Rado showed there is such a partition of the finite subsets of a denumerable set S, which shows in modern terminology that $\aleph_0$ is not a *Ramsey cardinal*.

and Rado proved early results for polarized partitions in this paper, since we will refer to that fact later.

Kurepa gave several talks on his results about graphs from May 1950:[179] his September 6, 1952 talk in Munich at the Kongress der Deutschen Mathematiker-Vereinigung was entitled *Über binäre symmetrische Relation*; his December 3, 1952 talk for a Colloquium of the Society of Mathematicians and Physicists of Croatia in Zagreb was entitled *O simetričnim relacijama i grafovima*; and his February 24, 1953 talk for the Faculty of Sciences in Paris was entitled *Sur les relations binaires.* We learn later that Kurepa had intended to submit his work on graphs to *Journal für die reine und angewandte Mathematik* after his talk in Munich, but postponed publication in the expectation that he would be able to generalize the results. He submitted a preliminary report [1953] prior to the talk in Paris, probably in connection with his December 3, 1952 talk.[180] He learned from G. Riguet[181] during his visit to Paris about the work of others in Ramsey theory, citing in his preliminary report (1) Ramsey's seminal paper [1930], (2) Erdős' follow-up [1942] to the Dushnik-Miller Theorem, (3) the joint paper of Erdős and Rado [1953] discussed below, and (4) a paper of F. Burton Jones [1952].

In a paper with Erdős, Rado introduced the basic version of the arrow notation, "relation (5) $a \to (b_1, b_2)^2$." He called it the *decomposition relation* between cardinals a, b_1, b_2, and defined it as follows [1953, 427]:

> For any set S we denote by $\Omega_2(S)$ the set of all sets $S' \subset S$ such that $|S'| = 2$. Then we say that (5) holds if, and only if, the following statement is true. Whenever
>
> $$|S'| = \alpha; \quad \Omega_2(S) = K_1 + K_2,$$
>
> then there is $S' \subset S$ and $\lambda \in \{1, 2\}$ such that
>
> $$|S'| = b_\lambda; \quad \Omega_2(S') \subseteq K_\lambda.$$
>
> The relation (5) is fundamental in many investigations in set theory. The authors hope to deal in another paper with its numerous interesting properties and generalizations. In the present note it only serves as a convenient abbreviation.

The partition approach turned out to be remarkably well-suited to generalization, and Erdős spoke on his work with Rado [1956, 427] on some of these generalizations on October 24, 1953 in New York during his invited hour address, *Combinatorial problems in set theory*, for the Eastern Sectional Meeting of the American Mathematical Society (cf. [Cohen, 1954, 21]).

[179] The details given in this paragraph were found in [Kurepa, 1954] and [Kurepa, 1959a].

[180] Kurepa described the December 1952 talk in a footnote [1953, 65] as a presentation "during the session of 23.10.1952 of the Slovenian Academy of Sciences and Art."

[181] Almost surely Kurepa is referring to Jacques Riguet, who used order in his study of binary relations, closures, and Galois correspondences [1948].

Rado [1954a] tackled the question of canonical polarized partition relations and proved two theorems, using his joint work with Erdős [1950] on canonical partitions and the approach to polarization in their [1952a]. Rado used the notation $[N]^n$ for the n-element subsets of a set N, which has been widely adopted. He wrote $[A_1, A_2, \ldots, A_\ell]^{n_1, n_2, \ldots, n_\ell}$ for the collection of all sequences $(X_1, \ldots, X_\ell)$ where each X_i is in $[A_i]^{n_i}$. The simplest case of the second theorem is Proposition B from page 74, which states that every infinite matrix all of whose elements are either 0 or 1 contains an infinite submatrix equal to one of the following four matrices: the matrix of all zeros; the matrix with zeros on and below the main diagonal and ones above it; the matrix of all ones on and below the main diagonal with zeros above it; and the matrix of all ones. Below, I have stated Rado's first theorem with a descriptive name for easy reference:

> *Rectangle Refining Theorem:* Given positive integers ℓ, $n_1, \ldots, n_\ell$, p, there is a positive integer $f = f_\ell(n_1, \ldots, n_\ell, p)$ which has the following property. If Δ is any partition of $[N, N_f, N_f, \ldots, N_f]^{n_1, \ldots, n_\ell}$ then there are sets N'_λ, such that N'_1 is infinite, and $N'_\lambda \in [N_f]^p$ for $1 \leq \lambda \leq \ell$, and
>
> $$\Delta(X_1, X_2, \ldots, X_\ell) = \prod_{\lambda}^{\lambda} \Delta_\lambda(X_\lambda) \qquad (X_\lambda \in [N'_\lambda]^{n_\lambda}),$$
>
> where each Δ_λ is a canonical partition.[182]

The displayed equation means that two sequences $\vec{X} = (X_1, X_2, \ldots, X_\ell)$ and $\vec{X}' = (X'_1, \ldots, X'_\ell)$ are in the same cell of the partition Δ of $[N, N_f, N_f, \ldots, N_f]^{n_1, \ldots, n_\ell}$ if and only if for each index λ, the components X_λ and X'_λ are in the same cell of Δ_λ.

In May 1954, Kurepa published his Math Review MR0058687 (15,410b) of [Erdős and Rado, 1953], in which the first half delineated the contents of the paper, and the second half listed references he felt should have been included, citing five of his own papers and Theorem 14 of [Hausdorff, 1908]. Of relevance to the partition calculus, he wrote "especially the proof and the result in the reviewer's paper [his Fundamental Relation proved in [1939]] $\cdots$ are connected with lemmas 3 and 4 of the paper under review." For infinite m, n, the contrapositive of the implication derived from the inequality for graphs in Kurepa's Theorem 0.1 of May 1950 ([1953], [1996, 399]) expressed in Rado's notation is $(m^n)^+ \to (m^+, n^+)^2$. This partition relation is closely related to the Erdős-Rado lemmas, since Lemma 3 is $a \to (b, a)^2$ for a regular limit cardinal a and $b < a$; and Lemma 4 is $a^+ \to (b, a^+)^2$ if a is infinite and b is the least cardinal such that $a^b > a$.

On November 11, 1954, Kurepa added a short note to [1953], [1996, 409]: "The results were obtained in [M]ay 1950 (and were to be presented in a talk in USA) when the author was anxious to give a new proof of a theorem he published earlier

[182] See [Rado, 1954a, 76]. The notation N_f denotes a fixed set with f many elements; N denotes an infinite set, not necessarily the natural numbers.

[v. Kurepa C.R. Acad. Sci. Paris **205** (1937), 1196-1198 "la relation fondamentale (1)" and Kurepa [1], [3, §20]."[183] That is, Kurepa added the announcement of his Fundamental Relation to the references and emphasized the starting point of the article was Corollary 0.2, his chain-antichain inequality for partial orders, as he had noted in the body of the paper. Recall that the first theorem of [Kurepa, 1953] gives an inequality on the cardinality of a graph in terms of the supremum of the cardinalities of the sizes of its complete subgraphs and the supremum of the cardinalities of the sizes of its independent subsets, and addresses the sharpness of this inequality. Kurepa prefaced its statement with the remark that the purpose of the paper was to prove the first theorem.

Notice the shift in the way Kurepa presented his work and its connection with graphs: Kurepa described the first theorem of May 1950 as new at the time he found it [1953], but in the Math Review of [Erdős and Rado, 1953], Kurepa has, in essence, said it is implicit in his Fundamental Relation.

Erdős and Rado [1956][184] generalized the decomposition relation of their [1953] paper from cardinals to order types, from partitions into two cells into many cells, and from 2-tuples to r-tuples for finite values of r. They even mentioned, but did not treat, "more general partition relations referring to partial orders" [Erdős and Rado, 1956, 430]. In this seminal paper, Rado introduced, in full generality, the paradigm shifting definition of the *ordinary partition symbol* [1956, 429]:[185]

> Fundamental throughout this paper is the *partition relation* $a \to (b, d)^2$ introduced in [6]. More generally, for any a, b_ν, r the relation
>
> $$(1) \qquad a \to (b_0, b_1, \ldots)^r_k$$
>
> is said to hold if, and only if, the following statement is true. The cardinals b_ν are defined for $\nu < k$. Whenever $|S| = a$; $[S]^r = \sum_{\nu<k} K_\nu$, then there are $B \subset S$; $\nu < k$ such that $|B| = b_\nu$; $[B]^r \subset K_\nu$.

[183] The references are to [Kurepa, 1939] and his book on set theory, *Teorija skupova*, published in 1951. Kurepa did make a lecture tour in the United States of America in 1950, as one can learn from [Kurepa, 1952, 97, footnote 1], but spoke on a different topic.

[184] The paper was received by the editors on May 17, 1955. The authors did not mention the results of Kurepa [1953] which appeared very late in 1954 or possibly in 1955. A later paper, [Erdős *et al.*, 1965], reported that Erdős and Rado were unaware of Kurepa's work when they submitted their paper.

[185] As Hajnal said in [Hajnal and Larson, 2010], "There are cases in mathematical history when a well-chosen notation can enormously enhance the development of a branch of mathematics and a case in point is the ordinary partition symbol." A contrasting view is provided by Rudin [1975a, 7], who, after starting a section entitled *Ramification arguments and partition calculus* with eight statements, the first four of which are arrow relations and the second four of which are inequalities: "The gibberish above is a set of really quite important theorems, the first four from the partition calculus and the last four from topology. The topological theorems are proved using partition calculus theorems or techniques which often means using tree type proofs." In the next paragraph, she continues: "In this chapter we first introduce the partition calculus notation. This notation is a stone wall barrier which must be crossed in order to make the rich literature of partition calculus available."

When discussing $a \to (b_0, b_1, \dots)^r_k$, let us call a the *resource* and the b_i's the *goals*, and call a set *homogeneous for color* i or *monochromatic for color* i if all its r-tuples are in the ith cell of the partition. Erdős and Rado abbreviated $a \to (b_0, b_1, \dots)^r_k$ to $a \to (b)^r_k$ in the special case with $b_\nu = b$ for all $\nu < k$. Partition relations in which all the goals are the same have come to be called *balanced*; if not all the goals are the same, they are called *unbalanced*. This distinction, already mentioned in special cases earlier, proved useful since proof techniques varied for the two types. They reviewed previous results, proved monotonicity results for the ordinary partition symbol and delimited the trivial cases (see Theorem 22, page 438). For example, with the arrow notation, Ramsey's Theorem is reduced to the compact statement $\omega \to (\omega)^r_\gamma$ for finite r and $1 \leq \gamma < \omega$. Sierpiński's partition generalizes easily to $2^{\aleph_0} \nrightarrow (\aleph_1, \aleph_1)^2$. Theorem I from [Erdős, 1942] is the statement that if $b > a^a$, then $b \to (a^+)^2_a$. The core of Rado's result from his paper with Erdős [1952a] on decompositions of the infinite subsets into two classes may be written $\kappa \nrightarrow (\aleph_0)^{\aleph_0}_2$, and their All Finite Subsets Problem may be written $\kappa \to (\kappa)^{<\omega}_2$, where here we are introducing a generalization of the ordinary partition relation that appears in [Erdős and Hajnal, 1958], which will be discussed below.

Erdős and Rado [1956] explored the partition calculus using the ramification method. They proved the following ordinal partition relation for $m, n, \ell \geq 2$: if $\ell \to (m, n)^2$, then $\omega{\cdot}\ell \to (m, \omega{\cdot}n)^2$ (see Theorem 25, page 440). They investigated the partition properties of λ, the order type of the set of real numbers, and proved that $\lambda \to (\alpha, n)^3$ for every $n < \omega$ and $\alpha < \omega_1$ (see Theorem 31, page 447). They proved that the same is true for any uncountable order type Φ such $\omega_1 \not\leq \Phi$ and $\omega_1^* \not\leq \Phi$, i.e. neither ω_1 nor its converse order ω_1^* is order-embeddable into Φ, and conjectured that every positive partition relation for λ generalized to such Φ, but in footnote 7 on page 443 they note that Specker disproved this conjecture. They also proved $\omega_1 \to (\omega + m, 4)^3$. Partition relations for triples would turn out to be particularly challenging. They focused attention on the question of whether $\omega_1 \to (\omega{\cdot}2, \omega{\cdot}2)^2$ after proving for $\alpha < \omega{\cdot}2$ that $\omega_1 \to (\alpha, \alpha)^2$. In Theorem 44, they reproved the Dushnik-Miller Theorem by reproving that $\omega_1 \to (\omega_1, \omega + 1)^2$ (see [Erdős and Rado, 1956, 459]), and more generally if κ is regular of uncountable cofinality, by proving that $\kappa \to (\kappa, \omega + 1)^2$.

A key result of this paper is their Theorem 39 (i) quoted below with the name by which it has come to be known ([Erdős and Rado, 1956, 467–468]):

Positive Stepping Up Lemma: If

(81) $$\ell \to (\alpha_0, \alpha_1, \dots)^r_k,$$

(82) $$|m| > \textstyle\sum_{\lambda<\ell} |k|^{|\lambda|^r},$$

then

(83) $$m \to (\alpha_0, \alpha_1, \dots)^{r+1}.$$

One corollary of Theorem 39, which includes various consequences of the Positive Stepping Up Lemma is the fact that for every positive r and finite sequence of ordinal goals $\beta_0, \beta_1, \dots, \beta_{m-1}$, there is a resource α sufficiently large so that $\alpha \to (\beta_0, \beta_1, \dots, \beta_{m-1})^r$ (see page 470).

The proof of the Positive Stepping Up Lemma involved a new concept. An *end-homogeneous set* for a partition $g : [m]^{r+1} \to k$ is a set on which the color assigned by g to an $(r+1)$-element set does not depend on the largest element of the set.[186]

Let us assume m is a cardinal, or more precisely, the inital ordinal of that cardinality. For each ordinal $x < m$, define a sequence $\langle f_\beta(x) \rangle$ by recursion: $f_0(x) = 0$, and if $f_\beta(x)$ has been defined for $\beta < \gamma$, then $f_\gamma(x)$ is the least element y of $x \backslash \{f_\beta(x) : \beta < \gamma\}$ with the property that for all r-element $u \subseteq \{f_\beta(x) : \beta < \gamma\}$ one has $g(u \cup \{y\}) = g(u \cup \{x\})$, if there is such a $y < x$, and otherwise $f_\beta(x) = x$ and the recursion stops. Let $\rho(x)$ be the index of the final element of the sequence. Note that by construction, the sequence is end-homogeneous for g.

Erdős and Rado use an induction argument to show that if $\rho(x) = \rho(z)$ and for all indices $\beta_0 < \dots < \beta_{r-1}$, $g(f_{\beta_0}(x), \dots, f_{\beta_{r-1}}(x), x) = g(f_{\beta_0}(z), \dots, f_{\beta_{r-1}}(z), z)$, then $x = z$. Note that if $\eta = f_\beta(x)$, then the same argument shows the sequence $\langle f_\zeta(\eta) : \zeta \leq \beta \rangle$ is an initial segment of $\langle f_\zeta(x) : \zeta \leq \rho(x) \rangle$. That is, these sequences form a tree under end-extension and the argument here is essentially a ramification argument.

A consequence of the induction arguments is that the number of such sequences of length σ is bounded by the cardinal of the set of functions $h : [\sigma]^r \to k$, which is at most $|k|^{|\sigma|^r}$. By the cardinality hypothesis that $|m| > \sum_{\lambda < \ell} |k|^{|\lambda|^r}$, for some x_0 one has $\rho(x_0) \geq \ell$ so the induction hypothesis may be applied to $S_0 = \{f_\beta(x_0) : \beta \leq \ell\}$ and the coloring $g' : [S_0]^r \to k$ defined by $g'(u) = g(u \cup \{x_0\})$, to get a subset S_1 of size α_i monochromatic for color i for g', and $S_1 \cup \{x_0\}$ is the required witness to the partition relation for the original coloring g.

In [Erdős and Rado, 1956] two special cases of what has come to be called the Erdős-Rado Theorem were derived, using weakenings of GCH to get around their lack of good notation for iterated exponentiation. From the assumptions that $2^{\aleph_\nu} \leq \aleph_n$ for $\nu < n$ and $\omega_n \to (\alpha_0, \alpha_1, \dots)^r_k$, they derived $\omega_{n+1} \to (\alpha_0 + 1, \alpha_1 + 1, \dots)^{r+1}_k$ (note that n, k need not be finite). From the assumptions that $|k| < \mathrm{cf}(\aleph_m)$, $r \geq 0$ and $2^{\aleph_\nu} \leq \aleph_n$ for $m \leq n < m + r$ and $\nu < n$, they derived $\omega_{m+r} \to (\alpha_0 + r, \alpha_1 + r, \dots)^{r+1}_k$.

The Positive Stepping Up Lemma is the key ingredient in the proof of the modern version of the Erdős-Rado Theorem. For non-triviality, let us assume that $k \geq 2$ and each $\alpha_i \geq 3$. If $\ell = \omega$, then the assumption that $\ell \to (\alpha_0, \alpha_1, \dots)^r_k$ means each $\alpha_i \leq \omega$, $k < \omega$, and in this case, Ramsey's Theorem already yields $\ell = \omega \to (\alpha_0, \alpha_1, \dots)^{r+1}_k$. So assume $\ell > \omega$. Then $|k| < \ell$, else, since $\ell^r = \ell$, there

[186]The concept of an end-homogeneous set appears in the proof of the Positive Stepping Up Lemma, but the name appears only later e.g. in Chapter IV, Section 15 of the Erdős-Hajnal-Máté-Rado compendium [1984]. The idea also goes under the name *pre-homogeneous* which was used by Kunen [1977].

would be a one-to-one mapping from $[\ell]^r$ into k. Also $\sum_{\lambda<\ell} |k|^{|\lambda|^r} = \sum_{\lambda<\ell} |k|^{|\lambda|} = \sum_{\lambda<\ell} 2^{|\lambda|} = 2^{<\ell}$. Note that $\sum_{n<\omega} 2^n = \omega$ yielding the following theorem.

Positive Stepping Up Lemma (modern form): For all infinite cardinals κ, all γ with $2 \leq \gamma < \kappa$, all finite r, and all cardinals $\langle \lambda_\nu : \nu < \gamma \rangle$, if $\kappa \to (\lambda_\nu)^2_{\nu<\gamma}$, then $(2^{<\kappa})^+ \to (\lambda_\nu + 1)^{r+1}_{\nu<\gamma}$.

Let $\exp_n(\kappa)$ denote n-times iterated exponentiation, that is, $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$. With this and the cardinal arithmetic above, we state below the modern version of their Theorem 39, obtained using the Positive Stepping Up Lemma repeatedly and starting from the clear fact that $\exp_0(\kappa) = \kappa \to (\kappa)^1_\gamma$ for $\gamma < \mathrm{cf}(\kappa)$.

Erdős-Rado Theorem (modern form): For every infinite cardinal κ, every finite $r \geq 2$, and all $\gamma < \mathrm{cf}(\kappa)$, $(\exp_r(2^{<\kappa}))^+ \to (\kappa + (r-1))^r_\gamma$.

Since $\kappa^+ \to (\kappa^+)^1_\kappa$, by the Positive Stepping Up Lemma, $(2^{<\kappa^+})^+ \to (\kappa^+)^2_\kappa$. Use $2^{<\kappa^+} = 2^\kappa = \exp_0(\kappa)$ to rewrite this partition relation as $(\exp_0(\kappa))^+ \to (\kappa^+)^2_\kappa$. Repeated application of the Positive Stepping Up Lemma yields the most frequently quoted version of the Erdős-Rado Theorem: $(\exp_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa$.

Erdős and Rado used the notation $[S_0, S_1, \ldots, S_{t-1}]^{r_1, r_2, \ldots, r_{t-1}}$ for the Cartesian product of the t sets $[S_i]^{r_i}$ from Rado's paper [1954a] and introduced the *polarized partition relation* (see [Erdős and Rado, 1956, 480]), denoted

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{t-1} \end{pmatrix} \longrightarrow \begin{pmatrix} b_{0,0} & b_{0,1} & \ldots \\ b_{1,0} & b_{1,1} & \ldots \\ \vdots & & \\ b_{t-1,0} & b_{t-1,1} & \ldots \end{pmatrix}^{r_0, r_1, \ldots, r_{t-1}}_k$$

for the assertion that whenever $|S_i| = a_i$ for $i < t$, and $[S_0, \ldots, S_{t-1}]^{r_1, \ldots, r_{t-1}}$ is partitioned into k classes, then there is some $j < k$ and sets $B_i \subseteq S_i$ of cardinality $b_{i,j}$ so that $[B_0, \ldots, B_{t-1}]^{r_1, \ldots, r_{t-1}}$ is a subset of the jth cell of the partition. They noted on page 480 that "the passage from our former type of partition relation i.e. the case $t = 1$, to the more general kind ... bears a certain resemblance to the process of polarization used in the theory of algebraic forms, which accounts for the name *polarized partition relation*." They go on to prove some results expressed with this notation but do not give an extended study of it.

On a trip to Israel, Erdős stopped in Switzerland where he visited Specker and shared some of the partition problems with him. During the visit Specker was able to prove that for all $m < \omega$, the ordinal partition relation $\omega^2 \to (\omega^2, m)^2$ holds. Erdős continued to Israel and on his return learned that Specker had a counter-example witnessing $\omega^3 \not\to (\omega^2, 3)^2$ (see [Specker, 1957] for both results).

In a paper [1958] received by the journal in 1957, Erdős and Hajnal proved the first positive result in the study of partitions of all finite subsets of a set.[187] They

[187] Erdős and Hajnal used $[S]^{<\aleph_0}$ to denote the collection of all finite subsets of S, where $[S]^{<\omega}$ is now used.

introduced the notation $\kappa \to (\alpha)^{<\omega}_2$ which means that for any coloring of the finite subsets of κ with two colors, there is a subset A of order type α such that for all $n < \omega$, f is monochromatic on the collection of all n-element subsets of A. They proved that if κ is a measurable cardinal, then $\kappa \to (\kappa)^{<\omega}_2$. Thus if there is a measurable cardinal, then there is a set S for which the answer to the All Finite Subsets Problem of Erdős and Rado [1952a] is negative. Erdős and Hajnal also include in their paper [1958] a proof of a result due to Fodor that for all cardinals κ smaller than the least inaccessible cardinal, $\kappa \nrightarrow (\kappa)^{<\omega}_2$ holds.

Erdős and Hajnal met Eric Milner[188] at the International Congress of Mathematicians held in Edinburgh in 1958. Hajnal [1997, 374] recalled that Rado interested Milner in partition problems at the meeting and Milner settled a problem about countable ordinals which he [1969] only published much later. Erdős then visited him and starting working with him in Singapore, and they continued to work together throughout his career. At the suggestion of Erdős, Milner spent a sabbatical year 1958–1959 at the University of Reading working with Rado, and moved to Reading in 1961.

The finite powers of ω are all *additively indecomposable*, i.e. they cannot be written as the sum of two strictly smaller ordinals. It is well-known that the additively indecomposable ordinals are exactly those of the form ω^γ.[189] For specified indecomposable β and finite m, it is possible to determine an upper bound for the resource α needed to ensure that the positive partition relation holds. In particular, Erdős and Milner showed that $\omega^{1+\mu m} \to (\omega^{1+\mu}, 2^m)^2$. This result dates back to 1959 and a proof appeared in Milner's thesis in 1962. See also [Williams, 1977, 165–168] where the proof is given via the following stepping-up result:

> *Ordinal Stepping Up Lemma*: Suppose that γ and δ are countable and k is finite. If $\omega^\gamma \to (\omega^{1+\delta}, k)^2$, then $\omega^{\gamma+\delta} \to (\omega^{1+\delta}, 2k)^2$.
>
> *Corollary* [Erdős and Milner [1972]]: If $m < \omega$ and $\mu < \omega_1$, then $\omega^{1+\mu\cdot\ell} \to (\omega^{1+\mu}, 2^\ell)^2$.

During the 1958-1959 academic year, Andrzej Mostowski and Alfred Tarski[190] gave a seminar on the foundations of set theory at the University of California,

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[188]Eric Charles Milner (May 17, 1928 – July 20, 1997) had an irregular start to his mathematical career, having gone from an undergraduate degree at the University of London to Malaysia in 1951 where he worked as a tin assayer for the Straits Trading Company, before joining the mathematics department of the University of Malaya in Singapore. Milner earned his doctorate as an external student at the University of London in 1963, with Richard Rado and Roy Davies serving as his examiners. Richard Guy, a former colleague from Singapore, became head of the department at the University of Calgary, where Milner became a professor in 1967 and continued for the remainder of his career, where, in addition to working on his own mathematics, he served as chair of the department, organized meetings and colloquia, and was a genial host for visits by the Hajnals and Erdős among many others. For more on his life and work, see [Nash-Williams, 2000] and [Hajnal, 1997].

[189]See Exercise 5 on page 43 of Kunen [1983].

[190]Tarski facilitated the expansion of the logic group at the University of California, Berkeley, in the 1950's. Leon Henkin was hired in 1953, having refused an earlier offer due to a loyalty oath previously required by the University of California; Robert Vaught arrived in 1958 (he

Berkeley. Among the topics were the results announced by Erdős and Tarski in [1943], i.e. five problems involving inaccessible cardinals and other structures including the partition property $\kappa \to (\kappa)^2_2$, the tree property (every tree with height κ all of whose levels have size less than κ has a branch), the existence of non-principal κ-complete prime ideal in the set algebra $\mathcal{P}(\kappa)$ (the measurability of κ), and a representation problem for certain Boolean algebras. Erdős proved that if $\kappa \to (\kappa)^2_2$, then κ is inaccessible (see [Erdős and Tarski, 1961, 59, footnote 5], where the result is attributed to [Erdős, 1942], in which it is implicit). Later they [1961] published proofs, in part based on the seminar, noting (pp. 51-52) "some of the proofs are in the form in which they were reconstructed by some members of the seminar, Andrzej Ehrenfeucht, James D. Halpern, and Donald Monk, and may differ from those originally found by the authors".[191]

Near the end of the decade, Kurepa [1959a] gave a full exposition of his distinctive approach to the partition calculus. Conceived in 1952 when he submitted a preliminary report [1953] discussed earlier in this section, he had postponed the completion until well after [Erdős and Rado, 1956] since he [1996, 411] "thought that the same result should hold for n-ary symmetrical relations". His paper started from his thesis and his Fundamental Relation, whose proof he arranged to have reprinted in [Kurepa, 1959b]. It gave a gentle introduction to his approach to trees and ramification. In §7, Kurepa proved that if $|S| > 2^{\aleph_\alpha}$ and f is a coloring of the pairs of S with finitely many colors, then there is an $X \subseteq S$ with $|X| > \aleph_\alpha$ on which f is constant. In the arrow notation, $\left(2^{\aleph_\alpha}\right)^+ \to ({\aleph_\alpha}^+)^2_n$ for $n < \omega$. The proof is by induction on n. In a brief remark at the end of §7, Kurepa said he had expected this positive argument to generalize to larger tuples, which he then treated in §8 and §9 using a different approach. He introduced a compact notation for exponentiation that allowed him to state theorems with iterated use of exponentiation in a compact way. The results up to and including §7 appear to predate the appearance of [Erdős and Rado, 1956], while those after, that is, results for larger tuples, appear to be influenced by it. He reported learning from Erdős in 1959 that Hajnal had shown that the positive results were optimal, and Kurepa had decided to publish his recent work together with a reprinting of his [1939].

5.2 Applications of Suslin lines

Interest in Suslin's Problem and ordered sets continued in the 1950's, and new equivalences and applications of Suslin lines were found. Ricabarra published a book on the subject described briefly at the beginning of §5; Bourbaki added

had been a student of Alfred Tarski, receiving his doctorate in 1954). The interdepartmental graduate degree program in Logic and Methodology was established in 1959. Anil Nerode, who visited Berkeley 1958-1959 spoke of the state of logic in the United States in the 1950's in his preface [2004] to a volume for a Tarski centennial conference, and [Feferman, 2004] described the expansion of the logic group at Berkeley in the same volume.

[191] Hajnal [1997, 356] reported that Monk wrote [Erdős and Tarski, 1961].

material on Suslin sets to the second edition of *Elements of Mathematics*. Yesenin-Volpin [1954] showed that Suslin's Hypothesis cannot be deduced in an axiom system which was used by Mostowski [1939].[192]

Kurepa [1950] announced his result [1952] that the Cartesian square of a Suslin line (a level-wise product) does not have the countable chain condition. In fall 1950, Kurepa spoke on his work on Suslin's Problem at Harvard University, the University of Michigan at Ann Arbor, and the University of California, Berkeley. The following January, he spoke on it in Lausanne [1952, 97, footnote 1]. Josef Novák[193] [1950] investigated Suslin's Problem with closed dyadic [binary] partitions[194] seemingly unaware of Kurepa's work on such partitions. He [1952] also showed that a necessary and sufficient condition for a linear continuum of cardinality κ to have a "an element of character c_{00}" (limit both from below and above of sequences of length ω) is that $\kappa < 2^{\aleph_1}$.[195]

Mary Ellen Rudin, publishing under her maiden name of Estill,[196] investigated E spaces, that is Moore spaces which do not have a countable dense set and do not contain uncountably many disjoint domains, i.e. open sets. In her thesis, Rudin [1950] constructed the first example of an E space satisfying the countable chain condition. Then she [1951] constructed an example which, in addition, is locally connected. Next she [1952] proved that the existence of a nonseparable locally connected Moore space satisfying the countable chain condition in which any two

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[192]Mostowski used his axiom system to show the independence of the Well-Ordering Principle from the Ordering Principle. This axiom system is closely related to the axiom systems of Bernays and Gödel. Jech [1967] described it as Gödel-Bernays set theory without the axioms of regularity and of choice.

[193]Josef Novák (April 19, 1905 – August 12, 1999) was a noted Czech topologist. He studied with Karl Menger in Vienna 1935–1936. He was a member of the Czechoslovak Academy of Sciences from its inception in 1952. For more on his life and work, see [Frolík and Zítek, 1975], [Frolík and Koutník, 1985], [Frič and Kent, 2000].

[194]A *closed dyadic partition* of a linearly ordered set C is a family P of closed intervals with the following properties: (1) the intersection of any pair of intervals is either empty, a singleton, or one of the intervals; (2) $C \in P$; (3) for every interval $X \in P$ there are X_1 and X_2 in P such that $X = X_1 \cup X_2$ and $X_1 \cap X_2$ is in P; and (4) the intersection of every decreasing set of intervals from P is either in P or a single element of C. That is, a dyadic partition gives a binary tree of intervals of the original linear order.

[195]However, as Bagemihl pointed out in his Math Review MR0056049 (15,17e), Hausdorff [1907] had already proved sufficiency.

[196]Mary Ellen Rudin (December 7, 1924 –) received her Ph.D. from the University of Texas at Austin in 1949. Her dissertation, supervised by R. L. Moore, was entitled *Concerning Abstract Spaces*. See [Albers and Reid, 1988] to learn more about her graduate student experience. Another early influence was F. Burton Jones, who had studied with Moore and was a professor at the University of Austin when Mary Ellen Rudin was a student. He moved to the University of North Carolina (UNC) the year after she moved to Duke University, and she attended weekly colloquia at UNC and talked frequently with Jones (see [Jones, 1993] for a description of their interaction and a glimpse of her working methods as she constructed a Dowker space from a Suslin line). It was at Duke that she met her husband Walter, a prominent analyst who was an instructor at the time in a neighboring cubical. Walter Rudin is known, for example, for his proof under CH there are $2^{\mathfrak{c}}$ many P-points (a type of ultrafilter), and for his textbook *Principles of Mathematical Analysis* [1953] with second and third editions in 1964 and 1976.

points can be separated by a countable set[197] if and only if there is a Suslin line.[198] Rudin had learned about Suslin's Problem from Moore.[199]

R. H. Bing[200] [1951] strengthened the notion of *normal* to *collectionwise normal*[201] and proved that a space is metrizable if and only if it is a collectionwise normal Moore space. He also showed that if there is a Q-set (defined in §4.2), then there is a separable non-metrizable normal Moore space (see Example E [1951, 182]).

C. Hugh Dowker[202] [1951] introduced the concept of countable paracompactness (every covering by countably many open sets has a locally finite refinement) and proved that for a topological space X, the property of X being countably paracompact and normal is equivalent to $X \times I$ being normal, where $I = [0, 1]$ is the unit interval. In footnote 4 [1951, 221], he asserted: "it would be interesting to have an example of a normal Hausdorff space which is not countably paracompact."

Rudin [1955] showed that if there is a Suslin line, the answer is negative by constructing from a Suslin space[203] a *Dowker space*, i.e. a normal space X whose product with the unit interval, $X \times [0, 1]$, is not normal.

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[197] Two points a and b are *separated by a countable subset* S of a space X if $X \setminus S$ can be partitioned into two disjoint clopen sets $X \setminus S = U \cup V$ with $a \in U$ and $b \in V$.

[198] This brief description of the early work of M. E. Rudin is informed by articles [Tall, 1993] and [Watson, 1993] from a conference in honor of Mary Ellen Rudin held June 26–29, 1991 in Madison, Wisconsin. Watson gives modern versions of some of her proofs and makes connections between this early work and Skolem functions, MAD families, the Pixley-Roy topology, and asserts that [Watson, 1993, 175] her use of an uncountable almost increasing family $\mathcal{F}$ of functions from ω to ω as a combinatorial structure each uncountable family of which is likely to possess "bad" behavior is an early ancestor of Todorcevic's Theorem 1.1 in [8] $\cdots$." The reference is to [Todorcevic, 1989].

[199] Tall [1993, 3] reported that the one time Moore lectured to Rudin's class was when he (mistakenly) thought that he had solved Suslin's Problem.

[200] R. H. Bing (October 20, 1914 – April 28, 1986) earned his Ph.D. in 1945 as a student of R. L. Moore. In 1979, Jones [1980, 2] recalled returning to Texas after the war as Bing was finishing his degree, and described him as "solving all the topological world's unsolved problems." Jones showed Bing his tin can example [a special Aronszajn tree with the tree topology], and told him about the metrization problem he could not solve.

[201] A collection $\mathcal{D}$ of subsets of a topological space X is *discrete* if every set $D \in \mathcal{D}$ is a subset of an open set disjoint from $\bigcup(\mathcal{D} \setminus \{D\})$. A space X is *collectionwise normal* if every discrete collection $\mathcal{D}$ of closed sets expands to a disjoint collection $\mathcal{U}$ of open sets, i.e. each $D \in \mathcal{D}$ is in one and only one $U \in \mathcal{U}$.

[202] Clifford Hugh Dowker (March 2, 1912 – October 14, 1982) grew up in Canada, earned his doctorate in 1938 with Lefschetz at Princeton University, and held a variety of academic jobs before working for the MIT Radiation Laboratory 1943–1946. Afterwords he held short-term positions at Tufts, Princeton and Harvard, then decided to leave North America during the period of McCarthyism, when several of his mathematical friends were harassed. He joined the faculty of Birkbeck College in England in 1950 where he remained until his retirement in 1979. For more on his life and work, see [Strauss, 1984].

[203] Rudin's definition of a *Suslin space* is a linearly ordered space S in which neighborhoods are segments which has the properties that S is not separable but every collection of disjoint segments is countable. Her only references were the paper of Dowker [1951] for the question and Suslin's statement of the problem [1920].

John L. Kelley[204] [1959] investigated the regular open algebra of a Suslin line and developed some Boolean algebraic equivalents of the existence of a Suslin line. His work built on that of [Maharam, 1947].

5.3 Ordered sets, structure and mappings

A question of Fraïssé [1948] inspired Sierpiński [1950] to work on the structure of subsets of the reals under order embeddability. He developed a technique for identifying a subset H of a continuum-size linear order E such that no order-preserving map from E to H is onto. He proved that the class of uncountable linear orders has a decreasing chain of length the continuum, an increasing chain of length the continuum, and an antichain of size the continuum. He indicated in a footnote that the construction of a rigid real type by Dushnik and Miller [1941, Theorem 3, p. 325] was an inspiration for his first key lemma.

John C. Shepherdson[205] [1951a] examined the structure of ordered sets by computing the *index*, the least ordinal number strictly greater than the ordinals of all well-ordered subsets. He was able to give a recursive characterization of the general form of an order type except when the index is a regular cardinal. He showed that if the index of an ordered set is not of the form ω^β or $\omega^\beta + 1$, then it can be expressed as the sum of ordered sets with smaller indices. The representation for sets whose index has normal form $\sum_{i<k} \omega^{\beta_i} n_i$ is quite complex. In Theorem 3.2, he proved that among ordered sets of which every proper initial segment has index greater than ω_β, there is a minimal order, $\mathbb{Z}_\beta$, which is the lexicographic order on the set of functions from ω to ω_β which are non-zero only finitely often. His analysis built on work of Hausdorff [1908] and Gleyzal [1940].

Shepherdson [1954] published his construction of a denumerable ramified set F with the properties that no maximal antichain of F meets every maximal chain of F and that no maximal chain of F meets every maximal antichain of F. This example provided an answer to two questions of Kurepa at the end of his paper giving the follow statement equivalent to a non-empty partially ordered set $(S, \leq)$ being finite: in every ramification $T(S)$ of S, every maximal antichain has non-empty intersection with every maximal chain.

Seymour Ginsburg,[206] a student Dushnik at University of Michigan, studied

[204] John Kelley (December 6, 1916 – November 26, 1999) did his undergraduate work and earned his master's degree at the University of California, Los Angeles, and obtained his doctorate in 1940 at the University of Virginia under the direction of Gordon T. Whyburn. He spent most of his career at the University of California, Berkeley. His textbook, *General Topology* [1955], based on lectures given at the University of Chicago 1946–1947, the University of California, Berkeley, 1948–1949 and Tulane University 1950–1951, has served generations of students. His autobiographical sketch [1989] is humorous and includes sketches both personal and mathematical of his education and work.

[205] J. C. Shepherdson was on the faculty of the University of Bristol from 1946–1991, when he reached emeritus status. He was elected to fellowship in the philosophy section of the British Academy in 1990 and became a fellow of the Royal Society in 1999. Set theorists may know him best for his work on inner models [1951b], [1952], [1953], and those in computability for his work with Howard Sturgis on the unlimited register machines.

[206] Ginsburg (1928-2004) became a well-known computer scientist.

real types and interpolation theorems for them in the 1950's (see [1953a], [1953b], [1955]). He investigated partitions of real types into various nice sets, and proved [1953b, 531, Theorem 3.3] that if f is an order-preserving map of E into itself with no fixed points, then E is the union of countably many pairwise disjoint sets all of which are mutually embeddable in one another. Ginsburg [1953b, 516] noted that he knew of no order type other than λ, the order type of the real numbers, that had a *similarity decomposition*, i.e. a partition of a set of that type into continuum many (pairwise disjoint) subsets all mutually embeddable with one another and with the original set.[207]

For every infinite cardinal κ, John B. Johnston [1956] constructed a partial order of cardinality 2^κ universal for partial orders of cardinality κ and dimension κ in his Theorem 3. Theorem 3 has an additional part generalizing the result of Mostowski [1938] that there is a countable partial order universal for countable partial orders.

Bjarni Jónsson [1956] gave a list of six axioms, which sufficed with the Continuum Hypothesis, to guarantee that structures satisfying the axioms have universal structures in all uncountable powers.

In a preliminary report [1957], Andrzej Ehrenfeucht[208] formulated two-player games as an organizing strategy,[209] inspired by Fraïssé's back-and-forth strategy for proving elementary equivalence of models.[210] Ehrenfeucht [1961] applied the technique to the analysis of properties of the arithmetic of ordinals.

The Dilworth [1950] chain decomposition theorem is the statement that if every antichain in a finite partially ordered set P has size $\leq k$, then P is the union of $\leq k$ many chains. In the proceedings of a symposium on combinatorial analysis held at Columbia University in 1958, Dilworth [1960] made connections between his decomposition theorem and Hall's Marriage Theorem [1935].

5.4 The Regressive Function Theorem

The seminal 1929 Alexandroff–Urysohn Regressive Function Theorem was steadily extended, getting to Fodor's 1956 widely applied version for stationary sets. We had already noted that Alexandroff and Urysohn [1929] showed that functions regressive on the set of infinite countable ordinals are constant on an uncountable

[207] Harzheim [2008] constructed a subset of the real numbers which does not order embed the real numbers but does have a similarity decomposition.

[208] Andrzej Ehrenfeucht (August 8, 1932 –) was a student of Andrzej Mostowski at the University of Warsaw, where he earned his master's degree in 1955. In 1961, he received his doctorate from the Institute of Mathematics of the Polish Academy of Sciences in Warsaw. After being an assistant professor at the University of California, Berkeley, and Stanford University, and an associate professor at the University of Southern California, he joined the Department of Computer Science at the University of Colorado in Boulder.

[209] For more on the early work of Ehrenfeucht on model theory, see [Vaught, 1997].

[210] Ehrenfeucht may have heard about Fraïssé's work from the mathematical logic colloquium in Paris in 1952 by Fraïssé titled *Sur les rapports entre la théorie des relations et la semantique au sens de A. Tarski*, since it is in the bibliography of [Ehrenfeucht, 1961] but otherwise unreferenced. The games Ehrenfeucht formulated are now called Ehrenfeucht-Fraïssé games.

set, and that Dushnik [1931] extended the result to regressive functions on the set of infinite ordinals smaller than a successor cardinal κ.

In a paper [1950, 140–141] received by the editors in November 1948, Erdős revisited Dushnik's version of the Regressive Function Theorem [1931] and proved that if Ω_k is an (uncountable) initial ordinal which is not cofinal to ω and f is a regressive function on the set of ordinals less than Ω_k, then there exists an ordinal β and a sequence α_i cofinal with Ω_k such that $f(\alpha_i) \leq \beta$. He noted that for regular Ω_k it is immediate from Dushnik's Theorem, and if Ω_k is cofinal with ω it does not hold for all such f. He pointed out that Dushnik's proof is easily adapted to prove the result for singular cardinals of uncountable cofinality, citing an oral communication but providing his own short proof.

As a tool for his investigation of Suslin continua with dyadic partitions, Novák [1950] proved a generalization of the Alexandroff–Urysohn Regressive Function Theorem:

> B being an uncountable closed subset of the set C of all countable ordinal numbers, let $f(x)$ be a single-valued transformation of B onto $A \subset C$ having the property that $f(x) < x$ for all $x \in B$. Then there exists a countable ordinal number $\alpha \in A$ and an uncountable subset $B^* \subset B$ such that $f(x) = \alpha$ for all $x \in B^*$.[211]

Walter Neumer[212] [1951] published a generalization of the Regressive Function Theorem that stated in modern language asserts that for all regular cardinals κ, any functions regressive on a stationary[213] subset of κ is constant on an set of size κ. For Λ an initial ordinal, he wrote $W(\Lambda)$ for the set of ordinals smaller than Λ. He called a subset $M \subseteq W(\Lambda)$ a *full part* [Vollteil] if it is similar to $W(\Lambda)$; he called it *closed* if it contained all limits of well-ordered sequences; and a closed, full part of $W(\Lambda)$ was called a *band* [Band]. In particular, he proved that if the complement of M contains no band and f is regressive on M, then there is a value that f takes Λ many times. Moreover, he showed the necessity of this condition, proving that if the complement of M does contain a band, then there is a regressive function for which every value is realized fewer that Λ many times. Together these results of Neumer characterize stationary sets of a regular uncountable cardinal Λ as those for which every regressive function is constant on a subset of cardinality

[211] Denjoy [1953] independently proved Novák's result.

[212] Walter Neumer (March 9, 1906 – January 25, 1981) received his doctorate in 1929 From Justus-Liebig-Universität Giessen where his advisors were Friedrich Engel (primary) and Ludwig Schlesinger (secondary). Most of his academic career was at Johannes Gutenberg-Universität Mainz.

[213] A subset of κ is *stationary* if every closed unbounded subset of κ meets it; Bloch [1953] is credited with the first use of the term in connection with the Regressive Function Theorem; Kanamori [2009, Chapter 1] credits Mahlo with the first use of this concept.

The word *stationnaire* was used by Lusin [1934] to mean a sequence which is eventually constant, in a monograph where he extended the result of Réne Baire that all decreasing sequences of closed sets are countable to show that all decreasing sequences which are continuous at limits of sets which are countable unions of closed sets are eventually constant.

Λ.[214] He went on to explore basic features of stationary sets, such as the fact that stationary co-stationary sets exist for regular cardinals greater than ω_1.

Gérard Bloch[215] also investigated stationary sets. He independently proved Neumer's Theorem for ω_1 in his Theorem I: A necessary and sufficient condition for a subset to be stationary is that every closed subset of its complement be at most denumerable (translated from the original French). Given a set A of countable limit ordinals, he used *suite distinguée* for a sequence of regressive functions $f_n(x)$ $(n \in \mathbb{N})$ defined on A such that, for all $x \in A$, $f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \ldots$ and $\lim_n f_n(x) = x$. He used this concept to prove that stationary sets can be split into disjoint stationary sets. In a brief postscript, he reported that Walter Neumer had some of the results for stationary sets (a reference to Neumer's paper [1951] was included), and that he, Bloch, had not known of Neumer's article when he submitted his note.

Géza Fodor,[216] published in [1955] his rediscovery of Dushnik's generalization of the Alexandroff–Urysohn Regressive Function Theorem.[217] The very next year, Fodor [1956] proved that for any regular uncountable cardinal κ, every function regressive on a stationary subset of κ is constant on a stationary subset. This result is known by various names, including the Regressive Function Theorem, Fodor's Lemma or Fodor's Theorem (see [Jech, 2003, 93], [Hajnal and Hamburger, 1999, 151]), and the Pressing-Down Lemma (see [Kunen, 1983, 80]). The special case for $\kappa = \aleph_1$ of this useful result also appeared in Spanish in 1958 in [Ricabarra, 1956].[218] Neumer [1958], apparently unaware of Fodor's [1956], also published a proof of this

[214] Hajnal and Hamburger [1999] refer to this characterization as *Neumer's Theorem*, give two different proofs of it, and discuss its relationship with the Regressive Function Theorem of Fodor (see below).

[215] His note was presented by Denjoy, and is the only item by him listed in Math Reviews.

[216] Géza Fodor (May 6, 1927 – September 28, 1977) was born in Szeged, Hungary, educated at the University of Szeged, and received a degree in 1954 equivalent to the American doctorate under the supervision of B. Szökefalvi-Nagy in the field of combinatorial set theory. Starting in 1954 he was a research associate and then an associate professor for Professor Laszlo Kalmár at the University of Szeged. He became a full professor in 1967 after receiving his Academy Doctor degree (like the Habilitation degree in German universities) and led the Department of Set Theory and Mathematical Logic from its creation in 1971. He was elected a corresponding member of the Hungarian Academy of Sciences in 1973 and served as Rector of the University of Szeged 1973 – 1967. This brief biography is based on [Editors, 1977], which also noted that Fodor was the originator of the question of whether a stationary subset of size κ could be split into κ disjoint subsets. For an overview of his mathematics in Hungarian, see Komjáth [198286].

[217] F. Bagemihl, in Math Review MR0076000, said Dushnik's result generalizes one in [Bachmann, 1955, 43].

[218] The international journal *Revista de Matemáticas Cuyana* was started by Mischa Cotlar who became director of the Instituto de Investigaciones Mathematicas de Cuyo, new at the time he arrived after earning his Ph.D. under Zygmund in 1953. In 1955, the institute was closed down by a military administration which claimed that it was a Communist organization financed by another Communist organization, UNESCO, according to testimony by Cora Sadowsky in conjunction with a conference honoring Mischa Cotlar held October 12, 2007 at the University of New Mexico in Albuquerque, New Mexico (see http://www.math.unm/conferences/10thAnalysis/resources/cotlar/sadosky.pdf accessed August 17, 2007). Ricabarra's paper was in an issue of the journal nominally for the year 1956, but only appeared in 1958.

important theorem.

6 1960-1970: FORCING, TREES AND PARTITIONS

The 1960's saw a sea-change in set theory with interactions with the growing field of model theory, the introduction of forcing, Martin's Axiom and diamond principles with multiple applications to combinatorial problems, and the development of wqo and bqo theory. Cohen's proofs that set theory was consistent with the negation of the Continuum Hypothesis and also with the negation of the Axiom of Choice set off a wave of new results. We mention Easton's work on the values the exponentiation function could have on regular cardinals, and Solovay's proof of the consistency of "all sets of reals are Lebesgue measurable" with Dependent Choice (DC).

Classical results for the partition calculus of cardinals were published in the middle of the decade in a massive paper that had been years in the making, and open questions about them were shared with the wider logic and set theory community. Suslin's Hypothesis was shown to be both consistent and independent of the usual axioms of set theory, and one might say a new section of Cantor's paradise was opened where realms were to be found representing different possibilities for Aronszajn, Kurepa, and Suslin trees.

The San Francisco Bay Area was a nexus for research during this decade with a wealth of graduate students, postdoctoral researchers and visitors in addition to the regular faculties of Stanford University and the University of California, Berkeley. Dana Scott was first on the Berkeley faculty and then the Stanford one.[219] Paul Cohen, Harvey Friedman,[220] and Ehrenfeucht were on the faculty at Stanford, where Jensen visited in the late 1960's with his student Adrian Mathias, who took a course with Scott. The following individuals (listed with their advisors) earned doctorates at Stanford:

1966	Platek	(Kreisel)	1968	Kunen	(Scott)
1967	Barwise	(Feferman)	1970	Mansfield	(Scott)

The mathematical life in Berkeley was enriched by the following visitors. Lévy was a visiting assistant professor 1959–1961; Jerzy Łoś visited 1959–1960 and 1962–1963;[221] and Hajnal visited for the calendar year 1964, lecturing on the partition calculus.[222] Fred Galvin held a post-doctoral position 1965–1968.[223] after finish-

[219] Dana Scott was a Miller Institute Fellow and assistant professor in 1960, was promoted to associate professor in 1962, and moved to Stanford University in 1963.

[220] Harvey Friedman was at Stanford 1967–1973, arriving as an assistant professor of Philosophy and promoted to associate professor in 1969. He spent 1970–1971 at the University of Wisconsin, Madison, and 1972–1973 as a Visiting Professor at the State University of Buffalo, where he was a Professor from 1973–1977. He has since been a professor at Ohio State University.

[221] Los [2000] gave a seminar jointly with Tarski in 1959–1960 and a seminar jointly with Jońsson 1962–1963.

[222] Hajnal noted [1997] that the persistent attendees of his 1964 lectures were Donald Monk, William Reinhardt and Jack Silver.

[223] From Berkeley, Galvin moved to the University of California, Los Angeles, and then in 1974

ing the research for his 1967 doctorate from the University of Minnesota under Jónsson.[224] The individuals listed below with their advisors earned doctorates at Berkeley:

1962	Gaifman	(Tarski)	1967	Reinhardt	(Vaught)
1962	Halpern	(Lévy, Scott)	1967	Prikry	(Silver)
1966	Silver	(Vaught)	1969	Laver	(McKenzie)
1966	Malitz	(Vaught)	1970	Baumgartner	(Vaught)
1966	McAloon	(Solovay)	1970	Mitchell	(Silver)
1967	Hechler	(Solovay)			

The Prague Set Theory Seminar was founded by Petr Vopěnka after the death in 1963 of his advisor Ladislav S. Rieger,[225] the first Czech mathematician to work in the foundations of set theory. Petr Hájek [1971] listed members of the seminar: Bohuslav Balcar, Lev Bukovský, Karel Hrbáček, Thomas Jech, Antonín Sochor, Petr Štěpánek, Petr Vopěnka, and Hájek himself. Hájek closed his paper with a 55-item bibliography of the Prague seminar.[226]

The decade included the first international set theory conference after the introduction of forcing. Officially it was the Fourteenth Annual Summer Research Institute sponsored jointly by the American Mathematical Society and the Association for Symbolic Logic, with financial support from the National Science Foundation. In the preface to the proceedings, Scott [1971, v] reported that the four-week conference held at the University of California, Los Angeles (UCLA), July 10 - August 5, 1967, featured tutorials by himself and Joseph Shoenfield (10 lectures each), together with individual contributions (usually in hour talks) at the rate of four per day but by the last week reduced to three per day, "as the strength of the participants had noticeably weakened." Two other conferences of the decade were the International Symposium on Topology and its Applications in Herceg-Novi in 1968, whose organizing committee was chaired by Kurepa, and the first British Combinatorial Conference held in Oxford in 1969, whose invited speakers included Erdős and Marc Kac, which signaled the growing sense of community among British combinatorial mathematicians.

We mention in passing that in this decade Rado [1964] constructed the Rado graph (the denumerable random graph), which is universal for countable graphs.[227] Michael Morley and Robert Vaught [1962] extended Jónsson's work on the exis-

to the University of Kansas.

[224] Bjarni Jónsson moved to Vanderbilt University in 1966.

[225] Ladislav Svante Reiger (1916–1963) received his doctorate from Charles University, Prague and in 1959 defended his Habilitation thesis, where the referee was Mostowski. Rieger founded the first seminar in Prague on mathematical logic in 1953 with a focus on Gödel's work on the consistency of the Axiom of Choice and the Continuum Hypothesis. For more on Reiger, see [Hájek *et al.*, 2008] and the thesis of E. Kozkíková, which is described as "a detailed monograph on L. S. Rieger (in Czech)".

[226] Karel Prikry is listed as participant in the Prague seminar by Hájek, Marek and Vopěnka in [2008].

[227] Rado's graph started a line of research which extends beyond the scope of this paper, for example with work by Shelah [1984b], [1990b], joint work with Komjáth [1995] and by Kojman [1998].

tence of universal structures, and Brian Rotman [1971] showed that most of Rado's results about the Rado graph can be deduced from the general model-theoretic framework of Jónsson, Morley and Vaught.

Cardinal transfer questions raised in this decade in model theory would turn out to be pivotal for questions about trees and about the partition calculus. Say a theory in a countable language with identity and distinguished unary predicate R *admits a pair* (α, β) if there is a model $M = \langle A, U, \cdots \rangle$ of T such that $|A| = \alpha$ and $|U| = \beta$, where U is the interpretation of R. Morley and Vaught [1962] proved that if T admits (κ, λ), then T admits $(\aleph_1, \aleph_0)$. Vaught restated the result as Theorem 14 and raised the following question in [1963, 308–309]:

> *Cardinal Transfer Question:* If T admits (κ, λ), what other pairs (κ', λ') must T admit?

Chang asked if Theorem 14 could be improved to the following question (see Vaught [1963, 309]): Does every model $\mathfrak{A}$ of T with $\overline{R_0^{\mathfrak{A}}} \neq \overline{\mathfrak{A}}$ have an elementary submodel $\mathfrak{B}$ with $\overline{\mathfrak{B}} = \aleph_1$ and $\overline{R^{\mathfrak{B}_0}} = \aleph_0$? The assertion that the answer to Chang's question is "yes" is now known as *Chang's Conjecture*. Vaught went on to connect the question with one in set theory: "A negative answer to the problem is an easy consequence of a positive answer to the following question: *Does there exist a family W of subsets of a set A of power $\aleph_1$ such that $\overline{W} = \aleph_2$ while, for any countable $B \subseteq A$, $\{X \cap B \mid B \in W\}$ is countable*?" Vaught pointed out that this question was a version of one raised by Kurepa [1942] (i.e. the questions of existence of a Kurepa tree) and quoted Ricabarra [1958, 344] as indicating it was still unresolved.

Chang proved the following theorem giving a partial answer under the Generalized Continuum Hypothesis [1965]:

> *Chang's Cardinal Transfer Theorem:* Assume GCH. If $\alpha > \beta \geq \omega$ are cardinals and a theory T in a countable language with a distinguished unary predicate R admits (α, β), then for all regular infinite cardinals δ, T admits (δ^+, δ).

We denote the cardinal transfer property "if T, R admits (α, β), then it admits (δ^+, δ)" by $(\alpha, \beta) \to (\delta^+, \delta)$.

We conclude this introduction with a brief discussion of the Normal Moore Space Conjecture. Robert Heath [1964] showed that there is a separable non-metrizable normal Moore space if and only if there is a Q-set, a result which some describe as the solution to the Separable Normal Moore Space Conjecture. During his 1967–1968 visit to the University of Wisconsin, encouraged by graduate student Franklin Tall, Silver showed the consistent existence of a Q-set (cf. [Tall, 1969], [Tall, 1977]). Recall that Rothberger [1948] proved that if the pseudointersection number is greater than $\aleph_1$, then Q-sets exist. Thus the combination of the work of Bing [1951] and Rothberger [1948] gives a consistency result. However, Rothberger's result had been forgotten in the sixties, and was rediscovered in the seventies by

David Booth. Tall, in his 1969 thesis, part of which is published in [1977],[228] focused attention on the "real question .. when is a normal space collection-wise normal?" According to Nyikos [2001, 1196], Tall, while a graduate student, proved the consistency of the statement that every first countable normal space is $\aleph_1$-collection-wise Hausdorff. His advisor, Rudin, observed that a corollary of Tall's result is the consistency of no Aronszajn tree being normal in the tree topology, so that in particular, it is consistent that the road spaces of Jones are not normal.

6.1 The Halpern-Läuchli Theorem

We start with the Halpern-Läuchli Theorem, a fundamental Ramsey-theoretic result for partitions of finite products of trees that generalized the rectangle refining theorem of Rado [1954a]. James D. Halpern[229] and Hans Läuchli[230] [1966] expressed their ground-breaking combinatorial result in terms of trees following a suggestion of a referee of their paper. The result was a key piece in the plan of Halpern and Lévy [1964] which they [1971] successfully carried out, to prove that in ZF, the Boolean Prime Ideal Theorem does not imply the Axiom of Choice. Halpern and Lévy [1971, 97] use Theorem 2 [Halpern and Läuchli, 1966, 362] in their proof, which is stated below in modern language, where a tree is *rooted* if there is a single element, called the *root*, in level 0:

> *Halpern-Läuchli Theorem:* For any finite sequence $\langle (T_i, <_i) \mid i < d \rangle$ of finitely branching rooted trees without treetops of height ω and any positive $q < \omega$ there is a positive integer n such that for any q-ary partition of the product $\prod_{i<d} (T_i \restriction n) = Q_0 \cup \cdots \cup Q_{q-1}$ there are $j < q$,

[228] Eric van Douwen reported in his Math Review MR0454913 (56 #13156) that the material in Tall's monograph is unchanged from his thesis except for some brief mentions of results obtained after the thesis. van Douwen went on to assert that this paper changed the general feeling of topologists that results proving consistency and independence from ZFC are not really part of topology, toward recognition that they are with the discovery of consistency and independence results about "down-to-earth objects like first countable normal spaces," the subject of the monograph.

[229] James Daniel Halpern was a student at the University of California, Berkeley, in the late 1950's and early 1960's. In his thesis, Halpern [1962] proved the independence of the Axiom of Choice from the Boolean Prime Ideal Theorem (BPI) over the theory ZF minus the Axiom of Foundation by showing BPI holds in a Fraenkel-Mostowski model in which the Axiom of Choice did not. The result was announced in an abstract in the *Notices of the American Mathematical Society* in 1961; a revised version was published in [Halpern, 1964]. His dissertation work was done under the direction of Lévy, who was visited Berkeley 1959-1961, and written up under the supervision of Dana Scott. See page 58 of [Halpern, 1964] for Halpern's remarks on the supervision of his work. See [Kanamori, 2004] for more information about Lévy, and, in particular, his interactions with Halpern. The proof uses a combinatorial lemma that Halpern calls the Rado Corollary (Scott is credited with the proof included in the paper) and is a generalization of Ramsey's Theorem which may be simply expressed as a finitary polarized partition relation. Halpern spent 1965–1966 at the Institute for Advanced Study, and 1974–1981 at the University of Alabama, Birmingham.

[230] Hans Läuchli (1933 - August 13, 1997) was a student of Ernst Specker at Eidgenössische Technische Hochschule Zürich (ETH) and received his degree in 1962 with a dissertation on *Auswahlaxiom in der Algebra.*

$h < n$, $\langle x_i \mid i < d\rangle$ and $\langle A_i \subseteq (T_i \restriction n) \mid i < d\rangle$ such that $\prod_{i<d} A_i \subseteq Q_j$ and for all $i < d$, x_i is on level h of T_i and every immediate successor of x_i has an extension in A_i.

Included below, in modern language, is their immediate corollary [1966, 363] about level products which may look more familiar to some readers.

Halpern-Läuchli Corollary: For any finite sequence $\langle (T_i, <_i) \mid i < d\rangle$ of finitely branching rooted trees without treetops of height ω and any positive $q < \omega$ there is a positive integer n such that for any q-ary partition of the product of the nth levels of the trees, $\prod_{i<d} T_i(n) = Q_0 \cup \cdots \cup Q_{q-1}$, there are $j < q$, $h < n$, $\langle x_i \mid i < d\rangle$ and $\langle A_i \subseteq T_i(n) \mid i < d\rangle$ such that $\prod_{i<d} A_i \subseteq Q_j$ and for all $i < d$, x_i is on level h of T_i and every immediate successor of x_i has an extension in A_i.

Most of the work to prove the above results is in the proof of Theorem 1, which I have relabeled but otherwise quote from [Halpern and Läuchli, 1966, 361–362].

Halpern-Läuchli Core: Let $\mathcal{T}_i = \langle T_i, \leq_i\rangle$, $1 \leq i \leq d$ be finitistic trees without tree tops and let $Q \subseteq \prod_1^d T_i$. Then either

a. for each k, Q includes a k-matrix or
b. there exists h such that for each k, $\left(\prod_1^d T_i\right) - Q$ includes an (h, k)-matrix.

A *finitistic tree without tree tops* is a finitely branching tree of height ω and no terminal nodes. An (h, k)*-matrix* is a product $\prod_1^d A_i$ where for each $1 \leq i \leq d$, $A_i \subseteq T_i$ and for some $\langle x_1, x_2, \ldots, x_d\rangle \in \prod_1^d T_i(h)$, for each $y \in T_i(k)$, if $x_i \leq_i y$, then there is some $z \in A_i$ with $y \leq_i z$. A *k-matrix* is a $(0, k)$-matrix, i.e. the special vector lists the roots of the trees.

In their proof of the Halpern-Läuchli Core Theorem, Halpern and Läuchli used metamathematical, heavily syntactic techniques. They comment "We have tried to eliminate the use of metamathematics without success and would welcome a simplification in this direction" [1966, 360].

In the late 1960's, Galvin (cf. [Erdős and Hajnal, 1974, 275]) proved that if you color the rationals with finitely many colors, then there is a subset of the same order type η which is at most *bi-colored*, i.e. realizes at most 2 colors. He conjectured that if you color r-element sequences of rationals with finitely many colors, then there are subsets $X_0, X_1, \ldots, X_{r-1}$, each of order type η so that $\prod_{i<r} X_i$ receives at most $r!$ colors. He proved this conjecture for $r = 2$,[231] and showed that for every positive r, the bound $r!$ cannot be improved. This conjecture, proved by

[231] The Sierpiński partition restricted to the rationals is a partition in which every subset of the rationals of full order type meets both cells of the partition.

Richard Laver in 1969 (cf. [Erdős and Hajnal, 1974, 275]), is expressed using a generalization of the polarized partition relation as follows:

$$\begin{pmatrix} \eta \\ \eta \\ \vdots \\ \eta \end{pmatrix} \longrightarrow \begin{pmatrix} \eta \\ \eta \\ \vdots \\ \eta \end{pmatrix}^{\overbrace{1,1,\ldots,1}^{r}}_{<\omega, r!}$$

As Laver later recalled, he [1984] proved the polarized partition relation essentially by identifying the rationals with the dyadic rationals which can be regarded as represented by the complete binary tree $({}^{<\omega}2, <_q)$ of all finite sequences of 0's and 1's under a suitable ordering $<_q$, and proving the theorem in this alternative setting. More generally, Laver showed that there is a function $f : {}^n\omega \to \omega$ so that for any $\vec{r} = \langle r_0, r_2, \ldots, r_{n-1} \rangle$, and any finite coloring of $\prod_{i<n}[\mathbb{Q}]^{r_i}$, there is an $\langle X_0, X_1, \ldots, X_{n-1} \rangle$ with each X_i a subset of $\mathbb{Q}$ of order type η, such that $\prod_{i<n}[X_i]^{r_i}$ realizes at most $f(\vec{r})$ colors.

Laver was unaware of the Halpern-Läuchli Theorem at the time, so in order to prove the polarized partition theorem, he proved a perfect subtree version of the Halpern-Läuchli Theorem which I name and state below using $\prod^A \vec{T}$ as an abbreviation for $\bigcup_{n\in A} \prod_{i<d} T_i(n)$:

> *Laver HL$_d$ Theorem:* If $\vec{T} = \langle T_i : i < d \rangle$ is a finite sequence of rooted finitely branching perfect trees of height ω and $\prod^\omega \vec{T} = G_0 \cup G_1$, then there are $\delta < 2$, an infinite subset $A \subseteq \omega$ and downwards closed perfect subtrees T_i' of T_i for $i < d$ with $\prod^A \vec{T}' \subseteq G_\delta$.

Call $\vec{X} = \langle X_i : i < d \rangle$ *n-dense* in $\vec{T} = \langle T_i : i < d \rangle$ if there is some $m \geq n$ such that $X_i \subseteq T_i(m)$ for all $i < d$ and for each $i < d$ and $t \in T_i(n)$, there is some $u \in X_i$ with $t \leq_i u$. For $\vec{x} \in \prod^\omega \vec{T}$, call $\vec{X}$ *$\vec{x}$-n-dense* in $\vec{T}$ if $\vec{X}$ is n-dense in the sequence $\langle (T_i)_{x_i} : i < d \rangle$ where $(T_i)_{x_i}$ is the subtree of T_i of all elements comparable with x_i. Most of the work of proving HD$_d$ for $d < \omega$ is in the following result, which I have named for easy reference and appears as Theorem 1 in [Laver, 1984, 386]:

> *Laver HL$_d$ Core:* Suppose that $\vec{T} = \langle T_i : i < d \rangle$ is a finite sequence of finitely branching tress of height ω with no terminal nodes, and $\prod^\omega \vec{T} = G_0 \cup G_1$. Then either
>
> a. for all $n < \omega$ there is an n-dense $\vec{X}$ with $\prod^\omega \vec{X} \subseteq G_0$, or
>
> b. for some $\vec{x} \in \prod^\omega \vec{T}$ and for all $n < \omega$ there is an $\vec{x}$-n-dense $\vec{X}$ with $\prod^\omega \vec{X} \subseteq G_1$.

6.2 Countable height trees

In 1960 Joseph B. Kruskal[232] published his celebrated tree theorem in an expanded version of his doctoral dissertation [1954] under the direction of Roger Lyndon at Princeton University:[233]

> *Kruskal's Tree Theorem* [1960]: If X is a wqo, then $T(X)$, the collection of finite trees with labels from X, is a wqo under tree embeddings $f : S \to T$ which respect the labels in the sense that for all $s \in S$, the label of s is less than or equal to the label of $f(s)$.

Independently, S. Tarkowski [1960] proved a theorem from which the above result follows. Tadashi Ohkuma announced [1960] and proved [1961] that under order-preserving maps, scattered trees[234] are linearly ordered under order preserving maps (they need not be injective). He also showed that every scattered tree is comparable with every tree under this ordering. He showed under CH that there are non-comparable trees.

Kruskal [1954] proved the closure of the family of wqos under finite unions and Cartesian products (quasi-ordered componentwise). He conjectured at the end of the paper that the set of all (connected) trees, finite or infinite, is a wqo.

Kruskal was motivated by a conjecture of Vázsonyi[235] that for any infinite sequence T_i of (finite) trees, there are two T_k and T_ℓ such that T_ℓ contains a homeomorphic image of T_k, or equivalently, T_ℓ is a subdivision of T_k.[236] Kruskal heard about the problem from Erdős.[237]

Connections between graphs and partial orders were investigated in the 1960's. Elliot S. Wolk [1962] came up with a characterization of when a graph is the comparability graph of a tree: on each path in between two vertices there is a vertex adjacent to both the initial and terminal vertex of the path. In 1962 Paul C. Gilmore and Alan J. Hoffman announced their characterization of when a graph is the comparability graph of a partial order [1964] at the International Congress of Mathematicians held in Stockholm in 1962: G is a comparability graph if and only if each of its odd cycles has at least one triangular cord (an edge which joins two consecutive edges of the cycle to form a triangle). Alain Ghouila-Houri [1962] gave an alternate description (and a proof for the finite case). Wolk [1965] proved that a graph is the comparability graph of a partial order if and only if all of its finite induced subgraphs are. Tibor Gallai [1967] gave a characterization in terms of forbidden subgraphs, which has proved very useful.

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[232] Joseph Bernard Kruskal (January 29, 1928 – September 19, 2010) was an instructor at Princeton and University of Wisconsin at Madison before becoming an assistant professor at University of Michigan in Ann Arbor in 1958. In 1959 he joined the technical staff at Bell Telephone Labs. He has held visiting positions at Yale, Columbia and Rutgers.

[233] Erdős is listed as a second advisor for Kruskal at the Mathematics Genealogy Project site.

[234] A tree is *scattered* if it does not embed the tree of all finite sequences of zeros and ones.

[235] Erdős dates Vázsonyi's Conjecture to 1937 in Math Review MR0123491 (23 #A816).

[236] We say S is a subdivision of T if S can be obtained from T by changing some of the edges of T into finite paths by insertion of vertices of degree two.

[237] Erdős is listed as a second advisor to Kruskal on the Mathematics Genealogy Project site.

Crispin St. J. A. Nash-Williams[238] became interested in quasi-orders in 1957, on learning of Vázsonyi's Conjecture. He set out to extend Kruskal's Tree Theorem to infinite trees, and, as a side benefit, he [1963] came up with a simpler proof of it. He generalized Ramsey's Theorem to the *Nash-Williams Partition Theorem*. In his notation, P stands for the set of positive integers and $A(I)$ stands for the set of ascending finite sequences of elements of $I \subseteq P$, including the empty sequence. A subset of $A(P)$ is *thin* if it does not include two distinct sequences s, t such that one is a proper initial segment of the other. With this notation in hand, we are ready to quote his theorem [1965b, 33]:

> *Nash-Williams Partition Theorem:* Let $m \in P$ and I be an infinite subset of P and $\{T_1, \ldots, T_m\}$ be a partition of a thin subset T of $A(I)$ into m disjoint subsets. Then there exists an infinite subset K of I such that $T \cap A(K)$ is contained in a single T_j.

Lemma 4, which is the statement for partitions into two parts, is the heart of the theorem, which then follows from a simple induction. Nash-Williams included a note that Erdős sketched an alternate proof using a finitely additive probability measure μ on the subsets of P that can be constructed using the Axiom of Choice so that $\mu(X) = 0$ if X is finite and $\mu(Y) = 1$ if Y is cofinite.

While the paper is particularly well-known for the Nash-Williams Partition Theorem, the main objective of the paper, according to the author, was the proof of the conjecture of Rado [1954b] that given a well-quasi-order Q, the collection $\mathbb{R}(Q)$ of sequences with domain an ordinal and range a finite subset of Q form a well-quasi-order under the ordering $f \leq g$ if and only if there is a one-to-one order-preserving function $\mu : \mathrm{dom}(f) \to \mathrm{dom}(g)$ such that for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha) \leq f(\mu(\alpha))$. Nash-Williams noted that Higman [1952] had proved that the collection of finite sequences from a wqo form a wqo; Rado [1954b] proved it for sequences of length less than ω^3; and both Kruskal (unpublished) and Erdős and Rado in their joint work [1959] proved it for sequences of length less than ω^ω. Nash-Williams [1965b, 34] also indicated that the proof used an idea underlying Theorem 3 of [Rado, 1954b], and that he was "indebted to Prof. R. Rado for some improvements in the presentation of this paper."

[238]Crispin St. J. A. Nash-Williams earned his Ph.D. from Cambridge University in 1959. The obituary by Welsh [2003] lists as his advisors Shaun Wylie and Davis Rees of Cambridge, and N. E. Steenrod of Princeton University, whom he met while on a fellowship at Princeton in 1956-57. Richard Rado as one of the readers of his 500 page dissertation. Welsh recounted Luise Rado's dismay at the amount of time her husband spent during a walking vacation reading Nash-Williams's dissertation. When Rado formally retired in 1975 (he remained active mathematically), Nash-Williams moved into his chair at the University of Reading. For a further introduction to the life and work of Nash-Williams, this obituary is warmly recommended, especially for the overviews of Nash-Williams's work on decompositions of infinite graphs, which is outside the scope of this paper. Nash-Williams liked both finite and infinite graph theory, and introduced one of his survey articles as follows: "One of my own reasons for becoming interested in graph theory was that I was intrigued by the possibility of developing non-trivial and fairly deep mathematics from a very simple concept. The ultimate illustration of this might be set theory, since sets are presumably the simplest and most basic of all mathematical structures and yet some aspects of set theory are both deep and substantial" [Welsh, 2003, 177].

Nash-Williams [1965a] went on to prove that infinite trees form a wqo, but to do so, he invented the notion of *better-quasi-order* or bqo, which have nice closure properties. The definition of better-quasi-order is involved. Maurice Pouzet[239] and Norbert Sauer[240] set the concept of better-quasi ordering in historical context as follows [2006, §1.3]:

> A basic result due to G.Higman, see [10], asserts that a poset P is wqo if and only if $I(P)$, the set of initial segments of P, is well-founded. On the other hand, Rado [24] has produced an example of a well-founded partial order P for which $I(P)$ is well-founded and contains infinite antichains. The idea behind the bqo notion is to forbid this situation: $I(P)$ and all its iterates, $I(I((I(P)))$ up to the ordinal ω_1, have to be well-founded and hence wqo.[241]

In 1968, as part of his thesis work with Ralph McKenzie,[242] Laver [1969], [1971] proved Fraïssé Conjecture I [1948] (stated near the end of §4.2) when he proved that if $(A_0, \leq)$, $(A_1, \leq)$, ... is a sequence of countable ordered sets, then for some $i < j$ $(A_i, \leq)$ isomorphically embeds in $(A_j, \leq)$. To prove this celebrated result, Laver used the Hausdorff hierarchy [1908], [Erdős and Hajnal, 1962a],[243] the foundational work on bqo theory by Nash-Williams [1965b], and a characterization of σ-scattered linear order types[244] by Galvin to prove that the σ-scattered types form a bqo. In the same paper, he proved Fraïssé Conjecture II that for any countable scattered order type τ there are most countably many order types up to equivalence under mutual embeddability that are embeddable in but not equivalent to τ.

In his thesis, Pierre Jullien [1969] gave a counter-example to Fraïssé Conjecture III: the set $E = \{\omega \cdot \omega^*, \omega^* \cdot \omega\}$ has the property that $\omega + \omega^*$ is embeddable in every order type which embeds both members of E, but $\omega + \omega^*$ is not embeddable in either of the elements of E.

A positive answer to Fraïssé Conjecture IV follows from work of Laver [1973]. See Chapter 10 of [Rosenstein, 1982, 178, 179, 196, 197] for a discussion of the solution to this conjecture.

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[239] Maurice Andre Pouzet received his doctorate from Université Claude Bernard, Lyon 1 in 1978 with Ernest Corominas as his advisor. He is known for his work in order, lattices and ordered algebraic structures.

[240] Nobert Werner Sauer received his doctorate in 1966 from the University of Vienna. A professor and long-time colleague of Eric Milner at the University of Calgary, Sauers interests range over set theory and universal algebra, especially Ramsey theory of partial orders and homogeneous structures.

[241] The citations [10] and [24] refer to Higman [1952] and Rado [1954b] respectively.

[242] Laver reputedly spent more time rock-climbing with McKenzie than talking mathematics with him. For Laver, rock climbing and mathematics were compatible. For example, he admitted to having proved a small theorem during an afternoon and overnight wait for a few minutes of assistance on a small ledge in Yosemite valley in the 1970's.

[243] Erdős and Hajnal [1962a] turned Hausdorff's type rings for scattered sets into a hierarchy.

[244] Recall that a scattered linear order is one that does not embed the order type of the rationals; σ-scattered linear orders are countable unions of scattered linear orders.

Galvin [1968b] explored partitions of the n-element subsets of the tree $C = {}^{\omega}2$ ordered lexicographically and with the topology that comes from identifying it with a subset of C^n, the n-fold product with the product topology. He proved what we will call Galvin's Partition Theorem for Perfect Sets: for $2 \leq n \leq 3$ and for any perfect set $P \subseteq C$, if $[P]^n$ is partitioned into finitely many open sets, then there is a perfect set (perfect subtree) $Q \subseteq P$ so that $[Q]^n$ meets at most $(n-1)!$ of the open sets. He conjectured that the theorem was true for all larger n.

Gerald Sacks spoke at the 1967 UCLA set theory conference on forcing with perfect closed sets. In his paper for the conference volume [1971, 335], he used the forcing to show the consistency of there being precisely two degrees of non-constructibility. He placed the origin[245] of the method in Cohen's forcing with finite conditions and Spector's [1956] construction of a minimal Turing degree. Sacks called the critical lemma which he used for proving that cardinals are preserved the *sequential lemma*, but noted in the footnotes that Mathias formulated the lemma more abstractly and called the result the *fusion lemma*.[246] Sacks noted that Shoenfield in an appendix to the 1967 Recursion theory seminar notes invoked the term "splitting" to describe the proof. Sacks cites the result in [Kleene and Post, 1954] of a minimal pair of upper bounds for the degrees of the arithmetic sets as the "germ of forcing with perfect conditions" and he references his joint paper with Robin Gandy [1967] where he used forcing with hyperarithmetic perfect closed sets to prove the existence of a minimal hyperdegree.

6.3 Suslin's Problem revisited

In the early days of forcing, Tennenbaum adapted Cohen's forcing to prove the relative consistency of the existence of a Suslin tree; Jech, working independently, gave an alternate proof; and Solovay and Tennenbaum proved the relative consistency of Suslin's Problem, i.e. the non-existence of Suslin trees.

According to Morley in his Math Review MR0215729 (35 #6564)), as early as 1964 Stanley Tennenbaum[247] announced that he was able to construct a model with a Suslin tree; Tennenbaum presented his proof in numerous seminars, but did not publish even an abstract prior to 1968, when his result appeared in

[245]See also the recursion-theoretic memories article [1999, 368–369] where he discusses the connections between forcing and recursion theory.

[246]Sacks expected Mathias's survey to appear in the second volume of the proceedings of 1967 UCLA set theory conference. It was distributed in 1968 as a preprint *A survey of recent results in set theory* from Stanford, and was later updated and published as [Mathias, 1979].

[247]Stanley Tennenbaum (April 11, 1927 – May 4, 2005) received a University of Chicago Ph.B. (an undergraduate degree) in 1945, and took graduate courses at the University of Chicago without earning a further degree. His academic career included many moves. Melvyn Nathanson [2008] recalled meeting Tennenbaum in fall 1964 when he was a visiting professor in the Department of Philosophy. Nathanson went to the University of Rochester for graduate work because Tennenbaum was teaching there at that time. In addition to his work in set theory, he is known for *Tennenbaum's Theorem* [1959]: there is no nonstandard recursive model of Peano arithmetic. A conference in his memory was held April 7, 2006 at the Graduate Center of the City University of New York.

[Tennenbaum, 1968]. In that paper, Tennenbaum, in his acknowledgment of support, mentioned summer 1963 support at Cornell University and listed two lecture series at which he presented the proofs of Theorems 1 and 2: Harvard University, January 27–29, 1964 and Symposium on Foundations of Set Theory, International Congress for Logic, Methodology and Philosophy of Science, Hebrew University, September 1, 1964. In Theorem 1 he proved the consistency of the existence of a Suslin tree together with GCH, and in Theorem 2 he proved the consistency of the existence of a Suslin tree with $\neg$CH. His reference for an equivalence of Suslin's Problem is [Miller, 1943]. Tennenbaum defined a tree to be a partial order in which the set of elements preceding any given element forms a chain. He listed Miller [1943] as his reference for the definition of Suslin tree and cited the original statement of the Suslin's Problem, Dowker's paper [1951] in which he posed the question of the existence of what have become known as Dowker spaces, Rudin's paper [1955], in which she showed the existence of a Suslin line implies the existence of a Dowker space, Cohen's article introducing forcing, and a forthcoming paper with Solovay, *Souslin's Problem II*, which is listed "to be submitted to *Fundamenta Mathematicae.*"

Jech [1967] independently constructed a proof of the consistency of the existence of a Suslin continuum using ∇-models[248] of Vopěnka [1965] and ideas from Yesenin-Volpin's paper [1954].

Next, background material on Solovay is given to set the stage for a discussion of the consistency of the non-existence of a Suslin tree by Solovay and Tennenbaum.

Robert Solovay[249] [2010b] recalled his explorations in logic and set-theory prior to his hearing of Cohen's work. He went to the University of Chicago with the clear idea that he wanted to be a theoretical physicist.

> Kaplansky did my entrance interview and I rattled off some 20 to 30 graduate texts I had worked through on my own. He clearly was unsure if I really had mastered the material in question but put me in two graduate courses with the comment that "if you really understand all that, these should be easy." I did, and they were.

Kaplansky went on to recommend that Solovay first get a Ph.D. in mathematics and then pursue his interest in physics. Solovay recalled how he came to know

[248]Mostowski in his Math Review MR0182571 of Jech's paper indicated how to translate the proof from ∇-models to forcing.

[249]Robert Martin Solovay received his doctorate in 1964 from the University of Chicago where Saunders Mac Lane was his advisor. He was a lecturer at Princeton University 1962–1964, and a visiting member of the Institute for Advanced Study 1964–1965. Then he moved to the University of California, Berkeley, where he spent most of his academic career until retirement from teaching in 1994. He was at the Rockefeller University 1967–1968, at the IBM Thomas J. Watson Research Center in New York 1974–1976, and a Sherman Fairchild Scholar at the California Institute of Technology 1976–1977. His mathematical interests include set theory, new foundations, algorithms, and quantum computing. He became a member of the National Academy in 1986. In 2003 Solovay received jointly with Gary Miller, Michael Rabin and Volker Strassen the 2003 Association for Computing Machinery Paris Kanellakis Theory and Practice Award for the Solovay-Strassen and Miller-Rabin randomized algorithms for testing primality.

more logic than a mathematician should know [2010b]:

> Somehow, or other, the senior faculty got the impression that (though obviously not completely lacking in talent) I didn't quite know what a proof was. ...
>
> Mac Lane was my thesis advisor and he gave me the "sage advice" that it would help if I studied some logic to help with this supposed defect. And I retorted to him that "I know more logic than it is decent for a mathematician to know."
>
> At the time, I had worked through much of Kleene's *Introduction to Metamathematics* and certainly knew the Gödel Completeness and Incompleteness Theorems as well as the Orange Monograph presentation of the results on $V = L$. I knew more than this since I figured out on my own that the Skolem-Lowenheim theorem was at the root of Gödel's proof of GCH in L, but there certainly were significant gaps in my knowledge that I filled in later.

Solovay had finished the research for his dissertation around June 1962 and that fall became lecturer at Princeton University. He attended Cohen's general audience lecture on forcing in Princeton on May 3, 1963, a critical turning point in his mathematical career. Here is his description [2010a] of how he became a set theorist:

> I had been interested in set theory (and in particular the question of whether $V = L$ was a theorem of ZFC) from my (brief) undergraduate career at Harvard. (I dropped out during my sophomore year.) Cohen's talk (which I attended) got me intensely interested in it again. Either at that talk or his subsequent talk at Berkeley (on July 4, 1963) he posed the question of proving the consistency with ZF of "All sets Lebesgue Measurable". (I don't think he mentioned DC. I actually independently rediscovered that (and the name!) by reading through Halmos' book on measure theory and seeing what forms of choice were needed for the positive results. For the most part, countable choice suffices. But it seemed to me at the time that DC was needed for the proof of the Radon-Nikodym theorem. (I certainly had no proof that it was, in fact needed.)) After many twists and turns, I did, of course, indeed succeed in solving that problem. And by that point I was deep into set-theory.[250]

By July 1964, Solovay [1964], [1970] had solved the problem described above and spoke on it at a July meeting in Bristol,[251] as he recalled in [Jan 2011]:

[250] Based on an interview with Solovay in July 1981, G. Moore [1988] reported that Cohen posed the problem of the consistency of ZF with every set of reals being Lebesgue measurable on July 4, 1963.

[251] Solovay announced the results in an abstract [1965] received by the editors in October 1964

> At the time when I had to submit the abstract of my talk for the Bristol meeting I only had the weaker result that there was a model of ZF + countable axiom of choice + "There is a translation invariant countably additive measure defined on the power set of the reals and which extends Lebesgue measure". But in the interval between submitting the abstract and going to Bristol, I derived the full result. As a minor point, I knew that the measure described in the earlier abstract definitely was not Lebesgue measure.[252]

The model he created in which all sets of reals are Lebesgue measurable had many other nice properties and became known as *Solovay's model*.

During the academic year 1964–1965, Tennenbaum visited to discuss set theory, and in particular, his attempts to prove the consistency of SH, the non-existence of Suslin trees. Solovay recalled their discussions and his role in the proof in an email to Kanamori [2011a] (below, c.c.c. is an abbreviation for the countable chain condition introduced in §2.3).

> His [Tennenbaum's] attempt/plan for the consistency of SH had the following ingredients:
>
> 1. it was to be an iteration in which at each step another Souslin tree would be killed.
> 2. The steps in the iteration were to be forcing with Souslin trees, . . .
> 3. Stan knew that forcing with a Souslin tree killed it and that the forcing was c.c.c.
>
> For much of the year, Stan was trying to prove that the iteration did not collapse cardinals. And he was considering iterations of length 2 and 3. My role was to passively listen to his proofs and spot the errors in them. With unjustified prescience, I kept saying that I was worried about "killing the same tree twice". Of course, there are now examples due to Jensen that show that forcing with the product of a Souslin tree with itself can collapse cardinals. But these weren't available then.
>
> At one of those meetings one of us (probably Stan) made progress and finally found reasonable conditions under which a two stage iteration did not collapse cardinals.
>
> Somehow this got me seriously thinking about the problem and by the time of our next meeting I had a proof of the theorem. This proof had the following ingredients which were new:

and, in [1970], noted that the main results of the paper were proved March–July, 1964, and were presented at the July meeting of the Association for Symbolic Logic at Bristol, England.

[252] G. Moore [1988], drawing on four interviews with Solovay in the period 1981–1984, reported that Solovay regarded the two weeks prior to the Bristol meeting as the most productive in his life. For more history of this result, see Moore's article.

(a) defining a transfinite sequence of forcing notions P_α where $P_{\alpha+1} = P_\alpha * Q_\alpha$.

This involves:

(a1) defining the operation $*$ where $P * Q$ is defined if Q is a poset in V^P;

(a2) defining what to do at limits.

(b) proving that if the component forcings are c.c.c. then the limit forcing is c.c.c.;

(c) exploiting the c.c.c.ness to see that if the length of the iteration has cofinality greater than ω_1 then all subsets of $\aleph_1$ in the final model appear at some proper intermediate stage;

(d) setting up the bookkeeping so that all Souslin trees are killed.

Solovay and Tennenbaum [1971] obtained the main results of their paper in June 1965, as they report in a footnote on the first page of their published paper. At the end of the paper, a brief note indicates that the paper was received by November 21, 1969. In footnote 2 on the first page of their article, Tennenbaum expressed regret that he had not mentioned, in [1968], Jech's work [1967]. In footnote 1, Solovay and Tennenbaum noted that Suslin's Problem had been studied by others leading to various formulations in terms of Boolean algebras and trees. In this discussion, they single out three papers: [Kurepa, 1952], [Miller, 1943], and [Sierpiński, 1948]. Solovay and Tennenbaum used a reformulation in terms of trees of the version of Suslin's Hypothesis due to Miller. They used the definition of tree used by Tennenbaum in [1968], but for the definition of *Suslin tree*, they extended the definition from Tennenbaum's [1968] by adding the condition that each element x has $\aleph_1$ elements in the tree above it, and in a brief appendix, in which they acknowledge that their reference to Miller is "not quite accurate," they show that if there is a tree without the additional condition, then there is one with it.

6.4 Martin's Axiom and diamond principles

The groundbreaking proofs of Tennenbaum [1968] and Solovay and Tennenbaum [1971] led to the extraction of combinatorial principles which opened the door to the application of set-theoretic methods to a wide variety of problems in set theory and set-theoretic topology.

Martin's Axiom

Around 1967, D. Anthony Martin[253] extracted the axiom which has become known as Martin's Axiom from work of Solovay for the Solovay-Tennenbaum paper. Rowbottom had independently suggested a similar axiom, but Solovay specified Martin in his paper with Tennenbaum and used the name *Martin's Axiom* for the following statement on page 233:

> *Martin's Axiom:* Let $\mathcal{P}$ be a reflexive partially ordered set satisfying c.a.c. Let $\mathcal{F}$ be a family of subsets of $\mathcal{P}$. Suppose that
>
> $$\max(\mathrm{card}(\mathcal{P}),\ \mathrm{card}(\mathcal{F})) < 2^{\aleph_0}.$$
>
> Then there is an $\mathcal{F}$-generic filter, G on $\mathcal{P}$.

Here *c.a.c.* is the *countable anti-chain condition*, i.e. the assertion that every antichain is at most countable, which is now universally referred to as the c.c.c. (sometimes written ccc), even though the terminology is less accurate. In what follows, we will write MA as an abbreviation for Martin's Axiom, and write MA_κ for Martin's Axiom restricted to families of dense sets of size $\leq \kappa$ together with the assumption that $\mathfrak{c} > \kappa$.

In the footnote to page 232, Solovay and Tennenbaum report on Martin's work:

> In addition to formulating his axiom, Martin observed that it could be proved consistent by the same techniques used by the authors to get a model of SH. He also noted that his axiom, in the presence of the inequality $2^{\aleph_0} > \aleph_1$, implied SH. This material is included here with his kind permission.

In their §7.2 they indeed observe that MA + ¬CH implies SH, and in the remainder of the paper they establish the consistency of MA + ¬CH.

Martin and Solovay [1970] made a case that an alternative to CH is needed in the presence of Cohen's proof that CH cannot be proved in ZF and the result of Lévy and Solovay [1967] that it cannot be proved even with the assumption of the existence of a measurable cardinal. Martin and Solovay pointed to Sierpiński's book *Hypothèse du Continu* [1934] with many consequences of CH which are not known to be decided by ¬CH. Then then proposed "axiom" **A**, now Martin's Axiom. Rowbottom is acknowledged as independently suggesting such an axiom, and some of Kunen's[254] results from his thesis are included in this substantial report on the first consequences of the new axiom. The main results of the paper [1970, 144] are collected into the following multipart statement:

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[253] Donald Anthony Martin (December 24, 1940 –) received a bachelor's degree from the Massachusetts Institute of Technology, was a Harvard Junior Fellow, but never earned a doctorate. He taught at Rockefeller and now has a joint appointment in mathematics and philosophy at the University of California, Los Angeles.

[254] Kenneth Kunen received his doctorate in 1968 from Stanford University where his advisor was Scott. He has spent his career at the University of Wisconsin, Madison, with Keisler, Miller and the Rudins. Proofs of a number of his results were published by others. See Kanamori [2011b] to learn more about his set-theoretic work.

*T*heorem. If **A** then

1. $2^{\aleph_0} > \aleph_1 \ \rightarrow$ Souslin's hypothesis [22];
2. If $\aleph$ is an infinite cardinal number $< 2^{\aleph_0}$, then $2^{\aleph} = 2^{\aleph_0}$;
3. If $2^{\aleph_0} > \aleph_1$, every set of real numbers of cardinality $\aleph_1$ is $\mathbf{\Pi}^1_1$ if and only if every union of $\aleph_1$ Borel sets is $\mathbf{\Sigma}^1_2$ if and only if there is a real t with ${\aleph_1}^{L[t]} = \aleph_1$;
4. The union of $< 2^{\aleph_0}$ sets of reals of Lebesgue measure zero (respectively, of the first category) is of Lebesgue measure zero (of the first category);
5. If $2^{\aleph_0} > \aleph_1$, every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable and has the Baire property;
6. $2^{\aleph_0}$ is not a real-valued measurable cardinal (see also [8]).[255]

Diamond principles

Tennenbaum's construction of a Suslin tree sparked others to work on related questions. In particular, Ronald Jensen, Jack Silver and Solovay analyzed his proof, according to Jensen in a private conversation with Larson and Mitchell in September 2008. Jensen [1968] announced the incompatibility of Suslin's Hypothesis with $V = L$. Here is a report from Larson's notes [2008]:

> While still at Bonn, Jensen thought that there must be a Suslin tree in L. His proof of the existence of a Suslin tree in L came out of Tennenbaum's forcing with finite trees, where the thing that made it work was that whenever you cut off at a limit α, every branch was generic over that and would hit every anti-chain in that model. Like many other proofs, this one occurred to him at night when he was trying unsuccessfully to fall asleep. He reported that he knew enough to think out the proof before falling asleep, and was able to get up the next morning and just write it up.
>
> Jensen analysed the Tennenbaum proof to see just what was needed combinatorially to construct such a tree and formulated his Diamond Principle $\diamondsuit$, which he then showed consistent by proving it held in L. Kunen independently extracted $\diamondsuit$ from Tennenbaum's proof, and Silver extracted $\diamondsuit^*$, a stronger principle. The principle $\diamondsuit^+$ was extracted from Solovay's construction of a Kurepa family.

Jensen was an Assistent at the University of Bonn from the time of his Ph.D. in 1964, through the time of his Habilitation in 1967, and until some point in 1969 when he became a professor at University of Oslo and simultaneously started

[255] The citation [22] refers to the Solovay-Tennenbaum paper, and [8] refers to Kunen's thesis [1968].

periodically lecturing at Rockefeller University. (See [Jensen, 1969b] for details of work on versions of the Diamond Principle and on Kurepa's Hypothesis asserting the existence of a Kurepa tree.) In an abstract outlining results on automorphism properties of Suslin continua, Jensen included an early published statement of his Diamond Principle [1969a, 576]:[256]

> $\diamondsuit$: There exists $\Gamma \subset \bigcup_{\alpha<\omega_1} {\omega_1}^{\alpha}$ such that $\Gamma \cap {\omega_1}^{\alpha}$ is countable for each α and if $f \in {\omega_1}^{\omega_1}$, then $\bigvee_{\alpha} f{\restriction}\alpha \in \Gamma$.

In other words, there is a family Γ of functions into ω_1 whose domains are countable ordinals such that for each $\alpha < \omega_1$, there are only countably many functions with domain α, and for every $f : \omega_1 \to \omega_1$, there is some α so that the restriction of f to α is in the family Γ.

This version of $\diamondsuit$ looks quite different from the general version introduced by Jensen [1972] in his master work on the constructible universe for each stationary subset $E \subseteq \kappa$:

> $\diamondsuit_\kappa(E)$: there is a sequence $\langle S_\alpha : \alpha \in E\}$ such that $S_\alpha \subseteq \alpha$ and for each $X \subseteq \kappa$, the set $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary in κ.

Jensen published a proof that if $V = L$ and κ is regular, then $\diamondsuit_\kappa(E)$ holds for every stationary set $E \subseteq \kappa$, and today we write $\diamondsuit$ for $\diamondsuit_{\aleph_1}(\omega_1)$.

To complete the overview of the basic diamond principles, we include the definitions of $\diamondsuit^+$ and $\diamondsuit^*_\kappa(E)$ for a stationary subset $E \subseteq \kappa$.

> $\diamondsuit^+$: There is a sequence $\langle \mathcal{S}_\alpha \subseteq \mathcal{P}(\alpha) : \alpha \in \omega_1\rangle$ of countable sets such that for all $A \subseteq \omega_1$, there is a closed unbounded set $C \subseteq \kappa$ such that the following properties hold:
>
> a. for all $\alpha \in C$, $A \cap \alpha \in \mathcal{S}_\alpha$;
>
> b. for all $\alpha \in C$, $C \cap \alpha \in \mathcal{S}_\alpha$.
>
> $\diamondsuit^*_\kappa(E)$: There is a sequence $\langle \mathcal{S}_\alpha \subseteq \mathcal{P}(\alpha) : \alpha \in E\rangle$ with $|\mathcal{S}_\alpha| \leq \alpha$ for all but a bounded set of $\alpha \in E$ such that for all $A \subseteq \kappa$, there is a closed unbounded set $C \subseteq \kappa$ with $A \cap \alpha \in \mathcal{S}_\alpha$ for all $\alpha \in C \cap E$.

Kunen, in work with Jensen [1969b], proved that $\diamondsuit^+$ implies $\diamondsuit$ by proving that $\diamondsuit$ is equivalent the following principle $\diamondsuit'$.

> $\diamondsuit'$: There is a sequence $\langle \mathcal{S}_\alpha : \alpha < \omega_1\rangle$ with each $\mathcal{S}_\alpha$ a countable set of $\mathcal{P}(\alpha)$ such that for each $X \subseteq \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha \in S_\alpha\}$ is stationary.

[256] The abstract was received by the *Notices of the American Mathematical Society* on December 2, 1968.

The version of $\diamondsuit$ of the Jensen abstract can be made to look more like $\diamondsuit'$ if one replaces the functions by their ranges, and shrinks to a closed unbounded set E of α where the ranges of functions in Γ with domain α are also subsets of α. Alternatively, think of the functions as subsets of $\omega_1 \times \omega_1$ and use a bijection between $\omega_1 \times \omega_1$ and ω_1 to obtain the function version from the usual one.

It is clear that $\diamondsuit^+$ implies $\diamondsuit^* = \diamondsuit^*_{\aleph_1}(\omega_1)$, which in turn implies $\diamondsuit'$, so $\diamondsuit$ follows from all of the other variations listed. Presumably it was known to Kunen and Jensen in 1969 that $\diamondsuit$ does not imply Kurepa's Hypothesis (KH), that there is a Kurepa tree, since the model of $\neg$KH constructed by Silver in the 1960's (see [1971b]) from an inaccessible cardinal satisfies $\diamondsuit$. These diamond principles were analyzed in notes [1969b] prepared by Jensen when he was at Rockefeller University in 1969. Most of the results from the notes appeared in Devlin [1973], although the definition of $\diamondsuit^+$ was changed from asserting the existence of a closed unbounded set C for whose elements $C \cap \alpha$ and $A \cap \alpha$ in $\mathcal{S}_\alpha$, to asserting the existence of an unbounded set B for whose limit points α one has $C \cap \alpha$ and $A \cap \alpha$ in $\mathcal{S}_\alpha$. The principle $\diamondsuit^*$ was not mentioned in Devlin's book, but it is of little import, since the implications $\diamondsuit^+ \implies \diamondsuit^*$ and $\diamondsuit^* \implies \diamondsuit'$ follow from the definitions of these principles. There is a remark in the Jensen notes [1969b] that $\diamondsuit^+$ is stronger than $\diamondsuit^*$ since the former implies that there are Kurepa trees and the latter does not. Jensen's proof that $\diamondsuit^+$ implies KH appears later in the notes, but a proof $\diamondsuit^*$ does not imply KH only appears in the 1970's (see §7.1 where the discussion of diamond principles continues).

6.5 More on uncountable trees

We start by discussing terminology, mainly from [Kunen, 1983], to facilitate our discussion of results for Aronszajn, Suslin and Kurepa trees. Recall that for regular κ, a κ-tree $(T, <)$ is a tree of height κ all of whose levels have cardinality $< \kappa$; in particular, an ω_1-tree is a tree of height ω_1 all of whose levels are countable. With this terminology for regular κ, a κ-Aronszajn tree is a κ-tree with no branches of length κ, a special κ-Aronszajn tree is a κ-Aronszajn tree which is the union of fewer than κ antichains, and a κ-Suslin tree is a κ-Aronszajn tree with no antichains of size κ (it is definitely not a special κ-Aronszajn tree since special κ-Aronszajn trees have antichains of size κ).

If $s, t \in T$ and $s < t$, then we say that s is a *predecessor* of t and t is a *successor* of s. Call T *well-pruned* if it is rooted and every element of the tree has successors on all higher levels. Any subset of a tree is a *subtree* under the induced partial order, though we note that the height, width, and level structure need not be retained. One can show that every κ-tree has a well-pruned subtree which is a κ-tree. A tree is *Hausdorff* if different nodes on the same limit level have different sets of predecessors. A tree is *full* if every non-maximal element has infinitely many immediate successors, and λ-*full* if every non-maximal element has λ-many immediate successors. Note that the concept due to Kurepa that we called *uniformly branching* has been refined to *full* and we have imposed the requirements that the trees of interest in this case be rooted and Hausdorff. Note that the notion of *normal ω_1-tree* from [Jech, 2003] is the same as full well-pruned Hausdorff ω_1-tree.

Haim Gaifman[257] and Specker [1964] proved that there are at least 2^{κ^+} many isomorphism types of κ-full well-pruned Hausdorff κ^+-Aronszajn trees if $\kappa^\lambda = \kappa$ for all $\lambda < \kappa$.[258] In their introduction they mention Kurepa's problème miraculeux 1 (First Miraculous Problem stated in §3.4), i.e. whether or not any two uniformly branching $\aleph_1$-Aronszajn trees are isomorphic, which they solve in the negative. Gaifman and Specker showed that κ-full well-pruned Hausdorff κ^+-Aronszajn trees could be represented by *sequential trees*, which are trees of transfinite sequences closed under initial segment and ordered by end-extension. They referred to both Aronszajn's construction (cf. [Kurepa, 1935]) and Specker's construction [1949] of these trees. The proof of existence of many non-isomorphic trees used the formation of new trees from known ones via a process much like a shuffle of their underlying sequences.

Toward the end of the decade, conferences and surveys helped share the growing body of results about trees.

Kurepa [1969] surveyed results related to Suslin's Problem, from its statement through the independence results of Tennenbaum and Jech, at the International Symposium on Topology and its Applications held in Herceg-Novi in the summer of 1968.

Rudin [1969] popularized the use of Aronszajn, special Aronszajn and Suslin trees with her *American Mathematical Monthly* article. In this expository article, she described the Suslin problem as sounding easy (see page 1113): "Anyone who understands countable and uncountable can 'work' on it." She went on to point out that there are standard patterns, standard errors and "there are standard not-quite-counter-examples which almost everyone who looks at the problem happens upon." After noting the growing collection of results about Suslin trees, she described her goals for the paper: describe the standard pattern (development as a way to construct a tree from a linear order), prove an elementary result (the construction of a Suslin line from a Suslin topological space) and give one of the not-quite-counter-examples (the construction of an Aronszajn tree). She concluded her article with a brief review of recent research, which in part documents some results of Jensen on the existence of a variety of Suslin trees of cardinality κ in the constructible universe.

Jech [1971] expanded a May 1969 talk on trees given at a conference in Ober-

[257] Haim Gaifman earned a master's degree at Hebrew University in 1958, and a doctorate at the University of California, Berkeley, in 1962, where Tarski was his advisor. He directed the program in History and Philosophy of Science at the Hebrew University while a professor of mathematics prior to moving to Columbia University in 1990 as a professor of philosophy.

[258] Gaifman and Specker call the trees of interest *normal κ-trees*, but for $\kappa = \aleph_0$, their definition matches the definition of *normal* ω_1-tree in [Jech, 2003, 114] except that they have added the condition that there is no branch of length κ^+. This paper may well be the source of the word *normal* as used by Jech. I have stayed away from *normal* for trees because of its use in topology, and because different authors have different definitions.

wolfach into a survey article. He included brief proofs of classical as well as recent results, so his paper gives a snapshot of what was known at the end of the 1960's about uncountable trees. He straightforwardly pointed out in footnote 1 that Kurepa's [1968] assertion[259] that κ^+-Aronszajn trees exist for every κ is false. Jech showed how to generalize the construction of ω_1-Aronszajn trees to larger cardinals, and in his footnote 2, pointed out that Rowbottom and Silver observed that the existence of κ^+-Aronszajn trees follows immediately from the existence of ω_1-trees using Chang's cardinal transfer property $(\omega_1, \omega) \to (\kappa^+, \kappa)$, which is known to be true for regular κ assuming GCH. Jech showed the necessity of the assumption of GCH. He briefly discussed the tree property, the generalization of König's Lemma to trees with small levels and large heights, and reported that Tarski and his school had shown that for inaccessible cardinals, the tree property is equivalent to several other properties including $\kappa \to (\kappa)^2_2$ and Π^1_1-indescribability and being weakly compact [Keisler and Tarski, 1964]. In terms of trees, κ has the tree property exactly when there are no κ-Aronszajn trees.

Jech noted that Silver [1966] (see also [1971b]) observed that every real-valued measurable cardinal has the tree property, gave a sketch of the proof, and concluded that since the statement $2^{\aleph_0}$ is real-valued measurable is consistent relative to some large cardinal assumption, so is the statement that the continuum has the tree property. This result also showed the consistency of the statement that not all cardinals with the tree property are weakly compact. Jech pointed out that a κ-complete λ-saturated non-principal ideal on κ for some $\lambda < \kappa$ is sufficient to get the tree property at κ.

Jech reported that Prikry [1968] proved the consistency of the statement $\aleph_{\omega+1}$ does not have the tree property relative to a large cardinal assumption. He started with a measurable cardinal κ in the ground model where κ^+ has an Aronszajn tree T built from bounded increasing sequences from $Q = {}^{<\omega}\kappa$, constructed a generic extension in which $\kappa = \aleph_\omega$, and all cardinals above κ are preserved. In this extension T is an $\aleph_{\omega+1}$-tree whose levels have power at most $\aleph_\omega$, and there is no $\aleph_{\omega+1}$ branch in T, since otherwise there would be an increasing sequence of that length in Q which has cardinality $|Q| = \kappa = \aleph_\omega$ in the extension.

Next, Jech turned to the question of the existence of a Suslin tree. He cited Kurepa [1937b], Miller [1943], and Specker [1949] as having made translations of the problem to one about trees. Jech discussed the model of [Tennenbaum, 1968] for the existence of a Suslin tree and his own construction [1967].

Jech sketched the Solovay-Tennenbaum proof of the consistency of non-existence of Suslin trees we discussed in §6.3, noting that the continuum in their model is $2^{\aleph_0} = \aleph_2$. Jech announced the remarkable proof by Jensen of the consistency of non-existence of Suslin trees with the Continuum Hypothesis discussed in §7.7.

[259] In his Math Review MR0255410 (41 # 72) of Kurepa's paper entitled *On A-trees*, N. C. A. da Costa, noted that the A-trees of the title of have been considered by the author on several occasions, specifically in [Kurepa, 1937a]. Hence the title reference is clearly to Aronszajn trees. In the review, da Costa states that Kurepa proved the existence of an Aronszajn tree on every successor cardinal without the use of the Continuum Hypothesis.

His citation of mimeographed notes at the very end of his bibliography for this landmark result, suggests that the announcement was a last minute addition.

In his discussion of Kurepa trees, Jech singled out as the first result that if ω_1 is inaccessible in L and if no ordinal between ω_1 and ω_2 is a cardinal in L, then there is a Kurepa tree; Jech indicated several people found this result and mentioned Lévy, Rowbottom [1964],[260] and Bukovský [1966]. Later, Rowbottom and Stewart[261] [1966] proved the consistency of the existence of a Kurepa tree relative to ZFC. Next, Jech sketched the proof by Silver [1971a] only published in 1971 that the non-existence of a Kurepa tree is consistent relative to the existence of an inaccessible cardinal. Jech then noted "recently," Solovay proved that there is a Kurepa tree whenever $V = L[X]$ for some $X \subseteq \omega_1$; Silver observed that from this result it follows that if ω_2 is not inaccessible in L, then there is a Kurepa tree, so the assumption of inaccessibility in Silver's proof cannot be dropped.

In the fourth and final section, Jech presented what he called the "cream of the article", namely the proof that if $V = L$ both Suslin and Kurepa trees exist. We have already discussed the existence of Suslin trees in L due to Jensen; Jech credited Solovay with the construction of a Kurepa tree.

Like a spring that bubbles up from many tiny sources building up to a river, many independently generated ideas and approaches firmed up into our notions of ω_1-tree, Aronszajn tree, special Aronszajn tree, Suslin tree and Kurepa tree.

6.6 Transversals and decidability

Micha Perles [1963] showed that Dilworth's chain decomposition theorem (see the end of §5.3) does not generalize to the infinite, using $\omega_1 \times \omega$ ordered coordinate-wise as his example of a partial order in which all antichains are countable but it is not the union of countably many chains.

H. A. Jung and Rado [1967] generalized Hall's Marriage Theorem to families of subsets of a set E which include a single infinite subset. They prove that a necessary and sufficient condition for a family $\langle A_i : i \in I\rangle^\frown\langle B\rangle$ of subsets of E with each A_i finite and B infinite to have a transversal is that $\langle A_i : i \in I\rangle$ have a transversal and B is not a subset of the union of all sets $A[T] := \bigcup_{i\in T} A_i$ over finite $T \subseteq I$ for which $|A[T]| = |T|$.

There were extensions of the generalization of Jung and Rado to allow finitely many infinite sets by several authors in the late 1960's and early 1970's. For example, here is the condition by Jon H. Folkman [1970] for a family $A = \langle A_i \mid i \in I\rangle$ of subsets of a set E having a transversal: A has a transversal if and only if the following condition is satisfied where for $J \subseteq I$, $A(J) = \langle A_j : j \in J\rangle$: for all $n \in \omega$ and $K \subseteq J \subseteq I$ such that $A(J) - A(K)$ is finite and $|J - K| \geq n + |A(J) - A(K)|$, there is a finite set $S \subseteq K$ such that $|A(T)| \geq n + |T|$ for all sets T such that $S \subseteq T \subseteq K$. Folkman gave the example of the set of denumerable ordinals each regarded as the set of its predecessors as an

[260] The reference to Rowbottom also includes a cryptic "(notes for 1967)" [Jech, 1971, 14].
[261] Stewart's proof is sketched in Jech's paper [1971].

example of an uncountable family which has no transversal (apply the Regressive Function Theorem). All three, [Brualdi and Scrimger, 1968], [Folkman, 1970] and [Woodall, 1972], gave equivalent characterizations, as shown in [Bry, 1981].

Milner became interested in transversals in the 1960's and wrote a series of papers on them, some coauthored with Erdős or Hajnal[262] or both. Of particular note is the triple paper [1968] in which they prove that given a sequence of κ^+ many disjoint sets, each of cardinality κ, there is a collection of κ^+ many almost disjoint transversals. L. Mirsky [1971] codified what was known about transversal theory up through the 1960's in his book of that name, with attention to infinite combinatorial problems using the Rado Selection Principle.

Next we consider an application of the classification of order types. Läuchli and John J. Leonard [1966] proved the decidability of the elementary theory of order through the analysis of the class M of order types such that $1 \in M$, M is closed under pairwise sum, multiplication by ω and by ω^*, and the shuffling operation σF for finite $F \subseteq M$, where σF is a linearly ordered set which can be partitioned into order type η many segments such that every segment has a type in F and between any two different types there are parts in each type from F. Läuchli and Leonard credit Skolem [1920], [1970, 132, Satz 2] with the uniqueness of the shuffling operation, i.e. between any two sets A and B with witnessing partitions $\langle C_\xi \in F : \xi \in \eta \rangle$ and $\langle D_\xi \in F : \xi \in \eta \rangle$ showing that A and B fit the definition of σF, there is an order-preserving map $\Phi : A \to B$ so that each C_ξ is taken to some D_ζ where C_ξ and D_ζ represent the same type from F. They contrasted their proof with that of Ehrenfeucht [1959], [1961] and note that a proof similar to theirs had been found independently by Fred Galvin. The Läuchli-Leonard approach recalls Hausdorff's analysis of countable linear order types in [1908], and Skolem cited this paper for terminology.

6.7 Partition calculus classics

The late 1950's and early 1960's were a fruitful time for those working in the partition calculus, saw the publication of the landmark paper, *Partition relations for cardinal numbers*, by Erdős, Hajnal and Rado [1965], and the proof of the Hales-Jewett Theorem, a result from finite combinatorics that would impact both finite and infinite combinatorics.

Let A be a finite alphabet (i.e. a set) and let v be a "variable" not in A. Let $\Sigma^* A$ be the set of all "words" over the alphabet A. A "variable word" is a word over $A \cup \{v\}$ in which v actually occurs. Alfred Hales and Robert Jewett [1963] used variable words to investigate partitions of H-fold products of an n-element set, a dimensional approach to partition theory. Their theorem implies van der Waerden's Theorem.

> *Hales-Jewett Theorem:* If $\Sigma^* A$ is finitely colored, there must exist a variable word $w(v)$ such that $\{w(a) \colon a \in A\}$ is monochromatic.

[262] Hajnal [1997, 374–375] recalled that his first joint paper with Milner was on transversals.

Before turning to a discussion of partition relations for cardinals, ordinals, and the reals, we remind the reader that partition results of Galvin for real types and rationals and of Laver for the rationals were discussed earlier in §6.1, since they will not be repeated in this section.

Early finite exponent results

Hajnal used the Continuum Hypothesis to show that the sharpening of Erdős and Rado of the Erdős-Dushnik-Miller Theorem could not be improved for $\kappa = \omega_1$ from $\omega_1 \to (\omega_1, \omega+1)^2$ [Erdős and Rado, 1956, Theorem 39] to $\omega_1 \to (\omega_1, \omega+2)^2$ [Hajnal, 1960, 283, Theorem 5]. The proof actually yields a bit more, namely $\omega_1 \not\to (\omega_1, \omega : 2)^2$, i.e. there is a coloring of the pairs of ω_1 which has no independent set of size ω_1 and also has no complete bipartite graph consisting of ω points in a set B all of which are below two points in a set C with all pairs between them joined. The main tool was a set mapping theorem (Theorem 1 [1960, 278]) he had proved using the Continuum Hypothesis in order to answer negatively a question of Erdős and Fodor. Hajnal also proved positive partition relations for ω_1 and for λ, the order type of the real numbers. For example, he proved that for any uncountable order type ϕ which embeds neither ω_1 nor ω_1^* (e.g. λ) that $\phi \to (\omega \cdot n, \alpha)^2$ for any $n < \omega$ and any countable ordinal α. Moreover, he proved (see Theorem 8 on page 290) that $\omega_1 \to (\omega \cdot n, \omega \cdot 2)^2$ for every finite n and stated as the simplest unsolved problems: Does $\omega_1 \to (\omega^2, \omega \cdot 2)^2$? Does $\omega_1 \to (\omega \cdot 3, \omega \cdot 3)^2$? His method of proof gives more (see [1960, 294]): $\omega_1 \to (\omega \cdot n, (\omega : \omega_1)_<)^2$,[263] where the $(\omega : \omega_1)_<$ stands for a complete bipartite graph on a pair of sets $B, C \subseteq \omega_1$with $B < C$, where B has type ω and C is uncountable, i.e. the graph of all pairs meeting both B and C.

Erdős and Hajnal took the approach of their [1958] a step further in [1962b] to prove that the least inaccessible cardinal κ satisfies the negative partition relation $\kappa \not\to (\aleph_1)_2^{<\omega}$, which we would now describe by saying the least inaccessible cardinal is not an *Erdős cardinal*.

Building on the work of the late 1950's seminar organized by Mostowski and Tarski that led to [Erdős and Tarski, 1961], William Hanf[264] [1964] showed that $\kappa \to (\kappa)_2^2$ implies the tree property on κ, and hence by [Erdős, 1942] (the attribution is given in [Erdős and Tarski, 1961, 59, footnote 5]), the two properties are equivalent for regular uncountable κ.

Hajnal [1964], responding to this work, proved that if κ is a regular cardinal

[263]The notation $\omega_1 \to (\omega \cdot n, (\omega : \omega_1)_<)^2$ comes from [Erdős *et al.*, 1984, 62].

[264]William P. Hanf worked for IBM 1958–1968. He was supported by IBM during the year he completed his degree in 1963 as a student of Tarski with a dissertation titled *Some Fundamental Problems Concerning Languages with Infinitely Long Expressions*. He was a visiting lecturer at the University of California, Berkeley, for two years before joining the faculty at the University of Hawaii in 1970, where the William P. Hanf Memorial Fund now supports cash awards for winners of the Problem of the Month for Manoa Undergraduates, according to the website http://www.math.hawaii.edu/home/ProblemMonth_Hanf.html downloaded August 17, 2010. For more information on Hanf, see the eulogy by Dale Meyers [1991].

which supports a linear ordering with no well-ordered or conversely well-ordered subset of length κ, then for all $r \geq 3$, $\kappa \nrightarrow (\kappa, r+1)^r$. As he pointed out, it is enough to prove this for $r = 3$, since if f witnesses the relation for 4 and $r > 4$, then the function g given by $g(\alpha_0, \dots, \alpha_{r-1}) = g(\alpha_0, \alpha_1, \alpha_2)$ witnesses the relation for r. He gave a simpler proof of Hanf's result[265] and noted that the four statements from [Erdős and Tarski, 1961] that inspired Hanf's work are equivalent, which he attributed to Hanf [1964] with prior work by [Erdős and Tarski, 1961] and [Monk and Scott, 1964].

Erdős and Hajnal [1962a] proved a partition relation for countable order types: $\theta \to (\theta, \omega)^2$ if and only if $\theta = \omega$, $\theta = \omega^*$ (the converse well-ordering) or θ is not scattered. A major tool was a hierarchy, the stratification of countable scattered order types into ranked layers as they are built up from 0 and 1 by sums indexed by ω or ω^* over sets of lower rank. In a footnote, Erdős and Hajnal share their belief that this approach must be in the literature, but that they have not found it. Indeed, this hierarchy is a repackaging of a type ring of Hausdorff [1908], now known as the Hausdorff hierarchy. While the Hausdorff type rings were for scattered sets of cardinality less than $\aleph_\alpha$, the modern Hausdorff hierarchy includes scattered sets of every cardinality.

Milner and Rado [1965] investigated an infinitary generalization for ordinals of the Pigeonhole Principle. In terms of the arrow notation introduced by Rado, given ordinals $\alpha_0, \alpha_1, \dots$, they sought the least ordinal α for which $\alpha \to (\alpha_0, \alpha_1, \dots)^1$ holds. Of particular note was the discovery of what came to be called the *Milner-Rado Paradoxical Decomposition* of ordinals $\beta < \omega_2$ witnessing the surprising failure of $\beta \to (\omega_1, {\omega_1}^2, \dots, {\omega_1}^n, \dots)^1_\omega$, More generally, for any uncountable cardinal κ and every $\beta < \kappa^+$, the negative relation $\beta \nrightarrow (\kappa^\omega)^1_\omega$ holds.

The Giant Triple Paper

The triple paper [1965] of Erdős, Hajnal and Rado, finally appeared.[266] As Hajnal recounted in his paper *Paul Erdős' set theory* [1997, 363] "Starting around 1957 or 58, we decided to write a triple paper with Rado on the partition calculus and the three of us set aside everything which we thought belonged there. Already in 1960 I visited Rado in Reading to work on the triple paper, carrying with me an almost completed manuscript." They succeeded in giving an almost complete discussion of the partition relation $\kappa \to (\lambda, \mu)^r$ for cardinal numbers in this massive paper under the assumption of GCH (theorems marked with (*) used this assumption). The Positive Stepping Up Lemma of [Erdős and Rado, 1956] was joined by the Negative Stepping Up Lemma: if $r \geq 2$, $\kappa \geq \omega$ and $\kappa \nrightarrow (\lambda)^r_{\nu<\gamma}$, then $2^\kappa \nrightarrow (\lambda_\nu + 1)^{r+1}_{\nu<\gamma}$, provided one of several situations holds as in the various parts of Lemma 5. Sections 13 and 14 (pages 118–129) include an analysis of counter-examples. The *Canonization Lemma* (Lemma 3 on page 110) was used to get positive results for singular strong limit cardinals using GCH. A sample is Theorem

[265] In a footnote Hajnal remarked that the original form of the proof was simplified by Hanf.

[266] Erdős reported that Hajnal used to refer to it as the "Giant Triple Paper" [Erdős, 1996, 121].

6 from page 114 (GCH): Let a be a singular cardinal of cofinality $a' = b^+$ and $c < a'$. Then for b' the cofinality of b, $a \to (a, (b')_c)^2$.

Among the variations of the partition symbol introduced in the Erdős-Hajnal-Rado classic paper is the square bracket partition relation (see 18.1, 18.2, 18.3 on page 144 of [1965]) $\kappa \to [\lambda]^n_\nu$, often used to express the existence of strong counter-examples. This partition relation holds if for every set S of cardinality κ and every $f : [S]^n \to \nu$, there is a subset $X \subseteq S$ of size λ so that the restriction of f to $[X]^n$ does not map onto ν; colloquially, f omits a color on $[X]^n$. For example, the Sierpiński partition [1933] may be expressed with this notation as $\omega_1 \not\to [\omega_1]^2_2$. Erdős, Hajnal and Rado assume GCH to prove in [1965, Theorem 17, 141] that for $\kappa = \lambda^+$, $\kappa \not\to [\kappa]^2_\kappa$.

Erdős and Hajnal quickly found an application of Theorem 17 to the problem of Jónsson in their [1966][267] where they show that under the assumption of GCH no successor cardinal is Jónsson. A *Jónsson algebra* is an algebra without a proper subalgebra of the same cardinality and a cardinal κ is *Jónsson* if there are no Jónsson algebras with universe of size κ. Keisler and Rowbottom [1965] proved that κ is Jónsson if and only if every structure for a countable first-order language with universe of size κ has a proper elementary substructure whose universe also has size κ. They showed that under the assumption of $V = L$, no cardinal is Jónsson. In Theorem 4 of [1966], Erdős and Hajnal prove that every algebra whose universe has cardinality a measurable cardinal has a proper subalgebra with universe of the same cardinality, i.e. every measurable cardinal is a Jónsson cardinal. They went on to use a set mapping theorem from the joint paper [1958] that came out of their first meeting to prove in Theorem 5 that for every infinite cardinal λ there is a Jónsson algebra of cardinality λ with one ω-ary operation.

This brief overview gives only a taste of this paper about which one of the authors (Erdős) [1996, 120] said: "Hajnal, Rado and I nearly completely settled $m \to (n, q)^2$ but the results are very technical and can be found in our joint triple paper [69] and in our book [68]."

All Finite Subsets Problem

Results related to the All Finite Subsets Problem of Erdős and Hajnal discussed in §5.1 were proved in the 1960's.

Frederick Rowbottom [1964] received his doctorate from University of Wisconsin, Madison, and in his thesis he used partition properties of the form $\alpha \to (\beta)^{<\omega}_2$[268] to investigate the constructible universe. Specifically he showed that the existence of a cardinal κ such that $\kappa \to (\omega_1)^{<\omega}_2$ implies $V \neq L$. He also showed that measurable κ are *Ramsey cardinals* in the sense that $\kappa \to (\kappa)^{<\omega}_2$ but are not the

[267] The paper by Erdős and Hajnal comes with the notation that it was "presented by A. Mostowski on August 27, 1965".

[268] Recall that the partition relation $\alpha \to (\beta)^{<\omega}_2$ holds if and only if for every coloring $c : [\alpha]^{<\omega} \to \{0, 1\}$ of the finite subsets of α with two colors there is a subset $X \subseteq \alpha$ of order type β and a function $f : \omega \to \{0, 1\}$ such that for all $n < \omega$, for all n-element subsets $A \subseteq X$, $c(A) = f(n)$.

least cardinals with this property. The proof involved showing that for a normal ultrafilter U over a measurable cardinal κ, $\kappa \to (U)^n_2$ for all $n < \omega$ by induction and using closure of U under countable intersections to get the desired result.[269] Rowbottom showed that, in the presence of a cardinal κ satisfying $\kappa \to (\kappa)^{<\omega}_2$, the reals of the constructible universe are countable.

Jack Silver[270] in Chapter 4 of his thesis [1966], [1970b] showed that $\kappa \to (\omega)^{<\omega}_2$ is consistent with $V = L$. He further proved that if $\kappa \to (\alpha)^{<\omega}_2$ and $\alpha < {\omega_1}^L$, then the statement $\kappa \to (\alpha)^{<\omega}_2$ is true in L. He drew as a corollary the conclusion that if $\kappa \to (\alpha)^{<\omega}_2$ for all countable α, then the same is true in the constructible universe. Silver, in his thesis [1966] and [1971b] showed the existence of $0^{\#}$ (named in [Solovay, 1967]) follows from the existence of a cardinal κ for which $\kappa \to (\omega_1)^{<\omega}_2$.[271]

6.8 Infinitary partition relations

During the academic year 1967–1968 (including the summer of 1968), Mathias, Silver, and the team of Galvin and Prikry proved seminal results on partitions of infinite subsets of ω. Adrian Mathias[272] proved the consistency of $\omega \to (\omega)^\omega$ in the absence of the Axiom of Choice and introduced what is now known as Mathias forcing. Galvin and Prikry[273] proved a remarkable generalization of the Nash-Williams Partition Theorem, generalized by Silver in the beginning of a chain of results that have led to a wide variety of applications. What follows is a mainly chronological account of how these theorems were proved.

Recall that Erdős and Rado, as early as 1935 and in their Example 1 of [1952a, 431], had looked at generalizations of Ramsey's Theorem to partitions of the infinite subsets of a set: they used a well-ordering of the collection of infinite subsets of an infinite set, say ω for concreteness, to construct a partition $\mathcal{P}(\omega) = K_0 \cup K_1$ such that neither cell contains all the subsets of an infinite subset of ω. Let us say family $\mathcal{F}$ of infinite subsets of ω has the *Ramsey property* or is *Ramsey* if there is an infinite subset $M \subseteq \omega$ such that either $[M]^\omega \subseteq \mathcal{F}$ or $[M]^\omega \cap \mathcal{F} = \emptyset$.

In fall 1967, Jensen was visiting Stanford and Scott had arranged for Mathias,

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[269] Drake singled out this property and its generalization to n-tuples for special attention in his book *Set Theory* [1974, 189]: "The most important property of normal measures for us is usually called the *Ramsey* property, by analogy with Ramsey's Theorem of ch. 2 §8."

[270] Jack Howard Silver completed his doctoral thesis at the University of California, Berkeley, in 1966 with Vaught, spent a year at the University of Wisconsin, Madison, and like his advisor, came to be a member of the faculty at Berkeley where he had earned his degree.

[271] See Kanamori's second chapter and Mitchell's chapter for information on $0^{\#}$.

[272] Adrian Mathias graduated from the University of Cambridge in 1965 and wrote his fellowship and Ph.D. dissertation in 1968, for which the bulk of the work was done in Bonn and Stanford University under Jensen's supervision. He taught at Cambridge for twenty years without obtaining tenure; he then spent ten years teaching and researching in various places [Berkeley, Warsaw, Oberwolfach, Caen, the CRM at Barcelona, and Bogota], and finally was elected to a tenured position in Réunion in 2000.

[273] Karel Prikry studied at Charles University in Prague, and went to the University of California, Berkeley, in fall 1965 with what he [2010a] described as the equivalent of a master's degree including a short publication. He earned his doctorate with Jack Silver at Berkeley in 1968. He has spent most of his career at the University of Minnesota.

as Jensen's student, to be a research associate. Here is how Mathias described the initial impetus to his work on partitions of the infinite subsets of ω [1977, 110]:

> The author's interest in the problem of refuting the relation $\omega \to (\omega)^\omega$ without the axiom of choice was aroused by [Harvey] Friedman during Scott's seminar on partition theorems conducted at Stanford in 1967.

In a letter to Odell dated November 11, 1997, Mathias listed a sequence of events as he recalled them on the question of refuting $\omega \to (\omega)^\omega$ without choice. The first two items of the list are omitted since their core content is covered by the above quote, to which we add the remark that the seminar was held in the fall quarter.

3. In response to Friedman's enthusiasm, Paul Cohen and, independently, Ehrenfeucht find proofs that all open sets are Ramsey. Cohen uses this result to answer another question of Friedman by building an infinite subset X of ω not recursive in any $Y \subseteq X$ with $X \setminus Y$ infinite.
4. Whilst searching the literature in another connection, I find Nash-Williams' proof of a similar result.
5. Friedman gives a talk at Berkeley, with Silver, then an associate professor, and Prikry, then a graduate student, in the audience, describing his attempts to build a non-Ramsey set without using choice. He sketches a proof that if one Sacks real is added to a model of ZFC, every new infinite subset of ω contains or is disjoint from an old infinite subset of ω. In a discussion after his talk, Prikry points out that Friedman's argument is erroneous, and I give a correct proof.[274]

Next we turn to Galvin and Prikry [1973] who apparently proved their theorem immediately on hearing Scott's idea that infinitary partition theorems should be possible for nicely definable partitions, which they may have learned from the above described lecture of Friedman. In an email to Galvin and Larson, Prikry [2010a] recalled that he and Galvin "came up with the proof the same day or night," and described the circumstances.

> Fred did it while he was brushing his teeth before going to bed (plus the few minutes before he fell asleep). I spent the whole night writing up the proof in my office in T4 since I never liked claiming a theorem before I wrote it up completely. I think it would have been in the fall 1967. My motivation was Scot[t]'s talk. The notion of Ramseyness for infinite sets has been around – it has to do with infinitary Jonsson algebras and so forth.

[274] The Nash-Williams Partition Theorem [1965b] is the likely candidate for what Mathias found in the literature. I have omitted an unmatched [from this item.

Galvin and Prikry both had arrived at the University of California, Berkeley, in fall 1965, Prikry as a graduate student and Galvin as an instructor.[275] Prikry [2010b] recalled that he had gotten to know Silver during the year that they had overlapped in graduate school. Silver had finished his degree in 1966 and had spent a year at the University of Wisconsin, Madison. Already in the summer of 1967, Prikry had came up with his cofinality changing forcing, now known as *Prikry forcing.*

Galvin and Prikry identified the collection of subsets of ω with the product space 2^ω with the product topology. Their one-page proof that open sets are Ramsey is elegant and uses the following definition. Given a subset $S \subseteq 2^\omega$, say $M \subseteq \omega$ *accepts* X if every $N \subseteq M$ for which $X < N$ has $X \cup N \in S$, and M *rejects* X if there is no $N \subseteq M$ such that N accepts X.[276]

Galvin and Prikry introduced the notion *completely Ramsey* for subsets $S \subseteq 2^\omega$ with the property that for every continuous function $f : 2^\omega \to 2^\omega$, the set $f^{-1}(S)$ is Ramsey. They showed that the collection of completely Ramsey subsets of 2^ω includes all open sets and is closed under complements and countably unions, concluding that Borel sets are completely Ramsey. After deriving several corollaries including Ramsey's Theorem and the Nash-Williams Partition Theorem, they concluded with counter-examples to possible extensions, including a nowhere dense set of measure zero which is not Ramsey.

In January, Galvin [1968a][277] had already stated a generalization of the Nash-Williams Partition Theorem which can be translated to "open sets are Ramsey". Recently Galvin [2010] commented on this abstract:

> Of course, I was reinventing the wheel in that abstract; "open sets are Ramsey" had already been proved ... Also, at the time I submitted the abstract, I wouldn't have recognized it in the form "open sets are Ramsey".

Galvin and Prikry [1973] noted that their initial proofs that Borel sets are Ramsey used induction on the ordinals. Their published proof in [1973] instead uses ideas of Nash-Williams [1965b], "as they were explained to us by Richard Laver".[278] Documentably by the end of March, Galvin and Prikry had proved that Borel sets are Ramsey, since Robert Soare quoted this result in an abstract [1968a] received April 1, 1968, and in the paper [1969], Soare expressed gratitude to Galvin for the prompt reporting of his results.[279]

[275] Prikry recalled getting to know Galvin: "We had offices next to each other and had somewhat similar schedules, so we ended each day together around 1 am at a pizza place on Euclid [Avenue]."

[276] Galvin and Prikry noted that the fact that open sets are Ramsey was independently discovered by Ehrenfeucht and Cohen among others (cf. Mathias' point 3 above), but was not published prior to their paper [1973] in *The Journal of Symbolic Logic.*

[277] Galvin's abstract [1968a] was received January 26, 1968.

[278] Galvin and Laver played speed chess as well as talking mathematics together.

[279] Soare submitted a preliminary report [1968b] which was received February 8, 1968 (it mentioned Galvin for the proof of a combinatorial result formulated by Jockusch) and published

Silver [1970a], building on work of Galvin and Prikry, proved that analytic sets are completely Ramsey, and showed that under the assumption of the existence of a measurable cardinal that every Σ^1_2 set is Ramsey. For $X \subseteq \omega$ and $S \subseteq [X]^\omega$, the family S is called X-*Ramsey* if there is an infinite set $Y \subseteq X$ such that either $[Y]^\omega \subseteq S$ or $[Y]^\omega \cap S = \emptyset$. A family $S \subseteq [\omega]^\omega$ is *completely Ramsey* if, for every finite $F \subseteq \omega$ and every $X \subseteq \omega \setminus F$, the set $\{Z \subseteq X : F \cup Z \in S\}$ is X-Ramsey.[280] Silver's proof started with the formula $\exists r \phi(r, s, t)$ defining an analytic family, used forcing to extend a given model of a fragment of set theory to one in which MA_{ω_2} holds, proved that the family defined by the same formula in the extension is completely Ramsey, and applied an absoluteness argument to conclude that the original family was completely Ramsey.[281]

Mathias [1997, items 14–15] recalled an important step in his proof:

> Somewhere around the end of June 1968 I realise that there would be a good chance of proving that in Solovay's model all sets are Ramsey, provided that the Jensen-Friedman adaptation of Prikry forcing had the property that every infinite subset of a generic set is also generic. This is my main claim to originality: that such a phenomenon might be possible had never previously been suggested. ...
>
> Two weeks later, I finally reach the concept that I called *capturing*, and with its help establish my "all subsets generic" property, prove that in Solovay's model all sets are Ramsey, prove that my notion of forcing has the strong Cohen-type property that a Mathias-generic set X of integers is not constructible from any $Y \subseteq X$ with $X \setminus Y$ infinite, and, as easy corollaries, give new proofs of Silver's theorems.
>
> I write all this up in four weeks, and the resulting document, nearly 170 pages of A4, mainly in manuscript, is submitted in August 1968 to Trinity College, Cambridge in the (unfulfilled) hope of obtaining a Fellowship there under Title A.

Prior to the submission of the above document, Mathias sent in an abstract [1968] to the *Notices of the American Mathematical Society* which was received July 18, 1968. He gave a description of the forcing, now known as Mathias forcing: a *condition* is a pair $\langle s, S \rangle$ where $s \cup S \subseteq \omega$, s is finite and S infinite, and $n \in s \wedge m \in S \to n < m$; a condition (t, T) is stronger than (s, S) if $s \subseteq t \subseteq s \cup S$ and $T \subseteq S$. He included the statement of a lemma that subsets of a generic set are themselves generic, but did not mention the Solovay model, about which nothing

in the April issue of the *Notices of the American Mathematical Society*. Soare spoke on his results for the April 16–20, 1968 meeting of the American Mathematical Society in Chicago, so he welcomed the timely update by Galvin which allowed him to strengthen his theorem from continuous partial functions to Borel measurable functions.

[280] In their paper [1973], Galvin and Prikry indicate that they are the authors of both notions of *completely Ramsey*, and that every family satisfying the version they used in their paper also satisfies the version used by Silver.

[281] Mathias [1997, 2] noted that Silver had these results prior to Mathias' departure from California to Germany at the end of the winter quarter, in March or April of 1968.

had been published at the time. Here is how Mathias stated his theorem in the abstract:

> *Theorem:* Con(ZFC + there is an inaccessible cardinal) implies Con(ZF + DC + $\omega \to (\omega)^{\omega}_{2}$).

He included a note that his work yielded easy proofs of Silver's results.

Mathias spent 1968–1969 as a visiting lecturer at the University of Wisconsin, Madison, where he interacted with Kunen, his student Booth, and M. E. Rudin. Ultrafilters, especially on ω, were a topic of discussion and Mathias also thought about MAD families. In June he traveled to Australia where, in a very productive spurt, he proved several theorems and came up with the notion of *happy family*[282] for families $A \subseteq \mathcal{P}(\omega)$ whose complement $\mathcal{P}(\omega) - A$ is a free ideal (i.e. it contains $\{n\}$ for all n)[283] and for any decreasing sequence $\langle X_n \colon n \in \omega\rangle$ of elements of A there is $X \in A$ such that, for all n, if $n \in X$ then $X - X_n \subseteq \{0, 1, \cdots, n\}$. Examples of happy families include the family of all infinite subsets of ω, the complement of the ideal generated by a maximal family of almost disjoint sets, and a selective ultrafilter.[284] Mathias generalized Silver's result, and we present below a formulation by Andreas Blass [2009, 2583, Theorem 7]:[285]

> If U is a selective ultrafilter and if $[\omega]^{\omega}$ is partitioned into an analytic piece and a co-analytic piece, then there is a set $X \in U$ such that $[X]^{\omega}$ is included in one piece.

While Silver [1970a] published his result that analytic sets are Ramsey reasonably promptly, the publication of the work of Galvin and Prikry [1973] took longer to appear,[286] As noted above, Mathias submitted an extended version of his initial proofs to Cambridge University for his fellowship and Ph.D. dissertation. The above results were included in the published version [1977], but includes additional results, including a 1971 proof modeled on the classical proof that all analytic sets

[282] The name *happy family* was suggested by Crossley.

[283] This formulation of *free* comes from Math Review MR0491197 (58 #10462) by Baumgartner. Todorcevic [2010] calls such families non-principal co-ideals.

[284] A non-principal ultrafilter U on ω is *selective* if every partition of ω into disjoint sets not in U has a selector (transversal) in U. Equivalently, U is selective if every function $f : \omega \to \omega$ becomes one-to-one or constant when restricted to some set in U. Kunen (cf. [Booth, 1970]) showed that selective ultrafilters are *Ramsey*, i.e. whenever $[\omega]^2$ is partitioned into two pieces, there is a set $X \in U$ included in one piece. Booth and Kunen (cf. [Booth, 1970]) showed several conditions on ultrafilters were equivalent, including selective and Ramsey. Mathias used Ramsey in his paper.

[285] The Blass statement is very close to one direction of the theorem stated but not proved by Mathias in [1972, 210] which appears in a volume whose preface by Hodges was written November 1971, and a proof of the theorem may be extracted from Section 4 of his [1977]. Mathias' Theorem 2.2 [1973, 412] is similar to the Blass statement but in the context of Prikry sequences and ultrafilters on measurable cardinals; it was proved during a June 1969 stay at Monash University, and the resulting paper was received by the journal in September 1969, and revised the following August. Mathias mentioned in [1977] that these two theorems were proved at the same time.

[286] The Galvin-Prikry paper [1973] was received by the journal on October 12, 1971.

of real numbers are Lebesgue measurable, which he was inspired to seek after a conversation with Moschovakis at the Cambridge Summer School in Logic.[287]

7 1970-1980: STRUCTURES AND FORCING

The talented individuals who passed through the San Francisco Bay Area in the 1960's enriched mathematical life there and took away new perspectives and techniques to share around the world. Kunen moved to the University of Wisconsin, Madison, joining M. E. Rudin and graduate students Tall and Booth. Galvin moved to the University of California, Los Angeles, in 1968 and to the University of Kansas in 1974. Laver had a post doc at the University of Bristol and spent time at UCLA before moving to the University of Colorado, Boulder, where he joined a large and friendly working group with Malitz, Monk, Mycielski, Reinhardt, and Walter Taylor. Mathias moved to the University of Cambridge with a fellowship at Peterhouse, and Jensen spent time in Oslo, Rockefeller, and Oxford. Baumgartner moved to Dartmouth College as a John Wesley Young Instructor, and joined the faculty at Dartmouth with a one-year leave of absence spent at Caltech. Mitchell had a post-doctoral position at the University of Chicago and spent a few years at Rockefeller with Hao Wang and Martin, before moving to Pennsylvania State University with its longstanding logic group, which included Jech and Mansfield.

The two problem papers by Erdős and Hajnal [1971b], [1974] were valuable both for their problems and for their comments indicating what had been proved. The papers grew out of the presentation by Erdős at the 1967 UCLA set theory conference, and were circulated in preprint form. The results from the late 1960's on partitions of infinite subsets of ω by Galvin, Prikry, Mathias, and Silver finally appeared and were extended by Ellentuck and others in the 1970's. A version of the Halpern-Läuchli Theorem was used to prove a Ramsey theorem for trees. Partition relations with ordinal goals received increased attention, with a meta-mathematical approach yielding the Baumgartner-Hajnal Theorem, that $\omega_1 \to (\alpha)^2_2$ holds for all $\alpha < \omega_1$. Interesting ordinal partition relations were proved, i.e. ones where the resource or set being partitioned has order type a non-initial ordinal. A series of results about square bracket partitions provided strong counter-examples to ordinary partition relations and revealed additional structural information about partitions.

The word "special" entered the language of mathematics for trees and partial orders which are countable unions of antichains. Consistency was shown for combinations of existence and non-existence of various kinds of trees. Embeddings of trees into the rationals and into the reals were studied, adding to the understanding of their fundamental structure. The structure of Suslin trees in terms of their order was examined in terms of their order embeddings. Milliken proved

[287]The Cambridge Summer School in Logic (Mathias was the secretary of the organizing committee, i.e. the main organizer) was my first international meeting. There I met Erdős, Hajnal, Kunen, Milner, Prikry, Shelah, Silver and many others for the first time.

a partition theorem for strongly embedded trees. Shelah constructed a Countryman line and increased understanding of linear orders arising as the lexicographic orderings of Aronszajn trees. Basis questions were examined for linear orderings which are not a countable union of well-orderings. The statement that all $\aleph_1$-dense sets of reals are isomorphic was shown to be consistent as was the statement that $2^{\aleph_0} = \aleph_2$ and there is a linear order of cardinality $\aleph_1$ universal for linear orders of size $\aleph_1$.

M. E. Rudin [1975b], in a history of attempts to prove the Normal Moore Space Conjecture through the mid-1970's, highlighted connections between Aronszajn trees and Moore spaces, a thread explored by William Fleissner [1975],[288] Devlin and Shelah [1979a], [1979b], Taylor [1981][289] and others. In 1980, Peter Nyikos [1980b] used the *Product Measure Extension Axiom* (PMEA) to prove the consistency of the Normal Moore Space Conjecture. Kunen had shown that the consistency of PMEA follows from the consistent existence of a strongly compact cardinal (see [Fleissner, 1984] for a proof). There is more to the history of the Normal Moore Space Conjecture but at this point the story moves away from our main themes. We encourage the interested reader to continue the history with the excellent paper by Peter Nyikos [2001] and learn about further consistency and independence results there.

7.1 Combinatorial principles

John Truss observed that König's Lemma is equivalent to the principle that every countable set of finite sets has a choice function, in a paper exploring consequences of further restriction on the type of branching allowed.

In addition to diamond principles, Jensen [1972] introduced square principles for the "purpose of carrying out inductions which would otherwise breakdown" (see page 271) for infinite cardinals κ:[290]

> $\square_\kappa$: There is a sequence $\langle C_\lambda : \lambda < \kappa^+ \rangle$ defined on limit ordinals $\lambda < \kappa^+$ such that
>
> (i). C_λ is a closed unbounded subset of λ;
>
> (ii). if $\mathrm{cf}(\lambda) < \kappa$, then $|C_\lambda| < \kappa$;
>
> (iii). if γ is a limit point of C_λ, then $C_\gamma = \gamma \cap C_\lambda$.

[288] The topologist Reed, in his Math Review MR0394583 (52 #15384) of Fleissner's paper [1975], discussed Jones' road space S [1953] which he said was introduced as a potential example of a normal, non-metrizable Moore space. Reed noted "The construction of S was discovered independently by N. Aronszajn, and it is known to set-theorists as a special Aronszajn tree," providing evidence that topologists were more familiar with the version of this construction from Jones.

[289] Taylor's article [1981] was received by the journal in 1979.

[290] I have changed the notation for cardinality of a set from $\overline{\overline{C}}$ to $|C|$ in the statement of $\square_\kappa$ for typographical simplicity.

Jensen proved in Theorem 5.2 that if $V = L$, then $\Box_\kappa$ holds for every κ. Moreover, he proved that if $V = L$ and κ is regular, then the weak compactness of κ is equivalent to *stationary reflection*: if $A \subseteq \kappa$ is stationary in κ, then $A \cap \beta$ is stationary in β for some $\beta < \kappa$ (see [1972, 287]). Combinatorial consequences of the existence of a *non-reflecting* stationary subset of κ, i.e. a stationary subset $A \subseteq \kappa$ such that $A \cap \beta$ is non-stationary for all $\beta < \kappa$, were quickly found. For example, under the assumption of $V = L$, Shelah[291] used a non-reflecting stationary set to prove in Lemma 3.1 of [1975] that if λ is a regular uncountable non-weakly compact cardinal, then there is a graph G on λ so that the *coloring number*[292] of the restriction of G to any bounded set is $\leq \aleph_0$ but the coloring number of G is uncountable.

The remainder of §7.1 is devoted Jensen's Diamond Principle and its variants, and many of the results were found by Keith Devlin.[293]

Devlin [1979b] proved that $\diamondsuit^*$ is strictly stronger than $\diamondsuit$. Recall there was a remark in the handwritten notes of Jensen [1969b] that $\diamondsuit^+$ was stronger than $\diamondsuit^*$ since the former implied the existence of a Kurepa tree and the latter did not, but it was somewhat cryptic, since no proof was give that $\diamondsuit^*$ failed to guarantee the existence of Kurepa trees. The missing piece was filled in by Devlin [1978]. Using an idea of Jensen and starting from an inaccessible cardinal, Devlin gave a forcing construction of a model of $\diamondsuit^*$ which has no Kurepa trees. He said on page 292 that Jensen had used the idea to "convince himself that $\diamondsuit^*$ does not imply KT, [the existence of a Kurepa tree], but never wrote up the proof."

In the 1970's two variants weaker than $\diamondsuit$ were introduced. Adam J. Ostaszewski[294] [1976] published his proof of the existence of a non-compact, hereditarily separable, locally compact, perfectly normal, countably compact space. While he said in the introduction to his paper that the construction was based

[291]Saharon Shelah received his 1969 doctorate from Hebrew University where Michael O. Rabin was his advisor. He is a professor at Hebrew University and also at Rutgers University. While his main mathematical interests are in logic, especially model theory and set theory, he has broad interests. Combinatorial set theory has benefited from his deep knowledge of the behavior of singular cardinals, his development of myriad forcing techniques, and his powerful problem solving capabilities. His output is dauntingly prodigious, with gems that sparkle for many different groups of mathematicians, including his more that 200 co-authors. He was the first recipient of the Erdős Prize in 1977; he received the Israel Prize for Mathematics in 1998, the Bolyai Prize in 2000 and the Wolf Prize in Mathematics in 2001.

[292]The *coloring number* of a graph G on λ is the minimal cardinal μ such that we can order the vertices of G as $\langle a_i : i < \ell_0 \rangle$ such that each a_i is joined to fewer than μ many a_j's for $j < i$.

[293]Keith J. Devlin received his Ph.D. from the University of Bristol in 1971, where his advisor was Frederick Rowbottom. He was an assistant professor at the University of Bonn 1974–1976, a reader in mathematics at Lancaster University 1977–1987, visiting associate professor at Stanford University 1987–1989, chair of the Department at Colby College in Maine 1989–1993, dean at Saint Mary's College 1993-2001, and has been at Stanford University since 2001. In addition to his academic mathematics, he is widely known for sharing mathematics with the public, was winner of the 2007 Carl Sagan Award for Science Popularization, and is known as "the Math Guy" on National Public Radio.

[294]Adam J. Ostaszewski received his doctorate in 1973 from the University of London where his advisor was Claude Ambrose Rogers. He is a reader in the Department of Mathematics at the London School of Economics where his current research focuses on mathematical finance.

on Jensen's $\diamondsuit$, he actually introduced an ostensibly weaker principle, $\clubsuit$, which sufficed for his proof (see [1976, 507]):

> $\clubsuit$: There is an increasing sequence of countable limit ordinals λ_α indexing a sequence $\langle s(\lambda_\alpha) : \omega \leq \alpha < \omega_1 \rangle$ where $s(\lambda_\alpha)$ is an ω sequence cofinal in λ_α, such that every uncountable subset of ω_1 contains one of them.

His paper includes a proof that $\diamondsuit$ is equivalent to CH + $\clubsuit$, which he attributes to Devlin in private correspondence (see also Devlin [1974][295]). Ostaszewski stated that it was not known whether $\clubsuit$ was consistent with the negation of the Continuum Hypothesis, and he pointed out that it is not difficult to show that it is false in the presence of MA + $\neg$CH.

At the end of the 1970's, Baumgartner and Shelah gave different proofs that $\clubsuit$ is weaker than $\diamondsuit$ by forcing over models of $V = L$ to get a model in which $\clubsuit$ holds but CH does not. Shelah [1980c, §5],[296] [1998, Chapter 1 §7] first added $\aleph_3$ Cohen subsets of ω_1 and then used finite conditions to collapse ω_1 to ω. Baumgartner, in an unpublished note,[297] added many Sacks reals side-by-side with countable support. See [Hrušák, 2001] for a proof of that $\clubsuit$ holds in this model.

Devlin and Shelah [1978] introduced the *Weak Diamond Principle*,[298] which states that for each $f : {}^{\omega_1 >}2 \to 2$ there is a $g \in {}^{\omega_1}2$ such that for any $h \in {}^{\omega_1}2$, the set $\{\alpha \in \omega_1 : f(h \restriction \alpha) = g(\alpha)\}$ is stationary. This principle is a consequence of Jensen's $\diamondsuit$, and Devlin and Shelah showed it follows from the inequality $2^{\aleph_0} < 2^{\aleph_1}$. Their motivating application was Whitehead's problem on abelian groups, and they also applied it to ladder systems.

Several people, including Baumgartner, independently showed that the Weak Diamond Principle is equivalent to the cardinal inequality $2^{\aleph_0} < 2^{\aleph_1}$ from which it had been derived. A proof appears in a paper by Taylor [1981, §3] which was received by the journal in 1979 and discusses weak diamond type statements and the Normal Moore Space Conjecture.

Next, the discussion turns to the diamond variations that live on stationary sets. Recall that for a cardinal λ and a stationary subset $E \subseteq \lambda$, $\diamondsuit^*(E)$ is the statement that there is a sequence $\langle W_\alpha \subseteq \mathcal{P}(\alpha) : \alpha \in E \rangle$ with $|W_\alpha| \leq \alpha$ for all but a bounded set of $\alpha \in E$ such that for all $X \subseteq \lambda$, there is a closed unbounded set $C \subseteq \lambda$ with $X \cap \alpha \in W_\alpha$ for all $\alpha \in C \cap E$. John Gregory [1976, 666] in Lemma 2.1 proved the following:

[295] Shelah [1980c] credits both Burgess and Devlin with the equivalence of $\diamondsuit$ with $\clubsuit$ + CH.

[296] The paper [Shelah, 1980c] was received by the journal December 24, 1978 and, in revised form, December 16, 1979. In an email of January 13, 2011, he recalled that he proved the result in 1977 shortly after learning of the problem from a list by Fleissner.

[297] Shelah and Džamonja [1999] briefly discuss the history of the problem, noting that Baumgartner's proof was given shortly after Shelah's. They list further extensions and answer questions of S. Fuchino and M. Rajagopalan on variants of the club principal.

[298] The principle, denoted Φ in [1978], was named the *Weak Diamond Principle* in Devlin's survey [1979b].

Gregory's Theorem: For regular cardinals κ and μ, if $\kappa^\mu = \kappa$ and $2^\kappa = \kappa^+$, then for the stationary set $E = E(\kappa^+, \mu)$ of all ordinals less than κ^+ of confinality μ, the principle $\diamondsuit^*(E)$ holds.[299]

Specifically, if GCH holds and $\mathrm{cf}(\kappa) > \mu$ for μ regular, then $\diamondsuit^*(E(\kappa^+, \mu))$. Thus, if GCH holds, for all regular uncountable cardinals κ, $\diamondsuit(\kappa^+)$ holds.

In his survey article [1979b], Devlin reported on Shelah's result [1977] that it is consistent with GCH to have disjoint stationary sets $E, F \subseteq \omega_1$ such that $\diamondsuit(E)$ is true and $\diamondsuit(F)$ is false.

7.2 Transversals and cardinal arithmetic

At the 1974 International Congress of Mathematicians in Vancouver, individuals mentioned in this history were well-represented among the invited speakers: Barwise, Hajnal, Jónsson, Milner (plenary), Rado, Shelah, Silver. In addition, Yiannis Moschovakis was an invited speaker, and Leo Harrington, Martin and Prikry attended (see [Shelah, 1976]).

Milner [1975] spoke on transversals and included a discussion on various infinite extensions of Hall's Theorem. Milner had started working on transversals in the 1960's. He and Shelah [1975] extended transversal theory to regular uncountable cardinals in a paper for the proceedings of the colloquium held at Lake Balatón in 1973 dedicated to Erdős on his 60th birthday. They showed that if κ is weakly compact, $\mathcal{F}$ is an indexed family of κ sets each of cardinality less than κ, and every subsequence of $\mathcal{F}$ of smaller cardinality has a transversal, then so does $\mathcal{F}$. They also proved that the converse of this statement holds in L. They went on to prove a negative stepping up theorem for transversals.

Shelah [1975], in the same volume, tackled the case for strong limit cardinals. For example, he showed (see Theorem 4.6) that if κ is a strong limit cardinal of cofinality $\aleph_0$ and $\lambda < \kappa$, and if $\mathcal{F} = \langle F_i \colon i \in I\rangle$ is a system of κ sets each of cardinality $< \lambda$, and if every subsystem of $< \kappa$ sets has a transversal, then so does $\mathcal{F}$. Shelah also gave a criterion of an inductive kind which provided necessary and sufficient conditions for an arbitrary family of countable sets to have a transversal.

At the Vancouver meeting, Silver [1975] spoke on his groundbreaking new result on the singular cardinals problem, that if $2^\nu = \nu^+$ holds on a stationary set of cardinals below κ, a singular cardinal of uncountable cofinality, then it also holds at κ. Since this result is discussed in Kojman's chapter, we only include here a few remarks about connections with individuals working in combinatorial set theory.

Several mathematicians were involved in finding alternative proofs and generalizations of Silver's result. An elegant and purely combinatorial proof was found independently by Prikry, Baumgartner, and Jensen. The proof by Baumgartner and Prikry [1976], [1977] used a result of Erdős, Hajnal, Milner [1968] as the base case.

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[299]Note that Gregory is using *confinality* as Hausdorff did, rather than the now standard *cofinality*.

Galvin and Hajnal[300] [1975], working independently of one another, proved what has become known as the Galvin-Hajnal Theorem.

> *Galvin-Hajnal Theorem* [1975, 492, Corollary 5]: Let $\aleph_\alpha$ be a singular strong limit cardinal of uncountable cofinality. Then $2^{\aleph_\alpha} < \aleph_\gamma$ where $\gamma = \left(2^{|\alpha|}\right)^+$.

This result is a corollary of their Theorem 1, which relied on the notion of a family $\mathcal{F}$ of *almost disjoint transversals* of a sequence $\langle A_\alpha : \alpha < \kappa\rangle$, where transversals f and g are *almost disjoint* if $\{\alpha : f(\alpha) = g(\alpha)\}$ is bounded. Their Theorem 2 is a stronger result, obtained by replacing the ideal of bounded subsets with the non-stationary ideal to define a well-founded partial order, $f <_{\mathrm{NS}_\kappa} g$ iff $\{\alpha : f(\alpha) \geq g(\alpha)\}$ is non-stationary. The rank function associated with this partial order is now known as the *Galvin-Hajnal rank*.

Theorem 2 has as corollaries Silver's result and its immediate generalizations. It stimulated further work[301] by Shelah [1980b, 56], who wrote that he was continuing [Galvin and Hajnal, 1975]. Shelah [1978b] applied the Galvin-Prikry ideas to Jónsson algebras, and he [1994, xi] said the following:

> My interest in cardinal arithmetic started in 1975, having heard the theorems of Silver, Galvin, Hajnal and others, on bounds, proved in ZFC, for cardinal exponentiation, and Jensen's discovery of the covering theorem. I felt then that I had come too late to the game (later I have felt to be alone actively interested in it.)

This story is continued in Kojman's chapter on cardinal arithmetic.

7.3 *Partition relations on cardinals and ordinals*

This section starts with a discussion of the Baumgartner-Hajnal Theorem and related results. Then the following topics are discussed in turn: ordinary partition relations in which the resource is a cardinal, ordinal partition relations, square bracket partition relations.

The Baumgartner-Hajnal Theorem

We begin with a special case:

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[300] The Galvin-Hajnal paper was received September 25, 1974, a mere month after the August 21-29, 1974 Vancouver meeting at which Silver spoke. Galvin and Hajnal thanked Silver and Prikry for communicating their results to them, and point out that they use Prikry's ideas (see [Baumgartner and Prikry, 1976]). For a nice exposition of their work, see the teaching blog of Andrés Caicedo or his forthcoming book.

[301] See [Jech, 1995], Kojman's chapter on cardinal arithmetic, and the final chapter of [Holz *et al.*, 1999] for more on the relationship between the Galvin-Hajnal theorems and Shelah's work on cardinal arithmetic.

Baumgartner-Hajnal Theorem for ω_1: For all $\alpha < \omega_1$ and $k < \omega$,

$$\omega_1 \to (\alpha)^2_k.$$

The full Baumgartner-Hajnal Theorem solved a series of questions raised by Erdős, Hajnal and Galvin[302] about partitions of pairs from ω_1 and partitions of pairs from λ, the order type of the real numbers. Before stating the full theorem, we review these problems and partial results leading up to it.

Problem 10 of [Erdős and Hajnal, 1971b] asks, under the assumption of GCH, whether $\omega_{\rho+1} \to (\xi,\xi)^2$ for all $\xi < \omega_{\rho+1}$ and all ρ, and Problem 11 asks a parallel partition relation for λ, the order type of the reals: does $\lambda \to (\alpha)^2_k$ hold for all finite k and countable α? In addition, Problem 10/A asks three specific instances of $\omega_1 \to (\alpha_i)^2_k$ for ordinals $\alpha_i \leq \omega^2$ and finite k, and Problem 11/A asks a parallel series of specific questions about instances of $\lambda \to (\alpha_i)^2_k$.

Hajnal [1960, Theorem 8] had shown that, in the presence of CH, $\omega_1 \to (\omega \cdot 2, \omega \cdot k)^2$ for every finite k. In a letter to Erdős and Hajnal cited in a note added in proof to [1971b], Galvin informed them that he had proved $\omega_1 \to (\omega \cdot 3)^2_2$. Prikry [1972] proved $\omega_1 \to (\omega^2, \alpha)^2$ for $\alpha < \omega_1$, which completely solved Problem 10/A, since for any finite sequence $\langle \alpha_i : i < k \rangle$ of countable ordinals, there is a countable ordinal α such that $\alpha \to (\alpha_i)^2_k$.

Laver [1975], during his time at Bristol 1969-1971, used $\mathrm{MA}_{\aleph_1}$ to prove a partition relation equivalent to

$$\omega_1 \to (\omega_1, (\omega : \omega_1))^2.$$

This means that every graph on ω_1 either has an uncountable independent set or has a pair of sets B and C where B has type ω, C has type ω_1, and all pairs $\{x, y\}_{\neq}$ with $x \in B$ and $y \in C$ are joined. Laver's proof was the first use of Martin's Axiom to prove a partition relation, and a forerunner of the proof of the Baumgartner-Hajnal Theorem.

Prikry [1972] in the Corollary on page 78 proved (in ZFC) a partition relation closely related to Laver's result from $\mathrm{MA}_{\aleph_1}$. Below I have named it and restated it in an equivalent form.

Prikry's Theorem: For all countable ordinals α, $\omega_1 \to (\alpha, (\omega : \omega_1))^2$.

By the time Erdős and Hajnal were correcting the proofs of their problem paper [1971b], Galvin had already shown that for every real type φ, the partition relation $\varphi \to (\alpha)^2_2$ holds for all countable α. Erdős and Hajnal, after stating Galvin's result, commented that all the known results for λ, the order type of the reals, hold for order types in which neither ω_1 nor $\omega_1{}^*$ is embeddable (let us temporarily call

[302] In a November 2010 telephone call, Baumgartner recalled visiting Galvin in his Berkeley office in the 1960's where he had three boxes of papers on the floor. Galvin would occasionally open one of the boxes and reveal a paper on Erdős type problems. Baumgartner said that for many years, Galvin was the one he went to in order to learn the status of such problems.

such types *pseudo-real types*). Erdős and Hajnal asked in Problem 12 if one can prove a partition relation of the form $\lambda \to (\theta_1, \theta_2)^2$ which does not hold for some uncountable pseudo-real type ψ and some order types θ_1 and θ_2.

Erdős and Hajnal review some of the history of the generalization of the question to order types in their follow-up problem paper [1974, 272], noting that the topic was extensively studied by Galvin, "and it is fair to say that even the general conjecture is due to him." Among Galvin's "old theorems" they list the implication that if $\varphi \nrightarrow (\omega)^1_\omega$, then $\varphi \nrightarrow (\omega, \omega+1)^2$. Consequently, the order types φ satisfying $\varphi \to (\alpha)^2_2$ for all countable α, must satisfy $\varphi \to (\omega)^1_\omega$.

This property of order types, singled out by Galvin, turned out to be the key:

> *Baumgartner-Hajnal Theorem for order types:* For all $\alpha < \omega_1$ and $k < \omega$, if $\varphi \to (\omega)^1_\omega$, then $\varphi \to (\alpha)^2_k$.

Since λ satisfies the partition property $\lambda \to (\omega)^1_\omega$, these results provided solutions to Problems 10/A and 11 of the Erdős-Hajnal problem paper [1971b].

Baumgartner and Hajnal [1973] used a meta-mathematical argument. They started by using Martin's Axiom to prove the desired partition relation and then used an absoluteness argument to show that the use of MA is inessential. Baumgartner and Hajnal cite Silver [1966], [1970b] as the source of the absoluteness argument. Baumgartner and Hajnal do not use MA directly, but instead cite two lemmas. Lemma 1 [1973], due to Solovay (cf. [1971]), is the fact that under MA_β, for any finite k and any indexed family $\{A(i,\xi) : i < k \wedge \xi < \beta\}$ of partitions $\bigcup_{i<k} A(i,\xi) = \omega$ of ω, there are an infinite set $X \subseteq \omega$ and a function $f : \beta \to k$ so that X is almost included in every $A(f(\xi),\xi)$. Lemma 2 [1973], due to Kunen [1968], is the fact that under MA_β, any family of at most β many functions from ${}^\omega\omega$ can be eventually dominated.

Galvin [1975] presented a combinatorial proof of the Baumgartner-Hajnal Theorem for ω_1 in the clear and elegant style for which he is known. He first isolated in his Theorem 1 a combinatorial statement about a fixed partition of 2-element chains of a partially ordered set stating that there is a family of its subsets with sufficient closure properties to derive the Baumgartner-Hajnal Theorem in both its cardinal and order type forms. In the same paper he used Theorem 1 to prove that if φ is a partial order type and $\varphi \to (\eta)^1_\omega$, where η is the type of the rationals, then $\varphi \to (\alpha)^2_k$ for all $k < \omega$ and $\alpha < \omega_1$. Note that the partitions under consideration are partitions of 2-element chains, rather than arbitrary 2-element subsets. This result led to increased interest in order types with this property. Galvin stated some open problems, including the question of whether $\varphi \to (\omega)^1_\omega$ implies $\varphi \to (\alpha)^2_k$ for all $k < \omega$, all $\alpha < \omega_1$ and all partial order types φ. He also generalized the result of Erdős and Rado [1956, 472] that $\omega_1 \to (\omega+1)^r_k$ for finite r, k, by replacing ω_1 by a partial order P such that $P \to (\omega)^1_\omega$.

Cardinal resources

While the classical work on ordinary partition relations for cardinals was essentially done in the sixties [Erdős *et al.*, 1965] and the Baumgartner-Hajnal Theorem came in the first half of the decade, additional results appeared in the 1970's.

Shelah [1973] used GCH to partially extend the Baumgartner-Hajnal Theorem to successor cardinals by showing that if $|\gamma|^+ < \kappa$ and κ is regular, then $\kappa^+ \to (\kappa+\gamma)^2_2$. Rebholz [1974] applied morasses to show that Shelah's result was optimal for successors of successor cardinals under GCH by demonstrating that

$$\aleph_{\alpha+2} \not\to (\aleph_{\alpha+1} + \aleph_\alpha)^2 \text{ holds if } V = L.$$

Consistently, the continuum may have rather different partition properties from ω_1. Kunen [1971] announced that if κ is real-valued measurable, then $\kappa \to (\kappa, \alpha)^2$ for all $\alpha < \kappa$. Thus, since Solovay [1971] established the consistency of the continuum being a real-valued measurable cardinal, $2^{\aleph_0} \not\to (2^{\aleph_0}, \alpha)$ cannot be proved for $\omega + 2 \leq \alpha < \omega_1$ without some hypothesis on the size of the continuum.

Ideals were used to prove other partition relations of the form $2^\kappa \to (2^\kappa, \alpha)^2$ for some $\alpha < \kappa$. A *κ-ideal* is a non-trivial ideal of subsets of κ which is κ-complete. Laver called a κ-ideal $\mathcal{I}$ *(λ, μ, ν)-saturated* if every family of λ many $\mathcal{I}$-positive sets (i.e. ones not in the ideal) has a subfamily of size μ such that the intersection of any subset of the subfamily of size at most ν is also $\mathcal{I}$-positive. Laver [1978b] proved that if there is a (κ, κ, γ)-saturated κ-ideal and $\beta^\gamma < \kappa$ for all $\beta < \kappa$, then $\kappa \to (\kappa, \alpha)^2$ for every $\alpha < \gamma^+$. He further showed how to force the existence of such ideals starting with a measurable cardinal, with a sample consequence being the consistency, relative to a measurable cardinal, of the partition relation $2^{\omega_1} \to (2^{\omega_1}, \alpha)$ for all $\alpha < \omega_2$.

Laver [1975, 1936] proved that if there is a Mahlo cardinal κ, then there is a forcing extension in which $2^{\aleph_0} = \kappa$ is weakly Mahlo, and $2^{\aleph_0} \not\to (2^{\aleph_0}, (\omega : 2))^2$.

Baumgartner [1976a, 434, Theorem 7.1] started with a model in which κ and λ are regular cardinals with $\kappa \leq \lambda$, and $\diamondsuit_\kappa$ if κ is not inaccessible, and forced to get $2^\kappa \geq \lambda$ while preserving cofinalities and arranging that in the extension the following partition relation holds for all regular $\mu \geq \kappa^{++}$ and every $\beta_0 < \kappa^+$:

$$\mu \to (\text{stationary } \mu, (\beta_0 : \text{stationary } \mu))^2.$$

Note the close relationship with Prikry's Theorem.

Recall that Hajnal [1960] had shown with CH that $\omega_1 \not\to (\omega_1, (\omega : 2))^2$, a refutation of a possible extension of the Dushnik-Miller Theorem. Baumgartner [1976a, 436, Theorem 7.2] used forcing to prove the consistency of $\kappa^+ \not\to (\kappa^+, (\nu : 2))^2$, where $2^\nu = \kappa^+$. Laver [1975, 1033] proved that if Martin's Axiom holds and $2^{\aleph_0} = \aleph_2$, then $2^{\aleph_0} \not\to (2^{\aleph_0}, (\omega : 2))^2$ (this result was later and independently found by Hajnal).

Stephen G. Simpson[303] [1970] came up with a model-theoretic proof of the Erdős-Rado theorem which was incorporated into the classic text *Model Theory*

[303]Stephen George Simpson (1945 –) received his doctorate in 1971 from the Massachusetts Institute of Technology where his advisor was Gerald Sacks. He is currently a professor at Pennsylvania State University.

by C. C. Chang[304] and H. Jerome Keisler[305] [1973].

Baumgartner [1975a] updated the Erdős-Rado Theorem by proving a canonical version of the Erdős-Rado Theorem: Write $\kappa \to *(\lambda)^n$ if for every f defined on $[\kappa]^n$, there is a finite set $\Delta \subseteq n$ and a subset $B \subseteq \kappa$ of order type λ such that $f(x) = f(y)$ for all subsets $x = \{x_0, x_1, \ldots, x_{n-1}\}_<$ and $y = \{y_0, y_1, \ldots, y_{n-1}\}_<$ of B for which $x_i = y_i$ for all $i \in \Delta$. Baumgartner proved that

$$\big(\exp_r(2^{<\kappa})\big)^+ \to *(\kappa + (r-1))^r,$$

using regressive functions together with ideas from the proof of the Erdős-Rado Theorem [1956].

Shelah [1975] solved Problem 3 of [Erdős and Hajnal, 1971b] by showing that if $n(i)$ $(i < \omega)$ is a sequence of integers such that $\aleph_\omega < \kappa_0 < \kappa_1 < \cdots$, where $\kappa_i = 2^{\aleph_{n(i)}}$, then $\sum\{\kappa_i \colon i < \omega\} \to (\aleph_\omega, \aleph_\omega)^2$. To do so, Shelah proved a Canonization Lemma, a special case of which is given below in a modern formulation.

> *Shelah's Canonization Lemma:* Suppose that $\langle \kappa_\xi : \xi < \mu \rangle$ is an exponentially increasing sequence[306] of infinite cardinals with $\kappa_0 \geq \tau, \mu, \omega$ for a cardinal $\tau \geq 2$. Then for any disjoint union $A = \dot{\bigcup}\{A_\nu : \nu < \mu\}$, any sequence $\langle F_\nu \subseteq \mathcal{P}(A_\nu) : \nu < \mu \rangle$, and any coloring $f : [A]^2 \to \tau$, if $|A_\nu| > 2^{\kappa_\nu}$ and F_ν sustains[307] A_ν over κ_ν for all $\nu < \mu$, then there is a sequence $\langle B_\nu \in F_\nu : \nu < \mu \rangle$ *weakly canonical* with respect to f with $|B_\nu| = {\kappa_\nu}^+$ for all $\nu < \mu$, i.e. $f(u) = f(v)$ whenever $u, v \in \big[\bigcup\{B_\nu : \nu < \mu\}\big]^2$ and $|u \cap B_\nu| = |v \cap B_\nu| \leq 1$ for every $\nu < \mu$.

Given a disjoint union $A = \dot{\bigcup}\{A_\nu : \nu < \mu\}$, a coloring $f : [A]^2 \to 2$ and an exponentially increasing sequence $\langle \kappa_\xi : \xi < \mu \rangle$ with $\kappa_0 \geq \mu, \omega$ and $|A_\nu| > 2^{\kappa_\nu}$ for all $\nu < \mu$, the sequence $\langle F_\nu : \nu < \mu \rangle$ where F_ν is the set of $X \subseteq A_\nu$ of size ${\kappa_\nu}^+$ which are homogeneous for f has the property that each F_ν sustains A_ν, so by Shelah's Canonization Lemma, there are $B_\nu \subseteq A_\nu$ of size ${\kappa_\nu}^+$ for $\nu < \mu$ such that $\langle B_\nu : \nu < \mu \rangle$ is weakly canonical with respect to f. Moreover it is *canonical with*

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[304]Chen Chung Chang (October 13, 1927 –) received his doctorate from the University of California, Berkeley, in 1955 where his advisor was Tarski. The title of his dissertation was *Cardinal and Ordinal Factorization of Relation Types.* He spent a postdoctoral year 1955–1956 at Cornell University where a spring seminar by J. Barkley Rosser led him to work on MV-algebras, moved to University of Southern California in Los Angeles the following academic year where he was an assistant professor, and in July 1958, joined the faculty of the University of California, Los Angeles, retiring in 1990. For more on his early career and working habits, see [Chang, 1998].

[305]Howard Jerome Keisler (December 3, 1936 –) received his doctorate from the University of California, Berkeley, in 1961 where his advisor was Tarski. He is now an emeritus professor at the University of Wisconsin, Madison. He is known for his work in model theory and non-standard analysis, especially for his calculus text book [1976] using non-standard analysis, which is currently available from his homepage.

[306]A sequence $\langle \kappa_\xi : \xi < \mu \rangle$ is *exponentially increasing* if $\xi < \nu < \mu$ implies $2^\xi < 2^\nu$.

[307]We say $G \subseteq \mathcal{P}(C)$ *sustains* C *over* κ if for every subset X of C of cardinality $(2^\kappa)^+$, there is a $Y \in G$ so that $Y \subseteq X$ and $|Y| = \kappa^+$.

respect to f since $f(u) = f(v)$ whenever $u, v \in \bigcup_{\nu<\mu} B_\nu$ and $|u \cap B_\nu| = |v \cap B_\nu|$ for all $\nu < \mu$.

Since μ is a cardinal, there is some subset $M \subseteq \mu$ of full cardinality such that $B' = \bigcup_{\nu\in M} B_\nu$ has the property that every 2-element subset of B' with two elements from a single block among $\{B_\nu : \nu \in M\}$ receive the same color. With this final reduction, the same color equivalence relation has a nice description with respect to the block structure: a 2-element set has the *same block color* if both elements come from the same block, and have the (ξ, ζ)-*block color* if one element is from the ξth block and the other from the ζth block. Clearly this approach can be extended to more colors, and one could ask that the coloring of pairs across blocks be induced by a canonical equivalence relation on μ, if that makes sense. It does not have the simplicity of the Erdős-Rado canonization of equivalence relations on 2-element subsets of ω, where any same color equivalence relation was equal to one on their list when restricted to some infinite subset of ω.

However, the point of the canonization lemma was not simply to show that the partition could be made canonical. Below we mildly paraphrase the theorem Shelah derived from his canonization lemma [1975] to make the names of the cardinals better match our statement of his canonization lemma:

> Shelah's Singular Cardinal Partition Relation: If $\mu \to (\mu)^2_2$, $\mu = \mathrm{cf}\,\lambda$, $\langle 2^\kappa : \kappa < \lambda\rangle$ is not eventually constant, but is eventually $\geq \lambda$, and $\chi = \sum_{\kappa<\lambda} 2^\kappa$, then $\chi \to (\lambda)^2_2$.

Ordinal partition relations

In the late sixties and early seventies, a variety of partition results of the form $\omega^n \to (\omega^m, k)^2$ for finite values of n, m, k were found by Galvin, Hajnal,[308] Milner, and Labib Haddad and Gabriel Sabbagh [1969b], [1969a], [1969c]. The article by Milner [1971] gave a good exposition of this work including its history, and even included a proof of $\omega^5 \to (\omega^3, 6)^2$, which was then unknown. While open questions remain, Milner's student, Eva Nosal settled more cases [1974], [1979]]:

> *Theorem:*
>
> 1. If $1 \leq \ell < \omega$, then $\omega^{2+\ell} \to (\omega^3, 2^\ell)^2$ and $\omega^{2+\ell} \not\to (\omega^3, 2^\ell + 1)^2$.
> 2. If $1 \leq \ell < \omega$ and $4 \leq r < \omega$, then $\omega^{1+r\cdot\ell} \to (\omega^{1+r}, 2^\ell)^2$ and $\omega^{r+r\cdot\ell} \not\to (\omega^{1+r}, 2^\ell + 1)^2$.

Galvin exploited Specker's counter-example $\omega^3 \not\to (\omega^3, 3)^2$ to the full. For ordinals α and β, α *can be pinned to* β, in symbols, $\alpha \to \beta$, if there is a mapping $\pi : \alpha \to \beta$ such that for every $X \subseteq \alpha$ of order type α, its image, $f[X]$, has order

[308] Erdős and Hajnal, in their problem paper [1971b, 21–22], attribute to Galvin and Hajnal independently a uniformization result which implies that for all $3 \leq k, n < \omega$, there is a least $f(k, n)$ such that $\omega^n \not\to (\omega^k, f(k, n))^2$.

type β.[309] Using pinning onto ω^3 in a paper with Jean Larson, he [1975] proved that the ordinal $\alpha = \omega^\beta$ satisfies the relation $\alpha \not\to (\alpha, 3)^2$ for every decomposable countable ordinal $\beta \geq 3$.

The first additively indecomposable countable ordinal not covered by the Specker and Galvin results is ω^ω. Chang proved [1972] that the ordinal ω^ω satisfies the partition relation $\omega^\omega \to (\omega^\omega, 3)^2$. This was the first major breakthrough in partition relations for countable ordinals since Specker's initial positive result for ω^2 and negative result for ω^3 in the mid-1950's. Chang's result was reported in the Erdős-Hajnal problem paper [1971b, 22] in a note added in proof and dated May 1970. Chang's 58-page paper [1972] in which this result appeared included a note that Milner had generalized the proof to show $\omega^\omega \to (\omega^\omega, m)^2$ for all $m < \omega$ and, added in proof, that Larson had a much shorter proof.[310] The $250 dollars that Erdős awarded for this result was the highest he had awarded to that time; Chang used the money to purchase a motorcycle.

Erdős and Hajnal [1971a] stimulated interest in partition relations for uncountable ordinals with a paper in which they investigated partition relations for small ordinal powers of uncountable cardinals. They asked in Problem 13 of their problem paper [1971b] whether the following partition relations hold in the presence of GCH:

$$\omega_1{}^2 \to (\omega_1{}^2, 3)^2 \text{ and } \omega_1 \cdot \omega \to (\omega_1 \cdot \omega, 3)^2.$$

Hajnal [1971] published elegant proofs from the Continuum Hypothesis that the answer is no by proving that

$$\omega_1{}^2 \not\to (\omega_1{}^2, 3)^2 \text{ and } \omega_1 \cdot \omega \not\to (\omega_1 \cdot \omega, 3)^2.$$

Larson[311] [1998] showed the partition relation is consistent with the failure of CH by proving it from the assumption that the *dominating number*[312] $\mathfrak{d}$ is ω_1. More generally, Hajnal showed that if $\kappa = \lambda^+$ is the successor of a regular cardinal and $2^\lambda = \lambda^+ = \kappa$, then $\kappa^2 \not\to (\kappa^2, 3)^2$ and $\kappa \cdot \lambda \not\to (\kappa \cdot \lambda, 3)^2$.

Baumgartner [1975b] then extended the first result to the successor κ of a singular cardinal λ for which $2^\lambda = \lambda^+ = \kappa$ by proving $\kappa^2 \not\to (\kappa^2, 3)^2$. He also showed that for a strong limit cardinal κ, the partition relation $\kappa^2 \to (\kappa^2, m)^2$ for finite m holds if and only if $(\mathrm{cf}\,\kappa)^2 \to ((\mathrm{cf}\,\kappa)^2, m)^2$ holds.

[309] Rotman [1970] introduced this notation for the notion used by Specker [1957] to transfer his negative partition result for the resource ω^3 to ω^n for all finite $n \geq 3$. Rotman proved that no countable ordinal pinned to a larger countable ordinal and conjectured that no ordinal could be pinned to a larger ordinal. In a surprising result on the structure of ordinals, Carlson [1984] showed that in the presence of the Continuum Hypothesis, $\omega_1^{\omega+2}$ pins to cofinally many ordinals in ω_2.

[310] See Larson's thesis [1972], [1973], and, for a particularly nice exposition, Neil Williams's book [1977, Chapter 7, Section 3].

[311] Larson remembers hearing Hajnal's beautiful proof from Galvin at UCLA in the early 1970's.

[312] A family $Y \subseteq {}^\omega\omega$ is *dominating* if for every function $f \in {}^\omega\omega$ there is a function $g \in Y$ such that $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is the minimum size of a dominating subfamily of ${}^\omega\omega$. For more on this cardinal invariant, see §5 of the chapter by Steprāns.

Kunen (cf. [Erdős and Hajnal, 1974, 274]) proved that if κ is real-valued measurable, then $\kappa^2 \to (\kappa^2, 3)^2$.

Baumgartner and Larson (cf. [Larson, 1975]) proved that if κ is weakly compact and $\alpha < \kappa$ satisfies $\alpha \to (\alpha, m)^2$, then $\kappa \cdot \alpha \to (\kappa \cdot \alpha, m)^2$.

Square bracket partition relations

While square bracket partition relations were introduced in the 1960's,[313] we have postponed discussion of this relation to the 1970's to consolidate the results of the two decades.

A nice positive result by Erdős, Hajnal and Rado is Theorem 22 (iii) [1965, 149] of which we state a special case that introduces a useful variant of the basic notation: For κ a singular strong limit cardinal of cofinality ω and $2 < \gamma < \omega$, $\kappa \to [\kappa]^r_{\gamma,2}$, i.e. for any coloring of the pairs of κ with γ colors, where $2 < \gamma < \omega$, there is a subset of size κ so that all but at most two colors are omitted.

Recall that we introduced, in §6.7, the following GCH negative result by Erdős, Hajnal and Rado Theorem 17 [1965, 145]:

$$\aleph_{\alpha+1} \nrightarrow [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}}.$$

Hajnal [1997, 365] called Theorem 17 a test result and noted that they later discovered that this "was known to Sierpiński in a different context." Erdős, Hajnal and Rado also proved stronger Theorems 17a, 18 [1965, 145–146].

In the proceedings of the 1967 UCLA set theory conference, Solovay [1971] proved a result with a square bracket consequence: every stationary subset of a regular cardinal κ can be split into κ many pairwise disjoint stationary sets. Thus if $\kappa > \aleph_0$ is regular then a witness that $\kappa \nrightarrow [\kappa]^\omega_\kappa$ can be obtained by taking a partition $S = \bigcup\{S_\alpha \mid \alpha < \kappa\}$ of a stationary set S into disjoint stationary sets, and defining the coloring $f : [\kappa]^\omega \to \kappa$ by letting $f(s)$ be the unique α such that $\sup(s) \in S_\alpha$.[314]

Erdős and Hajnal asked, in Problem 15 of their problem paper [1971b], if one can prove without the Continuum Hypothesis that $2^{\aleph_0} \to [\aleph_1]^2_3$. Blass [1972][315] and Laver[316] independently proved, without using CH, that $\aleph_1 \nrightarrow [\aleph_1]^2_3$. The Blass counter-example, in essence, superposed three linear orders of size $\aleph_1$: a well-ordering of length ω_1, the lexicographic ordering of the elements of an Aronszajn tree, $(T, <_{\text{lex}})$, and the usual ordering $<$ of the reals on a subset A of size $\aleph_1$ chosen such that $(A, <)$ is order-isomorphic to $(A, >)$ and such that every subset $Y \subseteq A$ has a countable subset $X \subseteq Y$ for which $X \cap [a, b) \neq \emptyset$ for all $a < b$ from Y. Blass noted that subsequently Galvin and Shelah [1973] improved the result to

[313] See §6.7 for the definition of the basic square bracket partition relation.

[314] This idea is used in Theorem 2 of the paper [1976, 370] by Galvin and Prikry who attribute it to Solovay and list it as unpublished.

[315] The paper was received April 25, 1971, and in revised form, February 25, 1972.

[316] Laver proved this partition relation in 1969 but did not publish it; it is attributed to him by Dow in his review [1989, 637] of several articles of Todorcevic.

$\aleph_1 \nrightarrow [\aleph_1]^2_4$, using the same partition. He also generalized the result to show that if κ is regular and accessible and there is a κ-Aronszajn tree, then $\kappa \nrightarrow [\kappa]^2_3$.

Recall that for a regular cardinal κ, a *κ-Suslin tree* is a tree of height κ all of whose chains and antichains have cardinality $< \kappa$. Jensen [1972, Lemma 6.6, 296] reported that Soare (corrected to Shore in the erratum) proved that if κ is a regular uncountable cardinal and there is a κ-Suslin tree, then $\kappa \nrightarrow (\kappa)^2_{r,\gamma}$ for $r < \gamma < \kappa$, and attributed the case $r = 2$ and $\gamma = 3$ to Martin. Erdős and Hajnal [1974, 274] reported that Richard Shore [1974] proved that if κ is regular and there is a κ-Suslin tree, then $\kappa \nrightarrow [\kappa]^2_\kappa$.[317] Since Jensen showed that if $V = L$, then for regular uncountable cardinals κ, there is a κ-Suslin tree if and only if κ is not weakly compact, Shore's result gives a characterization, under the assumption of $V = L$, of the regular uncountable cardinals for which $\kappa \nrightarrow (\kappa)^2_{r,\gamma}$ holds for $r < \gamma < \kappa$. Erdős and Hajnal [1974, 274] also reported that Shore proved that $\omega_2 \nrightarrow [\omega_2]^3_{\omega_1}$ holds in L.

Baumgartner [1975b] proved that for regular κ, if $\kappa^2 \to (\kappa^2, 3)^2$, then there is no κ-Suslin tree.

Galvin and Prikry [1976] gave a short proof of the equivalence of a cardinal κ being Jónsson and the square bracket partition relation $\kappa \to [\kappa]^{<\omega}_\kappa$, updating the result of Erdős and Hajnal [1966].

Galvin [1971] (see also [Erdős and Hajnal, 1974, 275]) proved that if $\varphi \geq \omega^\omega$ is an order type which embeds neither ω_1 nor the order type η of the rationals, then

$$\varphi \nrightarrow [\omega, \omega^2, \omega^2, \omega^3, \omega^3, \dots]^2_\omega.$$

All the order types in question are scattered, so Galvin used Hausdorff's result that the scattered order types are the closure of $\{0, 1\}$ under well-ordered and converse well-ordered sums. Galvin came up with an especially nice representation in terms of sets of finite sequences of non-negative integers and negative ordinals under the lexicographic order, with an additional property of being *good*, which will not be defined here. Then he used the coloring that is now known as the *oscillation map*: it takes a pair of sequences $\vec{a} = \langle a_0, a_1, \dots , a_{k-1}\rangle$, $\vec{b} = \langle b_0, b_1, \dots , b_{\ell-1}\rangle\}$, to the number of alternations between blocks of elements from $\{a_i : i < k\} \setminus \{b_j : j < \ell\}$ and blocks of elements from $\{b_j : j < \ell\} \setminus \{a_i : i < k\}$ as they occur in the list in increasing order of the union of these two sets. He proved that for finite positive n and k with $1 \leq k \leq 2n - 1$, any good subset of type ω^n contains a pair with k alternations. As reported in [Erdős and Hajnal, 1974, 275], the above partition relation negatively solved Problem 18 [Erdős and Hajnal, 1971b, 26], which asked if $\omega^\omega \to [\omega^\omega]^2_{\aleph_0}$ holds. Given that both η and ω_1 embed all countable ordinals, Galvin's result displayed above is optimal.

The discussion of square bracket relations is continued in the next subsection where we turn our attention to structural partitions, where the underlying set is

[317] In the book [Erdős *et al.*, 1984, 318], Jensen [1972] and Shore [1974] are both given credit for $\kappa \nrightarrow [\kappa]^2_\kappa$. Shore [1974] also credits Martin for the proof that for regular κ, $\kappa \to [\kappa]^2_3$ implies Suslin's Hypothesis, and in a footnote remarks that after Martin's discovery, both Jensen and Silver independently discovered the implication as well as the implication due to Shore that $\kappa \to [\kappa]^2_\kappa$ implies Suslin's Hypothesis.

either a tree or the set of rational numbers or real numbers, which have natural tree representations.

7.4 Ramsey theory for trees

While it was known in the 1960's that every real in a model obtained by adding a single Sacks real either contains or is disjoint from a ground model real, Baumgartner (cf. [Laver, 1984]) used HL_d for finite d (see §6.1) to show that the same is true when d side-by-side Sacks reals are added. He asked whether a theorem like HL_d is true for a product of infinitely many trees, because he was able to get the forcing consequence from such a statement.

In the late 1970's, Harrington (unpublished, see [Laver, 1984, 386] for the timing) gave an elegant proof of the Laver HL_d Core using forcing. Todorcevic [1995, 54–55] used Harrington's proof in showing that the appropriate forcing statement is equivalent to the Laver HL_d Core.

In his thesis [1979a], Denis Devlin generalized Galvin's result $\eta \to [\eta]^2_{<\omega,2}$ to the result $\eta \to [\eta]^r_{<\omega,t_r}$ and $\eta \nrightarrow [\eta]^r_{t_r}$, where $\langle t_n \mid n \in \omega\rangle$ is the sequence of tangent numbers, $t_n = \tan^{(2n-1)}(0)$, which starts $t_1 = 1$, $t_2 = 2$, $t_3 = 16$, $t_4 = 272$.[318]

Keith Milliken[319] [1975b], [1979] developed a very useful tool in partition theory for countable structures in his Ramsey theorem for strongly embedded subtrees.

Suppose that $(T, <_T)$ is a rooted, finitely branching tree whose levels have order type α and $S \subseteq T$ is such that $(S, <_S)$ is a rooted tree where $<_S$ is the restriction of $<_T$ to S. Let β be the order type of the levels of S. Then S is *strongly embedded in* T if there is an order-preserving embedding $\ell : \beta \to \alpha$ such that (1) if $s \in S$ is on level γ in S, then s is on level $\ell(\gamma)$ in T; and (2) if u is an immediate successor of $s \in S$ when s is regarded as a node of T, then there is v, an immediate successor in S of s, such that $u = v$ or $u <_T v$. For (uniformly) d-branching trees of limit ordinal height α, this is equivalent to the existence of a level preserving map from the d-branching tree all of whose branches have order type γ.

> *Milliken's Ramsey Theorem for Trees:* If $(T, <_T)$ is a finitely branching tree of height ω and, for some $1 \leq k < \omega$, the height k strongly embedded subtrees of T are partitioned into finitely many pieces, then there is a strongly embedded tree $S \subseteq T$ all of whose strongly embedded height k subtrees are in the same cell of the partition.

Milliken's proof used a pigeonhole principle which he described as a reformulation of the Halpern-Läuchli Theorem due to Laver in about 1969. He included a

[318] A sketch by Todorcevic of the proof of Devlin's result with some details for $n = 2$, $n = 3$ appeared in [1995, 45-47]. Vojkan Vuksanovič [2002] published a beautiful proof of it using Milliken's Ramsey Theorem for Trees (discussed later in the section). Devlin's result was generalized by Džamonja, Larson and Mitchell [Džamonja *et al.*, 2009] to κ-dense linear orders for suitably chosen uncountable cardinals κ. For $n > 2$, the critical numbers r_n for these κ-dense linear orders are larger than those for η.

[319] Keith Milliken received his doctorate from the University of California, Los Angeles, in 1975 where he was advised by Chang.

proof of it from the Halpern-Läuchli Theorem.

> *Milliken Pigeonhole Principle:* Suppose d and r are positive integers, and $\langle T_i : i \in d\rangle$ is a sequence of finitely branching trees each branch of which has order type ω. If
>
> $$\bigcup_{n\in\omega}\left(\prod_{i\in d} T_i(n)\right) = \bigcup_{j<r} C_j,$$
>
> then there must be a $k \in r$ and a sequence $\langle S_i : i \in d\rangle$ of subtrees with S_i strongly embedded in T_i for $i < d$ for which
>
> $$\bigcup_{n\in\omega}\left(\prod_{i\in d} S_i(n)\right) \subseteq C_k.$$

The Laver HL_d Theorem (see §6) is about perfect subtrees rather than strongly embedded trees. Both versions follow from the Laver HL_d Core, and, if starting from perfect trees, one can obtain perfect strongly embedded trees as well.

Blass [1981] used the Halpern-Läuchli Theorem to verify a conjecture of Galvin [1968b] by proving the following theorem (due to Galvin for $n \leq 3$):[320]

> *Blass Perfect Set Partition Theorem:* If P is a perfect subset of Cantor space and $[P]^n$ is partitioned into finitely many open pieces, then there is a perfect set $Q \subseteq P$ such that $[Q]^n$ intersects at most $(n-1)!$ pieces.

Blass noted that the result generalizes to partitions of perfect subsets of the real line. He fit his result into a wider context by pointing out that Mycielski [1964], [1967] had shown that any meager set or any set of measure zero in $[\mathbb{R}]^n$ is disjoint from $[P]^n$ for some perfect set $P \subseteq \mathbb{R}$ and derives the corollary that if $[\mathbb{R}]^n$ is partitioned into finitely many pieces that all have the Baire property or are all measurable, then there is a perfect set $Q \subseteq \mathbb{R}$ such that $[Q]^n$ meets at most $(n-1)!$ pieces. Blass noted that some limitation on the partitions is necessary by the Galvin and Shelah result [1973] that there is a partition of the pairs of real numbers into ω many pieces such that the set of pairs of every subset isomorphic to $\mathbb{R}$ meets all the pieces (in the partition calculus notation, $\lambda \nrightarrow [\lambda]^2_\omega$, where λ is the order type of the set of real numbers).

7.5 Hindman's Finite Sums Theorem

A Ramsey theorem for finite sums of a finite set was independently proved in the late 1960's by Jon Folkman (unpublished, see [Graham and Rothschild, 1971]), Rado [1970], and Jon Sanders [1968] which I will call the FRS Finite Sums Theorem, but in the textbook *Ramsey Theory* [1990] whose statement of the theorem I paraphrase, it is called Folkman's Theorem.

[320] Blass presented his perfect set theorem at a meeting on combinatorial set theory in Aachen in June 1976.

> *FRS Finite Sums Theorem:* If $\mathbb{N}$ is finitely colored, then there are arbitrarily large finite sets S such that the set of all non-repeating sums from S is monochromatic.[321]

This theorem is a corollary to Rado's Theorem for Linear Systems with Positive Integer Coefficients discussed in §3.2. It is also a corollary of the Graham-Rothschild[322] Parameter Sets Theorem [1971] for which I give a simplified version rephrased from [2006].

> *Graham-Rothschild Parameter Sets Theorem:* Suppose A is a non-empty finite alphabet. For all $m < k < \omega$ and positive r, there is a positive n such that whenever $[A]\binom{n}{m}$ is colored with r colors, there is a word $w \in [A]\binom{n}{k}$ such that $\{w\langle u\rangle : u \in [A]\binom{k}{m}\}$ is monochromatic.

Here, for $k = 0$, $[A]\binom{n}{k}$ is simply ${}^nA$ and, for $k > 0$, $[A]\binom{n}{k}$ is the set of all words w of length n over the enlarged alphabet $A \cup \{v_0, v_1, \ldots, v_{k-1}\}$ such that each variable v_i for $i < k$ occurs in w and the first occurrence of v_i precedes the first occurrence of v_{i+1} if $i + 1 < k$. Given a word w in $[A]\binom{n}{k}$ and a word u of length k on the alphabet $A \cup \{v_0, \ldots, v_k\}$, let $w\langle u\rangle$ be the word of length n such that for all $i < n$, $w\langle u\rangle(i) = w(i)$ if $w(i) \in A$, and $w\langle u\rangle(i) = u(j)$ if $w(i) = v_j$. Prömel and Voight [1990] in their discussion of parameter sets point out that the formal calculus of words, which we used in the above presentation, was introduced by Klaus Leeb [1973].

Graham and Rothschild [1971] gave many corollaries of their theorem including Schur's Theorem, Rado's Theorem for Linear Systems with Positive Integer Coefficients, Ramsey's Theorem, van der Waerden's Theorem, the Hales-Jewett Theorem, the FRS Finite Sums Theorem, and various theorems for affine and vector spaces. Graham and Rothschild were motivated by a problem of Rota about vector spaces and in a paper with Leeb [1972] they solve Rota's problem. In a survey of developments in Ramsey theory, Graham and Rothschild [1974] attribute to Leeb the notion of a category having the Ramsey property, extending the range of structures to which Ramsey theory is applied.

This brief account was informed by Hindman's account [2006] and the paper of Prömel and Voigt [1990]. For extensions and other perspectives, Todorcevic's book [2010] and the paper [2006] by Carlson, Hindman and Strauss are recommended.

At the end of their paper [1971, 290–291], Graham and Rothschild asked (see §9 Concluding Remark (ii)) if the various infinite versions of certain of their corollaries

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[321] For $S = \{1, 2, 5\}$, the non-repeating sums are $1, 2, 5, 1+2, 1+5, 2+5, 1+2+5$.

[322] Ronald Lewis Graham (October 31, 1935 –) received his 1962 doctorate from the University of California, Berkeley, where his advisor was Derrick Lehmer. He had nearly 30 joint papers with Erdős. In addition to being a mathematician of broad interests, he is juggler and a person who learned to flip a quarter so that it comes up heads on demand. There is a comprehensive archive of his papers at http://www.math.ucsd.edu/~ronspubs/ which also includes copies of some articles about him, including [Kolata, 1981] which was my source for his birth date.

Bruce Lee Rothschild (1941 –) received his doctorate from Yale University in 1967 where his advisor was Öystein Ore. He is a professor at the University of California, Los Angeles. To learn more about his mathematics, see [Hindman, 2006].

are valid, and specifically asked if the positive integers are 2-colored, does there always exist an infinite subset A such that the sums of all (non-empty) finite subsets of A are the same color? Hindman proved the infinite version and the remainder of this section is focused on Hindman's solution.

> *Hindman's Finite Sums Theorem:* For any positive integer k and any partition of the natural numbers into k classes, $\mathbb{N} = \bigcup_{i<k} A_i$, there are $i < k$ and an infinite subset $X \subseteq N$ such that the sum of every finite set of distinct members of X is in A_i.

Hindman gave a delightful account in [2005] of how he came to prove his theorem.[323] He reported that around 1971, Galvin asked Erdős whether there were any (downward) almost translation invariant ultrafilters on $\mathbb{N}$. Erdős asked the question of W. Wistar Comfort, who had been Hindman's dissertation advisor,[324] and Comfort passed the question on to Hindman. Along the way, (downward) almost translation invariant ultrafilter became (upward) almost translation invariant, a simpler problem, which Hindman tackled when he heard that Erdős would be visiting the Claremont Colleges at a time when Hindman was teaching at Cal State Long Beach. Hindman learned from Erdős that the question originated with Galvin, so Hindman telephoned Galvin to share the proof. Galvin congratulated him and let him know that he had answered a question different from the one originally asked. The point of the original question was that a positive answer would provide a simple proof of the truth of a conjecture of Graham and Rothschild which is now Hindman's Finite Sums Theorem. Hindman then proved that under GCH, the Graham-Rothschild conjecture implied the existence of a (downward) almost translation invariant ultrafilter, and "with a great deal more effort" [2005, 322] he established the validity of the conjecture by "an elementary but very complicated proof" [2005, 322] (see [Hindman, 1974] for Hindman's original proof of the finite sums theorem). Hindman continues [2005, 323]:

> At ... the end of 1972 ... Galvin's (downward) almost translation invariant ultrafilters were figments of the continuum hypothesis. Galvin wanted to know if they really existed, that is, whether their existence could be proved in ZFC. In 1975 he [Galvin] encountered Steven Glazer and asked him whether such ultrafilters could be shown to exist. Glazer immediately answered "yes".

Glazer went on to explain that (1) any compact right topological semi-group has idempotents; (2) $\beta\mathbb{N}$[325] has a natural operation $+$ extending addition on $\mathbb{N}$ which makes $(\beta\mathbb{N}, +)$ a right topological group; and (3) for $p, q \in \beta\mathbb{N}$ and $A \subseteq \mathbb{N}$, $A \in p + q$ if and only if $\{x \in \mathbb{N} \mid -x + A \in q\} \in p$. Thus an idempotent in $\beta\mathbb{N}$ is

[323] The account is part of a printed lecture presented at the international meeting of the Japanese Association of Mathematical Science on the receipt of its 2003 International Prize.

[324] Comfort reminisced about Hindman's graduate student days and their times together over the years in [Comfort, 2005].

[325] $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$, identifiable with the space of ultrafilters on $\mathbb{N}$.

a (downward) almost translation invariant ultrafilter. Galvin told Hindman this proof in fall 1975, which Hindman identified in [2005] as the time he fell in love with the algebraic structure of the Stone-Čech compactification of a discrete semigroup, whose structure and applications to Ramsey theory have been the focus of the remainder of his mathematical career.

Hindman's Finite Sums Theorem received multiple proofs. In addition to the original proof and the Galvin-Glazer proof (see [Comfort, 1977]) described above, Baumgartner [1974] published a short proof of Hindman's Finite Sums Theorem in the same journal and the same year as Hindman's original proof, and proved an equivalent version in terms of finite sets and their unions: if $f : [\mathbb{N}]^{<\omega} \to k$ is partition into finitely many parts, there is an $i < k$ and an infinite family S of disjoint non-empty finite subsets so that all finite unions of members of S receive color i. This theorem is sometimes call the *Finite Unions Theorem.*

7.6 Infinitary partition relations

We start this subsection with results on infinitary partition relations by Kleinberg and Kunen and those partition relations growing out of the work of Nash-Williams, Galvin-Prikry, Silver and Mathias.

Eugene Kleinberg[326] studied partition relations for infinite exponents, particularly those of the form $\kappa \to (\kappa)^\alpha$ for κ a cardinal and α an ordinal in his thesis (see [1970] based on his thesis). By a result of Kunen quoted in Kleinberg's paper in a note apparently added in proof, the Axiom of Determinacy (AD) implies that $\aleph_1 \to (\aleph_1)^\alpha$ for all $\alpha < \aleph_1$. Moreover Kleinberg proved that if λ is a regular cardinal smaller than κ and $\kappa \to (\kappa)^{\lambda+\lambda}$, then κ has a normal ultrafilter concentrated on ordinals of cofinality λ. The ultrafilter in this case is generated by the subsets of κ that are unbounded and closed under sups of increasing λ-sequences. Kleinberg went on to investigate the conditions under which infinite-exponent partitions admit homogeneous sets of measure one for these measures.

Kleinberg wrote a monograph [1977] on infinitary combinatorics in the theory ZF + AD + DC with an emphasis on infinite exponent partition relations, the measures these generate, and ultraproducts by such measures. For more on the story of infinitary partitions see the chapter by Paul Larson.

Erik Ellentuck [1974] gave a topological proof of Silver's result that analytic sets are completely Ramsey, a result that he incorrectly attributed to both Mathias and Silver,[327] by characterizing completely Ramsey sets as those having the Baire property with respect to a certain topology on $[\omega]^\omega$, now called the *Ellentuck topology*[328] or the *exponential topology*, and is also known as the *Vietoris topology*

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[326] Kleinberg was a student of Martin and graduated from Rockefeller University in 1969. He was on the faculty of Massachusetts Institute of Technology, and then of the State University of New York, Buffalo, where he is now emeritus.

[327] Recall Mathias [1977] gave new proofs and generalized Silver's work.

[328] A neighborhood (s, A) in the Ellentuck topology for finite part $s \subseteq \omega$ and infinite part $A \subseteq \omega$ with $\max s < \min A$ is the collection of infinite subsets $X \subseteq \omega$ with the property that $s \subseteq X \subseteq s \cup A$.

(cf. Todorcevic [2010]). Ellentuck characterized the Silver and Mathias proofs as using "relatively deep metamathematical notions involving forcing," while he characterized his own proof "using nothing more than the methods of Galvin-Prikry." The paper includes a postscript in which the observation of Galvin that the proof actually shows that all sets in the classical Lusin hierarchy are completely Ramsey. Independently, Louveau [1974] gave a similar topological proof of the theorems of Silver and Mathias; Louveau used a Ramsey ultrafilter to show analytic sets are Ramsey.

Milliken [1975a] proved a theorem that generalized both Silver's analytic sets are completely Ramsey theorem and Hindman's Finite Sums Theorem. He also gave a short proof of the theorem of Mathias that if $\mathcal{U}$ is a selective ultrafilter on ω and $R \subseteq [\omega]^\omega$, then R is completely $\mathcal{U}$-Ramsey. Taylor [1975], [1976] independently generalized Hindman's Finite Sums Theorem in the Finite Unions Theorem form and found canonical equivalence relations for the "same color" equivalence relation. Here is a modern statement of what has come to be known as the Milliken-Taylor Theorem. For finite sets $A, B \subseteq \omega$, let us write $A \ll B$ if and only $\max A < \min B$.

> *Milliken-Taylor Theorem:* For any finite partition $[\mathbb{N}]^r = C_0 \cup \cdots \cup C_{\ell-1}$, there is an infinite sequence $\langle a_n : n < \omega \rangle$ of natural numbers such that for some $i < \ell$,
>
> $$\left\{ \left\{ \sum_{i \in I_0} a_t, \ldots, \sum_{i \in I_{r-1}} a_i \right\} : I_0 \ll \cdots \ll I_{r-1} \in [\mathbb{N}]^{<\omega} \right\} \subseteq C_i.$$

Jacques Stern [1978] proved an extension of Silver's result to colorings of the infinite chains of a complete binary tree and applied his result to a problem of Brunel and Sucheston for real Banach spaces.

Hillel Furstenberg and Benjamin Weiss [1978] applied dynamical theorems to product spaces of countably many copies of a finite set to give new proofs of a series of combinatorial results including Rado's generalization of van der Waerden's Theorem and Hindman's Finite Sums Theorem.

Taylor [1978] investigated partitions of the pairs from the Cantor space ${}^\omega 2$. He ascribed to Galvin, Mycielski and Silver the statement that every partition of $[{}^\omega 2]^2$ into two pieces having the property of Baire has a homogeneous perfect set. He had learned about it from Baumgartner who proved it independently from Galvin, Mycielski and Silver, and Taylor noted that Burgess rediscovered it still later. Taylor proved that if $f : [{}^\omega 2]^2 \to \omega$ is a coloring for which $f^{-1}\{i\}$ has the property of Baire for all $i < \omega$, then there is a perfect set $P \subseteq {}^\omega 2$ such that either f is constant on $[P]^2$ or else f induces the same equivalence relation on $[P]^2$ as the discrepancy partition $\delta : [{}^\omega 2]^2 \to \omega$ defined by $\delta(\vec{x}, \vec{y}) = \min\{n : \vec{x}(n) \neq \vec{y}(n)\}$.

7.7 Structure of trees

The structure of trees of height ω_1 with countable levels and their embeddings continued to be an area of intense interest in the 1970's.

Embedding trees in the reals

Baumgartner, Jerome Malitz, and William Reinhardt[329] [1970] carried out an iterated c.c.c. forcing with finite conditions to prove the consistency of the statement that every Aronszajn tree and every tree-like order of size $\aleph_1$ is embeddable in the rationals, i.e. is $\mathbb{Q}$-embeddable. Baumgartner, Malitz and Reinhardt had shown in particular that MA + $\neg$CH implies that all Aronszajn trees are special.[330]

While they cited the paper [1937a] in which Kurepa constructed a special Aronszajn tree as an example of a paper on embeddings of trees into linear orders, they seemed unaware that he had shown the equivalence between being special and having an embedding into the rationals (see §3.4 for details). Most of the work for their consistency result is in Theorem 3 in which they show that their forcing satisfies the countable chain condition. In Theorem 4 they show under the assumption of MA + $\neg$CH that every tree-like partial order[331] T of cardinality less than the continuum in which every chain is countable is $\mathbb{Q}$-embeddable. The paper included announcements of a result by Laver of the existence of a tree of power the continuum which is $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable such that each uncountable subset contains an uncountable antichain, and a result by Galvin of the existence of such a tree which is not $\mathbb{R}$-embeddable (Baumgartner's thesis is given as a reference). Baumgartner, Malitz and Reinhardt pointed out that these examples show the independence of their result over ZFC, since in ZFC + $\neg$CH, it is not the case that every tree-like partial order without an uncountable chain is $\mathbb{Q}$-embeddable. Laver's example showed that one cannot hope to prove the consistency of the parallel embedding result for tree-like orders of size the continuum, and the Sierpiński poset (see §3.4) showed that one cannot hope to extend the embedding consistency result to arbitrary partial orders. The paper ended with a trio of open problems [1970, 1753]:

1. Is it consistent with ZFC to assume that for any partial order P, if card $P = \aleph_1$ and every uncountable subset of P contains an uncountable antichain, the P can be embedded in the rationals?

2. (Galvin, unpublished) Is it consistent with ZFC to assume that for any partial order P, if card $P = \aleph_2$ and if every $P' \subseteq P$ with card $P' \leq \aleph_1$ can be embedded in the rationals, then P can be embedded in the rationals?

3. (Baumgartner) Is it consistent with ZFC + $2^{\aleph_0} = \aleph_1$ to assume that every Aronszajn tree is embeddable in the rationals?

[329] Baumgartner, Malitz and Reinhardt were all students of Vaught, and Malitz and Reinhardt both had careers at the University of Colorado, Boulder.

[330] M. E. Rudin [1975b, 513] credited Kunen with having shown that MA + $\neg$CH implies that all Aronszajn trees are embeddable in the rationals, and the same is true for tree-like partial orders. This result is credited to Baumgartner in [Devlin, 1972]. In [Devlin and Shelah, 1978, p. 289], both Baumgartner and Kunen are given credit for proving that under MA + $\neg$CH, every Aronszajn tree is embeddable in the rationals.

[331] A partial order T is *tree-like* if the set of predecessors of each element is linearly ordered.

For future reference, we note that one can easily use their forcing to arrange for every Aronszajn tree to have a continuous strictly increasing embedding into the rationals.

Baumgartner [1970] announced two structural theorems (paraphrased below). The first theorem is that under MA + $\neg$CH, for every pair of full well-pruned Hausdorff trees $(S, \leq_S)$ and $(T, \leq_T)$ of power $\aleph_1$, there are countable sequences $\langle S_n \mid n \in \omega\rangle$ and $\langle T_n \mid n \in \omega\rangle$ of cofinal subtrees whose unions give back the original trees and which are term-by-term isomorphic. The second theorem is that if $V = L[A]$ for some $A \subseteq \omega_1$, then there are non-Suslin Aronszajn trees (a) which are $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable and (b) which are not $\mathbb{R}$-embeddable.[332]

Devlin, then a student at University of Bristol, heard about Baumgartner's results from Laver, who visited Bristol 1969-1971. After stating Theorem 1 with a review of some known results, in Theorem 2 [1972, 256], Devlin characterized what he called ω_1-trees (but we call full well-pruned Hausdorff ω_1-trees) that have strictly increasing embeddings into the rationals as the Aronszajn trees that are the union of countably many antichains, i.e. special Aronszajn trees. This result is a reiteration of one from [Kurepa, 1940a]. In the same theorem, he proved that if a full well-pruned Hausdorff ω_1-tree is $\mathbb{R}$-embeddable, then every uncountable subset contains an uncountable antichain. The proof used the fact that the union of the successor levels of any $\mathbb{R}$-embeddable tree is $\mathbb{Q}$-embeddable, which was also observed by Kurepa [1940a]. Devlin generalized both Baumgartner's result on the consistent existence of $\mathbb{R}$-embeddable non-$\mathbb{Q}$-embeddable Aronszajn trees and the Gaifman-Specker [1964] result that under CH there are $2^{\aleph_1}$ many non-isomorphic Aronszajn trees by proving that under $\diamondsuit$ there are $2^{\aleph_1}$ non-isomorphic $\mathbb{R}$-embeddable non-$\mathbb{Q}$-embeddable Aronszajn trees. Devlin (see pages 256–257) described his own approach as an adaptation of Baumgartner's argument "at the cost of some messy combinatorics" under the inspiration of arguments from Jech's paper [1972].

Jech [1972] investigated the automorphism group of ω_1-trees and used $\diamondsuit$ to proved the existence of $2^{\aleph_1}$ pairwise non-isomorphic Suslin trees. He also proved that it was consistent to assume the existence of $2^{\aleph_1}$ many pairwise non-isomorphic Kurepa trees.

Jensen on Suslin trees

Devlin and Håvard Johnsbråten[333] wrote on Jensen's work on Suslin's Problem in their [1974]. In the introduction, the authors noted that the first five chapters were based on a set of lecture notes written by Jensen in Kiel in 1969. The remaining five chapters were based on a set of notes written by Devlin in 1972, which were based on a set of notes written by Jensen some years earlier.

[332] Kunen [1980, 90] cited this abstract in his remark that $\diamondsuit$ implies that there is an Aronszajn subtree of the tree of one-to-one functions from countable ordinals into ω which is not special.

[333] Johnsbråten was a student of Fensted and studied with Jensen but did not complete his doctoral degree. He is now on the faculty of Telemark University College, in Norway.

The first half details Suslin's Problem, the connection with trees, and Jensen's $\diamondsuit$ and gives the construction of a Suslin tree first in L using the canonical well-ordering and then from $\diamondsuit$. Then $\diamondsuit$ is shown consistent both via forcing and via holding in the constructible universe. Various consistency results are proved assuming the consistency of ZF, including ZFC + GCH + $\neg$SH and ZFC + $\neg$CH + $\neg$SH.

In Chapters 4 and 5, the possible structure of Suslin trees was explored. Under $\diamondsuit$ there is a homogeneous Suslin tree, there is one with exactly ω_1 automorphisms, there is a non-reversible (i.e. not isomorphic to its converse) homogeneous Suslin tree with exactly ω_1 automorphisms, a reversible (i.e. isomorphic to its converse) homogeneous Suslin tree with exactly ω_1 automorphisms, a rigid[334] Suslin tree, and a rigid Suslin tree which is reversible. Jensen's proof that from $\diamondsuit^+$ there is a Kurepa tree is sketched.

In Chapter 6,[335] work of Solovay and Tennenbaum on the consistency of Suslin's hypothesis is given, along with an introduction to Martin's Axiom.

The remainder of the book is devoted to the remarkable proof of the consistency of Suslin's Hypothesis with CH. Jensen used a method of iterated forcing with conditions which were not countably closed but did not collapse cardinals and did not add reals. Todorcevic (cf. [Kurepa, 1996, 8]) called this result "one of the deepest applications of the method of forcing since its original invention".

Mix and match

Early in the decade, Jech [1972] asked about the existence of a rigid Aronszajn tree. We discussed the results of Jensen on rigid Suslin trees above. At the end of the decade, Uri Abraham[336] published his ZFC construction of a rigid Aronszajn tree[337] using non-isomorphic Aronszajn trees constructed by Gaifman and Specker [1964]. The same year, motivated by a question in Kurepa's thesis [1935], Todorcevic published his construction of a rigid Aronszajn tree.

By the end of the decade, the consistency of combinations of existence or non-existence of Suslin trees, existence or non-existence of Kurepa trees, together with either CH or MA + $\neg$CH were established.

Devlin had asked whether $V = L$ implies the existence of such a tree in 1969 (see [1983]). In 1971 Jensen (cf. Theorem 4 in [Devlin, 1974]) proved that if $V = L$, then there exists a Kurepa tree without an Aronszajn subtree. In the summer of 1980, while at the University of Toronto, Devlin [1983] came up with a simpler

[334]A structure is *rigid* if it has no non-trivial automorphism.

[335]Chapter 6 is essentially the only place where results appear that are not due to Jensen

[336]Uri Abraham, whose name is sometimes transliterated Avraham, received his 1979 doctorate [1979b] from Hebrew University where his advisors were Lévy and Shelah. Abraham has a joint appointment split between Computer Science and Mathematics at Ben Gurion University, Be'er-Sheva. His interests in set theory and logic include Boolean algebras, forcing and axiomatics as well as a variety of combinatorial topics. For a sample publication in computer science, see [1999].

[337]Baumgartner [1975c] knew of the existence of such a tree already in 1975, but the work was neither published nor widely publicized.

proof after talking with Fleissner about his work and Kunen's work on the Normal Moore Space Conjecture.

The *weak Kurepa Hypothesis* (wKH) asserts that there is a tree of height ω_1 of cardinality $\aleph_1$ which has more than $\aleph_1$ uncountable branches. Mitchell [1972] proved that the failure of the weak Kurepa Hypothesis is equiconsistent with the existence of an inaccessible cardinal. In his 1978 Master's Thesis, Stevo Todorcevic used techniques of [Mitchell, 1972] and [Devlin, 1978] to construct a model of MA + $\neg$wKH and did so in a way that allowed him to make the continuum any regular cardinal (see [Todorcevic, 1981c]). Todorcevic went on to derive a variety of topological consequences of MA + $\neg$wKH.

In a 63-page paper, Devlin [1978] reviewed known work and noted that models of CH had been constructed (a) with a Suslin tree but no Kurepa tree, (b) with a Kurepa tree but no Suslin tree, (c) with both a Suslin tree and a Kurepa tree, and that models of MA + $\neg$CH for the above possibilities had also been constructed. The two remaining possibilities were simultaneous non-existence of Suslin trees and Kurepa trees, either with CH or with MA + $\neg$CH. As for the latter, Devlin started with Silver's model of no Kurepa trees and iterated over it in such a way so as to produce a model of MA + $\neg$CH, adapting Silver's proof that σ-closed forcing does not add branches to control the addition of branches in this process, so that no Kurepa trees are added. As in any model of MA + $\neg$CH, all Aronszajn trees are special, so there are no Suslin trees. The last half of the paper is devoted to getting a model of CH with neither Suslin nor Kurepa trees, and Devlin used an idea of Jensen's for forcing the consistency of $\diamondsuit^*$ and adapted Jensen's proof of the consistency of no Suslin trees with CH.

Larger trees

Starting with a Mahlo cardinal, William Mitchell[338] [1972] constructed a model of set theory with no special $\aleph_2$-Aronszajn tree. He introduced the term *special* for κ^+-Aronszajn trees which are subtrees of the collection of one-to-one functions into κ whose domain is an ordinal $< \kappa^+$ ordered by end-extension. Mitchell's definition differs from our use of *special* for κ^+-trees that are the union of κ many antichains, which was introduced for $\kappa = \omega$ in §3.4 in the discussion of Kurepa's construction of a special Aronszajn tree.[339]

Mitchell [1972] proved that the non-existence of special $\aleph_2$-Aronszajn trees is equiconsistent with the existence of a Mahlo cardinal. He used a result, independently due to Silver and Rowbottom, which asserted the existence of a sentence

[338] William J. Mitchell (December 30, 1943 –) received his doctorate from the University of California, Berkeley, where Silver was his advisor. In [Mitchell, 1972, 22], where he published revised versions of his thesis work, he also thanked Baumgartner and Solovay for "helpful discussions and suggestions."

[339] Mitchell's definition of special recalls the original Aronszajn construction of a subtree of the one-to-one functions into ω. Such trees need not be special by the modern definition. Our definition (also used by Kunen [1980]) is based on Kurepa's initial construction of a special Aronszajn tree; the equivalence of being an ω_1-tree which is $\mathbb{Q}$-embeddable is used by Jech [2003].

which has a model of type (κ^+, κ) if and only if there is a special κ^+-Aronszajn tree. Since κ^+-Aronszajn trees are known to exist for regular κ with $2^{<\kappa} = \kappa$, it follows that any model of ZFC which has no special κ^+-Aronszajn tree fails to satisfy the transfer property $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$. Silver extended Mitchell's result by showing that for regular κ, there are no κ^+-Aronszajn trees if and only if κ is weakly compact in L. Mitchell's construction involved a mixed iteration of length κ adding Cohen reals, so that eventually $2^{\aleph_0} = \kappa$, and collapsing cardinals below κ to $\aleph_1$, so that eventually $\kappa = \aleph_2$, but κ retains enough of its weak compactness to guarantee that every tree on ω_2 of height ω_2 with all levels of size $\leq \aleph_1$ has a branch. For a model created in the presence of a weakly compact cardinal with iterated Sacks forcing and with no ω_2-Aronszajn trees, see [Baumgartner and Laver, 1979].

In his [1976], Gregory combines work of Jensen and Solovay with Gregory's Theorem discussed in §7.1 to construct higher Suslin trees. After stating Theorem 1.3 attributed to Solovay, Gregory includes commentary which concludes with the assertion of the existence of a non-reflecting stationary subset of κ from the existence of a $\square_\kappa$ sequence. The commentary is followed with a proof of the result of Solovay "given in place of a reference." Gregory states a multipart Theorem 1.10, attribute to Jensen,[340] which among other things says that if $\kappa^{<\kappa} = \kappa$ and there is a non-reflecting stationary set $E \subseteq \kappa^+$ for which $\diamondsuit(E)$ holds, then there is a κ^+-Suslin tree and it may be constructed with additional properties as well. Specifically, Gregory constructs an $\aleph_2$-Suslin tree using a non-reflecting stationary set consisting of ordinals of cofinality ω.

In his doctoral thesis [1979a], [1981d, Theorem 4.1, 240], Todorcevic[341] used CH + $\square$ to construct an ω_2-Aronszajn tree with no ω_1-Aronszajn subtrees and no Cantor subtrees.[342] Devlin asked if CH or GCH was sufficient. Todorcevic showed the answer was no by proving the consistency, relative to the existence of a weakly compact cardinal, of GCH holds and every ω_2-Aronszajn tree contains a complete binary ω_1-tree. He also showed that the proof of Theorem 4.1 generalized to show that under the assumption of $\square_\kappa$, there is a κ^+-Aronszajn tree with no λ-Aronszajn subtrees for every $\lambda \neq \kappa$ and no ν-Cantor trees for every ν. Moreover, in Theorem 5.5 [1981d, 249] he showed the consistency, via forcing over a model of $V = L$, that for every regular uncountable non-weakly compact cardinal κ, there is a κ-Suslin tree and there is a κ-Kurepa tree with no λ-Aronszajn subtrees for any λ and no ν-Cantor subtrees for any ν. Istvan Juhász and William Weiss [1978] had shown that a question of Sikorski[343] [1950] of the existence of a κ-metrizable κ-

[340] See the top of page 667 of [Gregory, 1976] for the attribution of Theorem 1.10 to Jensen.

[341] Stevo B. Todorcevic earned his doctoral degree in 1979 at the University of Belgrade with Kurepa as advisor and K. Devlin as outside reader. Devlin attended the defense; his encouragement led Todorcevic to visit Jerusalem where he sat in on Shelah's lectures on forcing.

[342] A tree is a κ-Cantor tree if it has height $\lambda + 1$ for some cardinal $\lambda \leq \kappa$, the first λ levels have size $\leq \kappa$, and the last level has size $> \kappa$.

[343] Roman Sikorski (July 11, 1920 – September 12, 1983) was a student of Mostowski, receiving his doctorate in 1949 and habilitation in 1950 at Warsaw University. He was a professor at Warsaw University 1952–1982, became a corresponding member of the Polish Academy of Sciences in 1962 and a full member in 1967. He was president of Polish Mathematical Society from 1965–1977. He is known for the Rasiowa-Sikorski Lemma. For more on his life, see

compact space of power μ was equivalent to the existence of a κ-Kurepa tree with μ branches and no κ-Aronszajn subtree. Their result combined with Todorcevic's Theorem 5.5 gives a consistently positive answer to Sikorski's question.

7.8 Linear and quasi-orders

We start with a review of results about wqos and bqos. Dick H. J. de Jongh and Rohit Parikh[344] [1977] proved that the ordinal length of a well-quasi-ordered set[345] can always be realized in a linear extension.

In a series of papers, Laver [1971], [1973], [1976], [1978a] explored the theory of bqos.[346] In an early example, Laver [1971] proved there are $\aleph_1$ many countable scattered linear orders with respect to bi-embeddability.[347] In a later example, Laver [1978a] extended the Nash-Williams [1965a] result that infinite trees form a bqo. Laver [1978a] called a tree $(T, \leq)$ σ-scattered if it is representable as $T = \bigcup_{n<\omega} T_n$ where each T_n is a scattered[348] downwards closed subtree of T, and denoted the class of these trees by $\mathfrak{M}$.[349] He proved that $\mathfrak{M}$ is a bqo under order-preserving and meet preserving maps (the meet of two elements of a tree is the largest element which is less than or equal to both of them). He observed that Aronszajn trees do not form a wqo under $\diamondsuit$ and raised the question of whether it is consistent that Aronszajn trees form a wqo.[350] Galvin had shown that the class of trees of height $\omega + 1$ is not well-quasi-ordered, which led him to ask the of whether σ-scattered trees are a wqo. Laver proved closure properties for bqos, proving that if Q is a bqo, then the collection of Q-labeled $\mathcal{M}$-trees is a bqo.

The theory of wqos and bqos has a wide variety of applications. For example, Dexter Kozen [1988] used one of Laver's wqo results [1976] to establish the finite

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[Anonymous, 1984].

[344] Dick de Jongh received his Ph.D. at the University of Wisconsin, Madison, with Kleene as advisor, and and Rohit Parikh received his Ph.D. at Harvard University with Hartley Rogers, Jr. and Burton S. Dreben as advisors.

[345] The *ordinal length* of a well-quasi-ordered set is the maximum of the order types of its linear extensions.

[346] Larson gave a graduate course on Laver's bqo work at UCLA when she was a Hedrick Assistant Professor (1972-1974), and remembers enjoyable conversations with him about bqo theory in the botanical garden.

[347] Arnold Beckmann, Martin Goldstern and Norbert Preining [2008] generalize Laver's method to extend this result of Laver by showing that there are only $\aleph_1$ many countable closed linear orders with respect to bi-embeddability with continuous monotone maps and apply their result to show there are only denumerably many Gödel logics. More specifically, they show that the set of countable closed linear orderings is better-quasi-ordered by strictly monotone continuous embeddability, and the result extends to the case in which the orders are labeled. They defined "countable closed linear order" as a countable closed subset of the real line.

[348] Recall from §6.2 that a tree T is *scattered* if it does not embed the tree of all finite sequences of zeros and ones. Note that Laver's definition departs from the usual usage of σ as a prefix to denote countable unions taken freely.

[349] Todorcevic [2007a] writes $\mathcal{S}$ for the class of scattered trees and $\mathcal{S}_\sigma$ for the class of countable unions of scattered trees, which he calls σ-scattered trees.

[350] Todorcevic answered Laver's question negatively in [2007a] in which he showed that there is an infinite, strictly decreasing sequence of Aronszajn trees (he used Lipschitz trees) under the weaker notion where $S \leq T$ if and only if there is a strictly increasing map from S into T.

model property for the μ-propositional calculus. Montalbán [2007] used theorems from [Laver, 1971] in his analysis, mainly from the viewpoints of computability theory and reverse mathematics, of equimorphism types[351] of linear orders.

Laver [1979] examined the question of which linear orders can be embedded in $({}^{\omega}\omega, <^*)$,[352] i.e. the functions with domain and codomain ω under eventual domination, in the absence of the Continuum Hypothesis. CH implies that every linear order of power continuum is embeddable in $({}^{\omega}\omega, <^*)$. Laver pointed out that it was well-known that one can force the continuum to be large and ω_2 not embeddable in $({}^{\omega}\omega, <^*)$. Kunen however noted that MA is consistent with the existence of a linear order of power $2^{\aleph_0}$ which is not embeddable in $({}^{\omega}\omega, <^*)$.[353] Laver [1979, 299] pointed out that "the problem with trying to inductively embed, using Martin's Axiom, any linear ordering of power $2^{\aleph_0}$ into $(\omega)^{\omega}$, is the possibility of creating in the course of construction a 'Hausdorff gap'." In connection with his work with W. Hugh Woodin on homomorphisms of Banach algebras, Solovay had asked if it is consistent that the continuum is large and every linear order of power continuum is embeddable in $({}^{\omega}\omega, <^*)$. Laver answered the question positively by showing that if $\kappa^{<\kappa} = \kappa > \omega$, then there is a c.c.c. generic extension in which $2^{\omega} = \kappa$ and the saturated linear ordering of power 2^{ω} is embeddable into $\langle {}^{\omega}\omega, < \rangle$. As in the consistency proof for MA, generic functions $f_{\alpha}\colon \omega \to \omega$ for $\alpha < \kappa$ are added iteratively so that all cuts in their ordering via $<^*$ are eventually filled.

Mitchell [1972] used trees to answer negatively a question of Malitz [1968] whether for regular cardinals κ, the existence of a linear ordering of power 2^{κ} with a dense subset of cardinality κ is true in ZFC.

In answer to a question of Erdős and Hajnal [1974, 281–282], Devlin [1974] proved that if GCH holds and there is a Kurepa tree with no Aronszajn subtree, then the following statement Φ holds: there is an order type of cardinality ω_2 which embeds neither ω_2 nor $\omega_2{}^*$, for which every subtype of cardinality ω_1 embeds one of ω_1 and $\omega_1{}^*$. Hence Φ is true in L, by this theorem combined with Jensen's construction in L of a Kurepa tree with no Aronszajn subtree, mentioned above in §7.7. In his thesis [1979a], [1981d], Todorcevic reduced Devlin's use of $V = L$ to $\square$. Recall that he had used $\square$ to construct an ω_2-Aronszajn tree with no ω_1-Aronszajn subtrees and no Cantor subtrees, as mentioned above in the subsection of §7.7 on larger trees. He showed in Theorem 7.1 [1981d, 254] that Φ is equivalent to the existence of either a Kurepa tree with no Aronszajn subtree or an ω_2-Aronszajn tree with no ω_1-Aronszajn subtrees and no Cantor subtrees. Thus he reduced the use of $V = L$ to the hypothesis $\square$ in Devlin's result. He also showed one can use forcing to construct a Kurepa tree with no Aronszajn subtree.

In 1971 Baumgartner announced his result [1973] that it is consistent that all subsets of the real numbers of cardinality $\aleph_1$ which are $\aleph_1$-dense[354] have the same

[351] Two linear orders are *equimorphic* if each can be embedded in the other.

[352] Laver used $(\omega)^{\omega}$ to represent this collection of functions. Currently, this work would be cast as a question about $\mathcal{P}(\omega)/\text{fin}$.

[353] To learn more about Kunen's result, see the chapter by Steprāns.

[354] A linearly ordered set $(P, <)$ is $\aleph_1$-*dense* if it has no first or last element and between any pair of points there are exactly $\aleph_1$ elements.

order type. Under the assumption of CH, given a pair of $\aleph_1$-dense sets A and B, he used diagonal arguments like those of Sierpiński [1950] and back-and-forth arguments to construct partitions of A and B into continuum many countable sets each dense in the appropriate one of A, B so that the following partial order satisfies the countable chain condition: P is the set of finite, order-preserving functions from A to B which send elements of the αth cell of the partition of A to elements of the αth cell of the partition of B, ordered by inclusion.

Baumgartner iterated this forcing to build a model in which all pairs of $\aleph_1$-dense subsets of the reals are isomorphic and $2^{\aleph_0} = \aleph_2$. He then observed that in this model, a singleton whose only element is an $\aleph_1$-dense set of reals is a basis for the set of all uncountable real types.

Baumgartner [1973] also posed the following two questions:

- Is it consistent to have all $\aleph_2$-dense subsets of the reals isomorphic?
- Does the statement that all $\aleph_1$-dense subsets of the reals are isomorphic follow from the assumption of Martin's Axiom and $2^{\aleph_0} > \aleph_1$?

Shelah [1980a] used oracle forcing to establish the consistency of $2^{\aleph_0} = \aleph_2$ together with the existence of a linear order universal in power $\aleph_1$. $\aleph_1$.[355]
In his paper Shelah recalled the Baumgartner [1973] proof of the consistency of all $\aleph_1$-dense sets of reals being isomorphic with a quick sketch and compared and contrasted his own proof with it. Shelah noted, without proof, that in a model in which $2^{\aleph_0}$ Cohen reals have been added, no such linear order universal in power $\aleph_1$ exists. These results complement the CH construction of a universal linear order of power $\aleph_1$ by Hausdorff in [2005]; see also [1908, 488], [2005, 243]. Shelah's consistency result should also be situated with earlier work of model theorists who had shown that saturated models are universal and that such models of cardinality λ exist for theories of cardinality $< \lambda$ when $\lambda = \lambda^{<\lambda}$ (see Chapter 5 of [Chang and Keisler, 1973]), and Shelah's Theorem 0.6 in Chapter VIII [1978a]: T has a saturated model of cardinality λ if and only if $\lambda^{<\lambda} + |D(T)|$ or T is λ-stable.

In a paper received by the journal in January of 1972, Baumgartner [1976b] discussed basis problems[356] for the class Φ of linear orderings which cannot be represented as the countable union of well-orderings. He set $\Phi_1 = \{\omega_1{}^*\}$, and let Φ_2 be the collection of uncountable real types. Baumgartner noted that Erdős and Rado [1956, 443] had asked if there were uncountable order types which embed neither ω_1 nor $\omega_1{}^*$ and do not embed an uncountable real type, and pointed out that Specker had answered the question in the positive.[357] Baumgartner

[355] Recall that a linear (partial) order U is *universal in power* λ if it is a linear (partial) order of cardinality λ that homomorphically embeds all linear (partial) orders of cardinality λ.

[356] Recall a basis B for a class $\mathcal{C}$ of linear orders is a set of linear orders in the class which has the property that for any element $(L, \lessdot)$ of $\mathcal{C}$ there is some $(K, <)$ in B which is order-embeddable into $(L, \lessdot)$.

[357] As discussed earlier, in footnote 7 on page 443, Erdős and Rado [1956] noted that Specker had disproved their conjecture after the submission of their paper. Baumgartner listed Specker's work as unpublished, and likely the Erdős-Rado paper was the source.

called types with these properties *Specker types* and let Φ_3 be the class of them. Baumgartner showed that every ordering witnessing a Specker type embeds a linear ordering obtained from an Aronszajn tree under a suitable ordering (such linear orderings are now called *Aronszajn lines*). Baumgartner reported that Galvin had asked whether or not $\Phi_1 \cup \Phi_2 \cup \Phi_3$ constitutes a basis for Φ, and Baumgartner identified a collection he called Φ_4 of order types φ such that every uncountable subtype embeds ω_1 but φ is not the union of countably many well-orderings. He answered Galvin's question negatively when he proved that Φ_4 is non-empty. Baumgartner also proved that Φ_4 is not a well-quasi-order, does not have a finite basis, and contains dense rigid types. Baumgartner pointed out a partition calculus application which was likely the motivation for Galvin's question: there is an ordering φ different from both ω_1 and λ (any ψ^* with $\psi \in \Phi_4$ works) such that $\varphi \to (\omega)^1_\omega$. If φ is such a type, then, by the Baumgartner-Hajnal Theorem for order types, $\varphi \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and $k < \omega$, so the extra work for that theorem is justified. Baumgartner noted out that $\Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4$ is a basis for Φ. Let N be the class of all continuous ordinal-valued strictly increasing functions whose domains are countable successor ordinals, and for $f, g \in N$, write $f \sqsubset_{\text{lex}} g$ if either f has g as a proper end-extension or f precedes g in the lexicographic order. His main technique was to represent order types as subsets of N under $\sqsubset_{\text{lex}}$. In Theorem 2.1, he showed that every type φ which fails to embed $\omega_1{}^*$ can be represented by $(S, \sqsubset_{\text{lex}})$ for some $S \subseteq N$ such that the tree $T(S) := \{f \in N : (\exists g \in S)(f \sqsubset_{\text{lex}} g)\}$ contains no path of length ω_1, and that if $|\varphi|$ is a cardinal of uncountable cofinality, then the desired S may be chosen as a subset of ${}^{<\omega_1}|\varphi|$. In Proposition 1.4, he showed that if $T(S)$ contains an Aronszajn tree, then the order type of $(S, \sqsubset_{\text{lex}})$ contains a Specker type. The paper closed with a list of problems, of which we give one below [1976b, 221]:

Problem 5:

1. Is ZFC + "Φ_3 has a finite basis" consistent?
2. Is ZFC + "$\Phi_2 \cup \Phi_3$ has a finite basis" consistent?

Note that the first of these problems asks for the consistent existence of a finite basis for Specker types, and implicitly asks for a finite basis for Aronszajn lines (sometimes called *A-lines*), since, as noted above, every ordering witnessing a Specker type embeds an Aronszajn line. Since any uncountable linear order which is the union of countably many well-orderings must embed ω_1, and by definition every linear order in Φ_4 embeds ω_1, the addition of ω_1 and its converse $\omega_1{}^*$ to a finite basis for $\Phi_2 \cup \Phi_3$ would give rise to a finite basis for all uncountable linear orderings. So the second of the above questions implicitly asks if it is consistent that there is a finite basis for uncountable linear orders.

In 1974 during the International Congress in Vancouver, Shelah [1976] constructed what is now known as a Countryman line,[358] namely an uncountable

[358]The name Countryman line refers to R. S. Countryman, Jr., whose unpublished typescript, dated to 1970 in [Todorcevic, 1984c], included the question of existence of such a type.

linearly ordered set, whose square, ordered lexicographically, is the union of countably many chains. Shelah first carefully and inductively constructed an Aronszajn subtree T of the tree $S = {}^{\omega_1>}\mathbb{Q}$ of all sequences of rationals, partially ordered by end-extension, which satisfied suitable constraints and then passed to a linear order imposed on T essentially by lexicographically ordering its nodes to obtain a Countryman line.

Shelah closed his paper with several observations, a sample of which follows. He noted that if $\lambda = \lambda^{<\lambda}$, then one could lift his result with λ and λ^+ in place of $\aleph_0$ and $\aleph_1$ by a similar construction. He observed that one can construct the tree so that it will be special. He commented that any uncountable linear order whose square is a countable union of chains is a Specker order (i.e. an Aronszajn line). Consequently it has cardinality $\aleph_1$, something Shelah reported that Countryman already knew.

Prior to Shelah's construction of a Countryman line, Galvin had observed that if I is Countryman, then for all $n > 2$, the set I^n can also by covered by countably many chains defined from those used to cover I^2 (Shelah and the referee observed this later), and that I cannot contain two anti-isomorphic uncountable subsets (Shelah and Abraham observed this later).

Shelah reported that Harvey Friedman[359] asked "for the existence of an infinite complete order I, such that any open interval of I is isomorphic to I, but I is not anti-isomorphic to itself." He then indicated how, with some care, one could construct a completion of his example with these properties.

Shelah called two linear orders *near* if they have isomorphic uncountable subsets and noted that one can use $\diamondsuit$ to prove that there are $2^{\aleph_1}$ pairwise not near orders each of which is uncountable and whose square can be covered by countably many chains.

In his ninth and final observation, Shelah conjectured the consistency of the following pair of statements [1976, 114]:

(A) For every Specker order I there is a J near to it which is an uncountable order whose square is the union of countably many chains.

(B) If I and J are both uncountable orders whose squares are the union of countably many chains, then I is near to J or its converse order J^*.

Shelah [2000a] later reflected back on his mid-seventies interest in the consistent existence of a finite basis for the uncountable linear orders, the necessity of such a basis having ω_1, $\omega_1{}^*$, a representative real type, and at least two Aronszajn lines because of his construction of a Countryman line which is not isomorphic to its reverse. This problem has become known as *Shelah's Finite Basis Problem* or *Shelah's 5-Element Basis Problem.*

[359] Shelah included in his references the problem paper [Friedman, 1975], but did not cite it in the body of the paper.

New perspectives were reached in the 1980's. Shelah [1980a], [1982b] introduced a generalization of countable chain condition forcings called *proper forcing.* A partial order P is *proper* if it preserves stationary subsets of $[\lambda]^{\leq\omega}$[360] for all uncountable cardinals λ.[361] which have the nice property that properness is preserved under countable support iteration. The use of iterated forcing spread, helped in part by a survey article by Baumgartner [1983a]. In particular, set-theoretic topologists used forcing in an increasing number of ways. Foreman, Magidor and Shelah [1988] introduced a maximal form of Martin's Axiom, called Martin's Maximum, and proved new consistency results that led to a paradigm shift in the understanding of large cardinal strength.

Todorcevic and Boban Veličković[362] [1987] reinterpreted many questions in combinatorial language leading to new perspectives, including giving an equivalence of $\mathrm{MA}_{\aleph_1}$ to a statement about c.c.c. partitions. Abraham, Rubin and Shelah and, separately, Todorcevic, introduced distinct open coloring axioms. New ways of obtaining Suslin trees were discovered, and partition theorems for products of trees moved into the transfinite.

Conferences promoted interactions between set theorists and the communities built around set-theoretic topology, order theory and (finite) combinatorics. For example, the survey by Baumgartner [1982] on uncountable linear order types, based on a talk at the *Ordered Sets* conference held in Banff in 1981,[363] consolidated understanding of an area that had been spread out in the literature; another new resource for people interested in order was the book *Linear Orderings* by Joseph Rosenstein [1982].

Boolean algebras as a field of study matured with the appearance of the three-volume *Handbook of Boolean Algebras*: Sabine Koppelberg edited the first volume [Koppelberg, 1989], and Monk and Robert Bonnet edited the second and third volumes [1989a], [Monk and Bonnet, 1989b]. Connections were made between the study of Boolean algebras and pcf theory in the following decade (see Kojman's

[360] The the set of all countable subsets of λ is variously denoted in the literature by $[\lambda]^{\leq\omega}$, $\mathcal{P}_{\omega_1}(\lambda)$ and $\mathcal{P}_{\aleph_1}(\lambda)$. See [Jech, 2003] for the concepts of closed unbounded subsets and stationary subsets of $[\lambda]^{\leq\omega}$.

[361] See [Jech, 2003] and [Abraham, 2010] for more on proper forcing.

[362] Boban Veličković received his doctorate in 1986 from the University of Wisconsin, Madison, where his advisor was Kunen. He was a Bateman Research Instructor at Caltech 1986–1987, a C. Morrey Assistant Professor at the University of California 1988–1989 and 1990–1991, with the intervening year as a postdoctoral fellow at the Mathematical Sciences Research Institute in Berkeley, was an NSERC International Fellow at York University 1992–1994, an associate professor 1994–1995 before becoming a professor at University of Paris VI, where he is the head of the logic group.

[363] This two-week conference organized by Ivan Rival was, as Duffus [2003] points out in his obituary of Rival, a "significant point in the creation of an 'order theory community'... with links to lattice theory, combinatorics, set theory, and computer science". Subsequent conferences included *Graphs and Order*, a NATO Advanced Study Institute held in Banff in 1984, *Combinatorics and Ordered Sets*, an AMS-IMS-SIAM Joint Summer Research Conference in Arcata, California in 1985.

chapter).

Fraïssé published *Theory of Relations* [1986] bringing together work on order and work on relational structures. It included in the first eight chapters a brief review of set theory, fundamental results for the theory of linear, partial and quasi orders, especially well-quasi-orders, widely applied results in Ramsey theory including Ramsey's Theorem, the Nash-Williams Partition Relation, the Dushnik-Miller Theorem, the Erdős-Rado Theorem and Galvin's work on canonical partitions for pairs of rationals. The second half of the book was devoted to relational structures and included Fraïssé's method of building countable universal structures through a process of amalgamation of finite structures that leads to what is now known as the *Fraïssé limit*.

The *Handbook of Set-Theoretic Topology*, edited by Kunen and Vaughan [1984], popularized many techniques and results of set theory in the set-theoretic community and made them more readily available to young researchers in both areas. Juhász, in his article on cardinal functions, encouraged other topologists to follow him in using forcing directly rather than combinatorial consequences in their work. Baumgartner, in his expository article on applications of proper forcing aimed at set-theoretic topologists, facilitated the spread of this powerful technique and introduced the *Proper Forcing Axiom* (PFA), a generalization of Martin's Axiom in which the role of c.c.c. partial orders is taken on by proper partial orders. Models of PFA are non-CH models which have nice combinatorial structure.[364] Kunen focused on ideals in his hand book article on random and Cohen reals, and showed that their effects differ markedly on the meager ideal and the null ideal. Weiss discussed versions of Martin's Axiom, its influence on partial orders, and variants of the axiom where the partial orders considered are restricted, an area of research that continued. The volume also presented problems and results of set-theoretic topology in a way that invited others to work in the area. For example, there were reports on the Normal Moore Space Conjecture and work coming out of that line of research (see the chapters by Fleissner, Nyikos, and Tall). Judith Roitman's article, *Basic S and L*, was designed to introduce the reader to an area she described as one of the most active in set-theoretic topology for the preceding ten years, whose key questions are the following.

(S) Is every hereditarily separable space hereditarily Lindelöf?

(L) Is every hereditarily Lindelöf space hereditarily separable?

A space is *separable* if it has a countable dense set and *hereditarily separable* if every subspace is separable. A space is *Lindelöf* if every open cover has a countable subcover and *hereditarily Lindelöf* if every subspace is Lindelöf. Roitman was the first to formulate the S- space and L-space problems in terms of partitions [Roitman, 1978].

Of particular significance for this chapter is the survey by Todorcevic [1984c] of work on trees and linearly ordered sets from set-theoretic, topological and algebraic

[364]Todorcevic showed in [1984b] that PFA implies $\mathfrak{c} = \aleph_2$.

perspectives. Elegantly written and comprehensive for its time, it includes many results from Kurepa's thesis and later works that had been largely inaccessible due to their appearing in journals not widely distributed.

8.1 Set-theoretic topology

In the 1970's, set-theoretic topology as a discipline became increasingly visible. Rudin was an invited speaker at the 1974 International Conference of Mathematicians[365] in Vancouver where she spoke on the normality of products,[366] and she gave a series of expository talks at the CBMS Regional Conference held in Laramie, Wyoming in August 1974 (see Rudin [1975a]).

Rudin [1972] used a Suslin tree to produce the first example of an *S-space*, a hereditarily separable space which is not hereditarily Lindelöf. It was well-known that a Suslin line is an example of an *L-space*, a hereditarily Lindelöf space which is not hereditarily separable.

In 1980, W. Weiss arranged for Todorcevic to come to North America for the six-week summer school called Settop held in Toronto during July and the first half of August. The summer school featured three two-week sessions each with a ten-lecture series by a single speaker: Baumgartner spoke on forcing (see his *Handbook of Set-Theoretic Topology* exposition of proper forcing [1984]); Devlin spoke on the constructible universe and combinatorial principles derived from L; Juhász spoke on topology. Todorcevic shared a room with Abraham. Both had proved the existence of rigid Aronszajn trees (see [Abraham, 1979a] and [Todorcevic, 1979b]) in response to a question by Jech [1971]. While they were at the conference, Abraham and Todorcevic [1984] proved the consistency of MA + $\neg$CH + there exists a first countable S-space.[367] This theorem answered a question of Hajnal and Juhász that the existence of a first countable S-space does not imply the existence of a first countable L-space, since by a result of Szentmiklóssy, none exist in models of MA.

After the Settop conference, Todorcevic visited Dartmouth College for a few months during the academic year 1980-1981.[368] During that time he [1981a], [1983a] showed the consistency with $\mathrm{MA}_{\aleph_1}$ of the statement "every regular hereditarily separable topological space is Lindelöf", i.e. the consistency of "there are no

[365] Tall attributed the invitation to Rudin's construction [1971] of a Dowker space without using a Suslin tree. See [Rudin, 2008] for recollections of this discovery.

[366] Rudin [2008] expressed her perspective on normality as follows: "I admit I most enjoy tearing down structures: proving that reasonable sounding conjectures are false: constructing counterexamples. Normality is my thing since it seldom has the strength to guarantee your space is not miserable is some other way."

[367] Footnote 1 of [Abraham and Todorcevic, 1984] locates this result in time and space. Roitman [1984] referenced handwritten notes by Abraham and Todorcevic titled *MA and strong properties of S-spaces* and dated 1980. Justin Moore [2006b] constructed a ZFC example of an L-space, showing that the S-space and L-space problems are surprisingly different.

[368] See the note of gratitude to Baumgartner and thanks to Galvin for a "stimulating correspondence concerning a class of problems about strong partition relations on ω_1, a small part of which is considered in this paper" in [Todorcevic, 1983a, 705].

S-spaces", which fulfilled the expectations of Hajnal and Juhász [1974, 837] who asserted that "it is quite likely that the existence of such spaces is independent of the usual axioms of set theory. In fact, Martin's axiom + the negation of CH is a candidate to imply that no such spaces exist."[369]

At the end of the decade, Todorcevic published a monograph, *Partition Problems in Topology* [1989]. He observed that proof techniques developed for solving the S-space problem and the L-space problem turn out to be useful in many other problems in general topology, writing "this is so because Ramsey-type theorems are basic and so much needed in many parts of mathematics and (S) and (L) happen to be Ramsey-type properties of the uncountable most often needed by the topologist" [Todorcevic, 1989, 1].

A key part of Suslin's Problem is the chain condition, an important concept for the 20th Century with many close ties with infinite combinatorics, so the book *Chain Conditions in Topology* [1982] by Comfort and Negrepontis was a welcome new resource. They gave background in infinite combinatorics, in particular, examined generalized chain conditions useful for products of spaces among other topological concepts. Specifically, the book includes a proof of Kurepa's result that the product of two Suslin continua does not satisfy the countable chain condition, and the Galvin construction, under the assumption of $2^\kappa = \kappa^+$, of pairs of partial orders on κ^+ with all antichains of size at most κ but whose product has an antichain of size κ^+. Comfort and Negrepontis worked largely in ZFC, with occasional use of the Generalized Continuum Hypothesis.[370]

In 1941, Marczewski and Knaster wrote in the *Scottish Book* what it means for a topological space T to have the property (S) (of Suslin): every family of disjoint sets, open in T, is at most countable. Then, under his original name of Szpilrajn, he inscribed a problem in the *Scottish Book* (see [Mauldin, 1981, 265–266]) which I have named and quoted below:

> *The Szpilrajn Problem:* Is the property (S) an invariant of the Cartesian product of two factors?

Recall that Kurepa [1952] showed that the square of a Suslin line fails to have the countable chain condition. Thus if the product of two c.c.c. partial orders is always c.c.c., then there are no Suslin trees.

[369]Gary Gruenhage, in his 1981 Math Review MR0588816 (81j:54001) of *A survey of S- and L-spaces* by Juhász [1980], stated "The question of the existence of S- and L-spaces is one of the most important open problems in set-theoretic topology." Rudin also surveyed the subject in [1980]. In her article for the *Handbook of Set-Theoretic Topology*, Roitman [1984, 297] referred to the S-space and L-space existence problems as "a natural outgrowth of Suslin's hypothesis and of investigations of properties of metric, semi-metric, and Fréchet spaces," and attributes the modern form independently to Countryman and to Hajnal and Juhász, without giving references. Todorcevic [1989, 12–13] briefly discussed the history of the problem, and pointed to a series of papers by Hajnal and Juhász [1967], [1968], [1969] from which an early modern statement of the problem may have arisen.

[370]In his Math Review MR0665100 (84k:04002), Blass expressed regret that the book did not discuss independence results.

Galvin [1980] proved that if λ is a regular cardinal and the square bracket partition relation $\lambda \to [\lambda]^2_3$ holds, then for any two partially ordered sets satisfying the λ-chain condition, their product also satisfies the λ-chain condition.[371] $\mathrm{MA}_{\aleph_1}$ implies that the product of c.c.c. partially ordered sets is also c.c.c. (see Theorem 2.24 in [Kunen, 1980, 61].) Galvin also proved that if κ is an infinite cardinal such that $2^\kappa = \kappa^+$, then there are two partially ordered sets satisfying the κ^+-chain condition whose product does not. The two papers of Todorcevic [1985b], [1986b] contain the first ZFC examples of pairs of partial orders which satisfy the κ-chain condition but whose product does not, in particular for $\kappa = \mathrm{cf}(2^{\aleph_0})$, $\kappa = \mathfrak{b}$ (the bounding number[372]) and $\kappa = (2^{\aleph_0})^{+\omega+1}$. Shelah [1988a] gave such examples for every successor cardinal $> \aleph_1$.

8.2 Partition relations

The authors, Erdős, Hajnal and Rado, of the classic paper [1965] on the partition calculus were joined by Attila Máté in writing a compendium [1984] on the topic. It is the place to go for the definitions of the various partition symbols; care was taken to use modern notation and to express what is known without the assumption of the GCH. In the preface, the authors delimit the scope [Erdős *et al.*, 1984, 6]:

> There are many interesting assertions that are consistent with set theory without the axiom of choice but contradict this latter, and there are many important theorems of set theory plus some interesting additional assumptions, e.g. the axiom of determinacy, that is known to contradict the axiom of choice. We did not include any of these; unfortunate though this may be, we had to compromise; we attempted to discuss infinity, but had to accomplish our task in finite time.

Cardinal resources

Todorcevic [1983a] proved the consistency of a strong partition relation for ω_1, of which $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$ is a special case. More specifically, Todorcevic introduced the partition relation $\omega_1 \to (\omega_1, (\omega_1; \mathrm{fin}\ \omega_1))^2$, used forcing to prove the consistency of this partition relation with $\mathrm{MA}_{\aleph_1} + 2^{\aleph_0} = \aleph_2$. He derived multiple consequences, the most well-known of which is the consistency of the non-existence of S-spaces. More pertinent to this chapter is that fact that from the assumption of $\mathrm{MA}_{\aleph_1}$ and $\omega_1 \to (\omega_1, (\omega_1; \mathrm{fin}\ \omega_1))^2$, he proved in Theorem 4 [1983a, 716] that for every partition $[\omega_1]^2 = K_o \cup K_1$, either there is an uncountable subset $A \subseteq \omega_1$ all of whose pairs are in K_0 or for every $\alpha < \omega_1$, there are $B, C \subseteq \omega_1$ such that B has order type α, C is uncountable, and all pairs $\{\beta, \gamma\}_<$ with $\beta \in C$ and

[371] Recall the Galvin and Shelah [1973] had shown that $\lambda \to [\lambda]^2_3$ does not hold for $\lambda \in \{\aleph_1, 2^{\aleph_0}, \mathrm{cf}(2^{\aleph_0})\}$.

[372] A family $Y \subseteq {}^\omega\omega$ is *unbounded (under eventual domination)* if for every function $f \in {}^\omega\omega$ there is a function $g \in Y$ such that $g \not\leq^* f$. The *bounding number* $\mathfrak{b}$ is the minimum size of an unbounded subfamily of ${}^\omega\omega$. For more on this cardinal invariant, see §5 of the chapter by Steprāns.

$\gamma \in B \cup C$ are in K_1. This consistent partition relation is considerably stronger than its easily stated consequence $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$. These results shed light on possible consistent extensions of the Baumgartner-Hajnal Theorem for ω_1, that $\omega_1 \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and $k < \omega$.

Laver [1982], starting from a huge cardinal, forced to get a model in which ω_1 carries an $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal (cf. §7.3) and derived interesting consequences for the partition calculus, including the consistency of

$$\omega_2 \to (\omega_1 + \omega_1 + 1, \alpha)^2.$$

Kanamori [1986] surveyed partition relations of the form $\kappa^+ \to (\alpha)^2_\delta$ for regular cardinals κ, with an emphasis on $\delta = 2$. First he reported that Donder [1982] had used Jensen's covering lemma to show that if $\kappa > \omega_2$ is a successor cardinal with κ^- as its cardinal predecessor, $2^{\kappa^-} = \kappa$ and $0^\#$ does not exist, then $\kappa^+ \not\to [\kappa : \kappa^-]^2_\kappa$ (recall the colon notation in a partition relation was defined in §6.7). Next, given a weakly compact cardinal κ, Kanamori [1986] noted the ordinary partition relation $\kappa^+ \to (\kappa : \eta)^2_2$ holds by applying the square bracket polarized partition relation

$$\begin{pmatrix} \kappa^+ \\ \kappa \end{pmatrix} \to \begin{bmatrix} \eta \\ \kappa \end{bmatrix},$$

which was stated without proof by Chudnovsky [1975], with proofs supplied by [Wolfsdorf, 1980], Shelah, and [Kanamori, 1982]. Kanamori [1986, 156] then asked:

Question 2.2: If κ is weakly compact, does $\kappa^+ \to (\kappa + \kappa)^2_2$?

Kanamori also considered successors of measurable cardinals, but found that the existence of a sufficiently saturated ideal sufficed for his purposes. Building on a definition in the cardinal resources subsection of §7.3, call a κ-ideal $\mathcal{I}$ a *Laver ideal* if every family of κ^+ many $\mathcal{I}$-positive sets has a subfamily of size κ^+ such that the intersection of any subset of the subfamily of *fewer* than κ sets is also $\mathcal{I}$-positive. Then, if $\kappa^{<\kappa} = \kappa$ and there is a Laver κ-ideal, then for every $\alpha < \kappa$:

$$\kappa^+ \to (\kappa + \kappa + 1, \alpha)^2.$$

Since measurable cardinals carry such ideals, they satisfy the positive partition relation. Laver [1982] had only outlined a proof in his paper of the consistency of the existence of such ideals; Kanamori [1986] rediscovered the proof, and included a proof with permission from Laver. At the end of the paper, Kanamori noted that the positive partition relation could consistently hold at a non-weakly compact inaccessible cardinal, since it can be verified that a saturated ideal constructed by Kunen [1978] on such a cardinal is a Laver ideal. Kanamori [1986, 161] focused attention on the boundary of knowledge about these partition properties:

Question 2.5: If κ is measurable, does $\kappa^+ \to (\kappa + \kappa + 2)^2_2$?[373]

[373] Foreman and Hajnal have proved that the answer is yes in Theorem 39 of their [2003], which gives $\kappa^+ \to (\rho)^2_m$ for finite m and ρ below a large bound.

Todorcevic [1986a] showed the consistency relative to the existence of a weakly compact cardinal of the partition relation $2^{\aleph_0} \to (2^{\aleph_0}, \alpha)^2$ for all $\alpha < \omega_1$, and indicated how to modify the argument to obtain the consistency of $2^{\aleph_1} \to (2^{\aleph_1}, \alpha)^2$ for all $\alpha < \omega_2$. He also gave a forcing argument for cardinals κ of uncountable cofinality that $\kappa \to (\kappa, \omega + 2)^2$ may hold. The context for this relation includes Hajnal's proof [1960] that under CH, $2^{\aleph_0} \not\to (2^{\aleph_0}, (\omega\colon 2))^2$ and consistency results in [Baumgartner, 1976a], [Laver, 1975] that this result holds under stronger hypotheses, for various values of the continuum.

Milner and Prikry [1986] used MA + ¬CH to prove

$$\omega_1 \to (\omega + m, 4)^3.$$

By absoluteness arguments like those used in the metamathematical proof of the Baumgartner-Hajnal Theorem, the use of MA + ¬CH may be eliminated. This positive partition relation was the first significant progress on partitions of triples of countable ordinals since the paper of Erdős and Rado [1956].

Ordinal partition relations

Erdős continued to focus attention on partition relations of the form $\alpha \to (\alpha, m)^2$ through offering money. In 1985, he [1987] offered \$1000 for a complete characterization of those countable ordinals α for which $\alpha \to (\alpha, 3)^2$. Ordinals α, countable or not, which satisfy this partition relation have come to be called *partition ordinals* (see [Schipperus, 1999]).

Larson [1980] used the combinatorial structure of a lexicographic representation of the ordinal ${\omega_1}^{\omega+1}$ to prove a partition relation for an ordinal with infinite exponent:

$${\omega_1}^{\omega+1} \not\to ({\omega_1}^{\omega+1}, 3)^2$$

In connection with a 1985 meeting held at Humboldt State University, Baumgartner and Hajnal [1987] proved a variety of ordinal partition relations involving simple ordinal products of cardinals, some of which are listed below:

- If κ is a regular cardinal and $2^\kappa = \kappa^+$, then $(\kappa^+)^2 \not\to (\kappa^+ \cdot \kappa, 4)^2$.
- If κ is a singular cardinal of cofinality τ and $2^\kappa = \kappa^+$, then $\kappa^+ \cdot \tau \not\to (\kappa^+ \cdot \tau, 3)^2$ (i.e. $\kappa^+ \cdot \tau$ is not a partition ordinal).
- If κ is a regular cardinal and $\kappa^{<\kappa} = \kappa$, then $(\kappa^+)^2 \to (\kappa^+ \cdot \kappa, 3, 3)^2$.

In a 1987 meeting held in Toronto, Baumgartner [1989] focused attention on partition ordinals by showing that the existence of uncountable partition ordinals is consistent. Under the assumption of MA + ¬CH, he proved two simple products were partition ordinals and asked the simplest open question suggested by these results:

- $\omega_1 \cdot \omega \to (\omega_1 \cdot \omega, 3)^2$.
- $\omega_1 \cdot \omega^2 \to (\omega_1 \cdot \omega^2, 3)^2$.
- *Question:* Is it consistent that ${\omega_1}^2 \to ({\omega_1}^2, 3)^2$?

Shelah and Lee J. Stanley [1987] gave proofs of several partition relations for ordinal products of cardinal numbers and we list some representative examples.

- Assume CH. Then $\omega_2 \cdot \omega \to (\omega_2 \cdot \omega, 3)^2$.
  This is a solution to Problem 13 [Erdős and Hajnal, 1971b, 24].
- It is consistent to have $\neg$CH + $\omega_2 \cdot \omega \not\to (\omega_2 \cdot \omega, 3)^2$.
  The proof used finite conditions.
- It is consistent to have GCH + $\omega_3 \cdot \omega_1 \not\to (\omega_3 \cdot \omega_1, 3)^2$.
  The proof used historic forcing.[374]
- Assume that the existence of a weakly compact cardinal is consistent. Then so is GCH + $\omega_3 \cdot \omega_1 \to (\omega_3 \cdot \omega_1, k)^2$ for all $k < \omega$.
- *Question:* Does $\mathfrak{c} \cdot \omega \to (\mathfrak{c} \cdot \omega, 3)^2$?

In his thesis, Tadatoshi Miyamoto [1988, 67-94] gave a morass construction of a partition witnessing $\omega_3 \cdot \omega_1 \not\to (\omega_3 \cdot \omega_1, 3)^2$, which had earlier been proved by forcing by Shelah and Stanley [1987]. The existence of such a construction had been conjectured by Stanley and Daniel Velleman among others. Miyamoto's morass was a simplified morass, but had some extra structure, similar to linear limits. During a July 1988 meeting in Oberwolfach, Stanley and Velleman tried to understand Miyamoto's proof and, in the process, they came up with their own construction of a witness to the negative partition from a simplified $(\omega_2, 1)$-morass with linear limits, but used a complete amalgamation system in place of the rest of Miyamoto's extra structure. Later they learned that Charles Morgan had independently found essentially the same proof which they published together in [1991]. Assuming $2^{\aleph_1} = \aleph_2$, it is known that if both $\aleph_2$ and $\aleph_3$ are successor cardinals in L, then there is such a simplified $(\omega_2, 1)$-morass with linear limits. It follows that, assuming $2^{\aleph_1} = \aleph_2$, the positive partition relation $\omega_3 \cdot \omega_1 \to (\omega_3 \cdot \omega_1, 3)^2$ holds only if either $\aleph_2$ or $\aleph_3$ is inaccessible in L.

Square bracket partition relations

Todorcevic [1981d] developed methods for constructing uncountable trees while controlling the type of their subtrees and applied his tree results to prove consistency results for the partition calculus. In particular, he proved from $\square_\kappa$ the

[374]The process of historicization of naïve conditions was introduced in [Baumgartner and Shelah, 1987] to prove consistency results for superatomic Boolean algebras, without use of the name. The Shelah-Stanley paper was the first to introduce it officially according to Rosłanowski and Shelah [2001].

stepping up negative square bracket partition relations as given below when $\kappa > \omega$ is regular, $2 \leq r < \omega$, $r < \lambda_0 \leq \kappa$, $\omega < \lambda_\xi \leq \kappa$ and λ_ξ regular for $0 < \xi < \nu$:

$$\kappa \not\to [\lambda_\xi]^r_{\xi<\nu} \quad \text{implies} \quad 2^\kappa \not\to [\lambda_\xi + 1]^{r+1}_{\xi<\nu}\ .$$

The statement of this square bracket result was added in proof to the Erdős, Hajnal, Máté, Rado [1984, 323] compendium.

In the early 1980's the expectation was that ZFC results like the Galvin-Shelah result [1973] that $\lambda \nrightarrow [\lambda]^2_\omega$ for λ the order type of the reals and $\aleph_1 \nrightarrow [\aleph_1]^2_4$ discussed above would not extend all the way to a GCH result that $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ (see Theorem 17 in [Erdős *et al.*, 1965, 145]). Thus the proof of the partition relation $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ by Todorcevic [1987] was a wonderful shock that introduced a wide audience to the walks on ordinals and the oscillation function.

Todorcevic was a Miller Fellow at the University of California, Berkeley, when in May 1984 his came up with a new proof of the existence of a Countryman line.[375] A key ingredient was the method of walks on ordinals, which Todorcevic devised walking on stairs in one of the libraries on the Berkeley campus, trying to think of a way to attach something finite to a pair of countable ordinals. We think of ladders as a way to climb up. Imagine having a nice ladder system on ω_1, say a $\diamondsuit$-sequence $\langle S_\alpha : \alpha < \omega_1 \wedge \alpha \text{ limit}\rangle$. Descending sequences of ordinals are finite. One can "walk down" from a large δ to a smaller γ in stages $\delta_0 = \delta, \delta_1, \ldots, \delta_k = \gamma$, where at each stage $i + 1$, you move from δ_i to the least ordinal δ_{i+1} in $S_{\delta_i} \setminus \gamma$. But if $\delta = \gamma + \omega$, the set $S_\delta \setminus \gamma$ is a subset of $\{\gamma + n : n < \omega\}$, and if $\gamma \notin S_\delta$, then the recursive definition breaks down in an easily fixable way: from $\eta + 1$, walk to η. Todorcevic called a sequence $\vec{C} = \langle C_\xi : 0 < \xi < \kappa\rangle$ a *c-sequence* if each C_ξ is a closed unbounded subset of ξ. In practice, $C_{\eta+1} = \{\eta\}$ for every successor ordinal. If the cofinal subsets for limit ordinals consist of successor ordinals, then the walk from a limit ordinal to a smaller one has at least one intermediate step.

Todorcevic obtained the partition relation $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ in September 1984, lectured on it in the Berkeley seminar that fall, wrote up the notes of his lectures and circulated them in January 1985.

Todorcevic produced colorings of the pairs of countable ordinals with countable ordinals that had large range on every uncountable set, for three notions of large: contains a club, contains a cobounded set, is all of ω_1. He showed that from one such mapping, one could define mappings of the other kinds, and the same interdefinability was true for mappings from $[\omega_1]^2$ to ω whose range on every uncountable set is all of ω.

To define a function from pairs of countable ordinals into ω that realizes all colors on every uncountable subset, Todorcevic used the *oscillation function* osc : ${}^\omega\omega \to \omega \cup \{\omega\}$ defined by letting $\mathrm{osc}(x, y)$ be the cardinality of the set of all n for which $x(n) \leq y(n)$ and $x(n + 1) > y(n + 1)$. Recall that Galvin [1971] had

[375] Recall that the original construction of a Countryman line by Shelah [1976] also involved construction of an Aronszajn tree. This is not an accident since given any Countryman line L, every countably branching partition tree (development) of intervals under $\subseteq$ whose leaves are singletons must be an Aronszajn tree. See Baumgartner's survey [1982] to be discussed in §8.6.

used an instance of this mapping as discussed in the subsection of §7.3 on square bracket partition relations. Todorcevic [1984a] had used this function to define interesting partitions of nice subsets of the Baire space ${}^{\omega}\omega$. His article [1988b] for the proceedings for the Logic Colloquium 1986 held in Hull is an exposition of the use of the oscillation function, and he proved that on any unbounded subset of the monotonically increasing functions in ${}^{\omega}\omega$ linearly ordered under eventual domination attains all possible values.

In his concluding remarks, Todorcevic [1987, 287] wrote: "The key idea of our coloring can roughly be stated as follows: If the set of interesting places is stationary then in any unbounded set we can find $\alpha < \beta$ such that walking from β to α along the C_ξ's we pass through at least one interesting place." He pointed out that several papers had already been written using the methods of his circulated notes including [Hajnal *et al.*, 1987], [Shelah, 1988b], [Shelah, 1990a], and [Shelah and Steprāns, 1987] (see [Todorcevic, 1987, 288]). The applications of the ideas in this seminal paper are manifold; see the densely packed book by Todorcevic [2007b] for some of them.

Shelah, in a paper entitled *Was Sierpiński right? I* [1988b] built on the work of Todorcevic to show the consistency of $2^{\aleph_0} \to [\aleph_1]^2_3$ relative to the existence of an Erdős cardinal. Shelah's result answered a question of Erdős and Hajnal in the negative: the Sierpiński partition $2^{\aleph_0} \not\to [\aleph_1]^2)2$ cannot be generalized from two colors to three colors in ZFC.

Shelah [1989] proved that it is consistent that for every order type θ and cardinal κ, there is an order type ψ such that $\psi \to [\theta]^2_{\kappa,2}$.

8.3 Structural partition relations

In a paper dedicated to the celebration of the 80th birthday of Deuber, Rado [1986] published a new proof of the the Canonical Ramsey Theorem for $[\mathbb{N}]^n$, showing that it is a corollary of the original Ramsey Theorem. Deuber [1989] surveyed results on partition regular linear systems that flowed from Rado's dissertation [1933], keeping this research area alive. Interesting generalizations to other types of partitions and the equivalence relations arising from them were proved, but because they tend to be difficult to state, no others will be included in this history.

Confirming a conjecture of Galvin [1975],[376] Todorcevic [1981b], [1985a] simultaneously generalized the Baumgartner-Hajnal theorem that for linear orders, $\varphi \to (\omega)^1_\omega$ implies $\varphi \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and $k \in \omega$, and the Galvin [1975] partial order result that $P \to (\eta)^1_\omega$ implies $P \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and $k \in \omega$.

> *Todorcevic Poset Partition Theorem:* $P \to (\omega)^1_\omega$ implies $P \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and $k \in \omega$.

The first step in the proof was a reduction from partition problems for partial orders $(P, <_P)$ to partition relations for trees via passage to one of σP and $\sigma^* P$,

[376] In a brief note, Todorcevic [1985a, 4] thanked Galvin for many valuable communications concerning the problems listed in Galvin's paper [1975].

where σP is the collection of all increasing maps from ordinals into P and $\sigma^* P$ is the collection of maps from σP whose domain is a successor ordinal. Both σP and $\sigma^* P$ are trees under end-extension. The operation σP was first considered by Kurepa [1935], who observed that σP is a tree and constructed his first Aronszajn tree as a subtree of $\sigma\mathbb{Q}$. The hypothesis $P \to (\omega)^1_\omega$, when $P = T$ is a tree, becomes the condition that T is non-special. The second step consisted of showing that for non-special trees T of cardinality less than the pseudointersection number $\mathfrak{p}$, the desired partition relation holds. The final step is elimination of the hypothesis on the cardinality of the trees via a forcing argument.

The paper includes proofs of generalizations of classical partition relations to higher cardinals, trees and partial orders. For example, the Dushnik-Miller Theorem for $\aleph_1$ is generalized to non-special Aronszajn trees: non-special tree $\to$ $(\text{non-special tree}, \omega+1)^2$. Also, the classical partition theorem $\mathfrak{c}^+ \to (\omega_1+2)^2_{\aleph_0}$ is generalized, for partial orderings P satisfying $P \to (\omega_1)^1_{\mathfrak{c}}$, to $P \to (\omega_1+1)^2_{\aleph_0}$.

In response to a question by Baumgartner, Laver [1984] generalized HL_d for finite d to HL_ω, a perfect subtree version of the Halpern-Läuchli Theorem for infinitely many trees. The notation here was introduced in §6.1.

> *Laver HL_ω Theorem:* If $\vec{T} = \langle T_i : i < \omega\rangle$ is a sequence of rooted finitely branching perfect trees of height ω and $\prod^\omega \vec{T} = G_0 \cup G_1$, then there are $j < 2$, an infinite subset $A \subseteq \omega$, and downwards closed perfect subtrees T'_i of T_i for $i < \omega$ with $\prod^A T'_i \subseteq G_i$.

In addition to proving this theorem, Laver showed that it is equivalent to the following two statements:

1. If $\mathbb{P}_\kappa$ is the partial ordering for adding κ side-by-side Sacks reals with countable supports, then every subset of ω in the extension contains or is disjoint from some infinite subset of ω in the ground model.

2. If $\langle f_i : i < \omega\rangle$ are continuous functions from the Hilbert cube $[0,1]^\omega$ into $[0,1]$, then there exist non-empty perfect sets $P_i \subseteq [0,1]$ for $i < \omega$ and an infinite subset $A \subseteq \omega$ such that, on $\prod^A P_i$, $\langle f_i : i \in A\rangle$ is monotonic (and uniformly convergent).

Continuous colorings were examined. After noting that an open coloring of pairs from X can be regarded as a continuous function on $X \times X$ (minus the diagonal), Abraham, Rubin and Shelah observed [1985, 153] that "it seems thus natural to examine these partition theorems for general continuous functions. We did not investigate these questions thoroughly ..." and concluded with a theorem which we omit. Suppose X and Y are topological spaces and X is Hausdorff and τ is a topology on X^n.[377] Let $D(X) = \{(x_0, x_1, \ldots, x_{n-1} : (\exists i < j < n)(x_i = x_j)\}$. A coloring $c : [X]^n \to Y$ is *continuous* if, when viewed as a symmetric function

[377] A natural topology τ on X^n is the topology generated by the sets $[O_0, O_1, \ldots, O_{n-1}] = O_0 \times O_1 \times \cdots \times O_{n-1}$ where $\{O_0, O_1, \ldots, O_{n-1}\}$ is a family of pairwise disjoint open subsets of X.

on $X^n \setminus D$, it is continuous with respect to the topology τ on X^n. We are most interested when X is a subset of the reals or a related topological space, e.g. $\mathbb{Q}$, Baire space ω^ω, and Cantor space 2^ω. For colorings c of pairs of reals, being continuous means that to determine a finite amount of information about $c(x, y)$ one only needs a finite amount of information about x and about y. Todorcevic [1985b] proved the existence a continuous coloring witnessing $\aleph_1 \to [\aleph_1]_{\aleph_0}$,[378] strengthening a result of Galvin and Shelah [1973].

Furstenberg [1981] investigated applications of recurrence in ergodic theory and topological dynamics to combinatorics and number theory, giving proofs of multi-dimensional versions of van der Waerden's Theorem on monochromatic arithmetic progressions, Szemerédi's theorem on arbitrarily long arithmetic progressions in sequences of integers with positive density, Hindman's Finite Sums Theorem and Rado's generalization of van der Waerden's Theorem. Taylor [1982] gave a combinatorial version of the Furstenberg-Weiss topological proof of van der Waerden's Theorem.

For a rooted finitely branching tree T without treetops, Milliken [1981] introduced a tree topology on the family $\mathrm{Str}^\omega(T)$ of strongly embedded subtrees of T. He generalized Silver's theorem to prove what we will call *Milliken's Complete Ramseyness for Trees Theorem* by showing that if $R \subseteq \mathrm{Str}^\omega(T)$ is analytic in the tree topology on $\mathrm{Str}^\omega(T)$, then it is completely T-Ramsey, i.e. there is a strongly embedded subtree S of T such that the collection of its strongly embedded subtrees is contained in or disjoint from R. He generalized Ellentuck's approach to Silver's theorem and used his own work on strongly embedded subtrees of finite height [1979].

In [1984], Timothy Carlson[379] and Simpson proved *dual Ramsey theorems*, i.e. Ramsey theorems about the set of all partitions. For $k \leq \omega$, let $(\omega)^k$ be the set of all partitions of ω into k parts. Their *Dual Ramsey Theorem* is the statement that if $k < \omega$, $(\omega)^k = C_1 \cup \cdots \cup C_n$ and each of the C_i's is Borel, then there is some i and some k-cell partition X such that all its coarsenings lie in C_i, where the relevant topology is the product topology on $2^{\omega\times\omega}$ and the partitions are regarded as equivalence relations on ω, i.e. subsets of $\omega \times \omega$. Their *Dual Galvin-Prikry Theorem* is obtained by setting $k = \omega$. Their *Dual Ellentuck Theorem* states that a subset D of $(\omega)^\omega$ is *Ramsey* if and only if it has the property of Baire and D is *Ramsey null* if and only if it is meager, where the relevant topology is the Dual Ellentuck topology.

At the end of the introduction to their paper they include some historical remarks which indicate that Simpson originated the idea that dual Ramsey theorems should be provable with a series of conjectures in August 1981 after conversations with Leeb,[380] which became their Theorem 2.2 (a generalization of the Dual Ram-

[378] See J. Moore [2000] for the attribution of this continuous coloring result to Todorcevic [1985b].

[379] In 1978, Carlson earned his doctorate at the University of Minnesota, Minneapolis where Prikry was his advisor. Carlson is now a professor at the Ohio State University.

[380] Recall that in §7.5 the discussion following the statement of the Parameter Sets Theorem included the Graham and Rothschild [1974] report that Leeb was the one who looked at the

sey Theorem), the Dual Ellentuck Theorem, and two more theorems more technical than I have chosen to state. Simpson was able to prove the Dual Ramsey Theorem in the special case in which the partitions have exactly three cells and his proof used Hindman's Finite Sums Theorem. Simpson also isolated the combinatorial core (Lemma 2.4), which we here call the Key Lemma, that would suffice to prove these theorems.[381] The introductory sections lists the following theorems as Simpson's "chief" sources of inspiration: the Galvin-Prikry Theorem, The Graham-Rothschild Parameter Sets Theorem, and the Paris-Harrington Theorem [1977]. Simpson's reductions of these theorems to the Key Lemma are included in Sections 2–5. Section 2 presents the proof of the Dual Ramsey Theorem from the Key Lemma. (See also the contemporaneous independent work by Prömel and Voigt [1985a] which strengthens the Carlson-Simpson Lemma 2.3 from Borel partitions to partition's with the property of Baire.) Section 3 gives corollaries of the Dual Ramsey Theorem, including Ramsey's Theorem, the finite version of the Graham-Rothschild Parameter Sets Theorem, the Laver HL_d Theorem and the Laver HL_ω Theorem; the first two presumably were found by Simpson, and the last two were explicitly found by Carlson. Section 4 introduces the dual Ellentuck topology, presents the proof of the Dual Ellentuck Theorem from the Key Lemma, and derives the Ellentuck Theorem as a corollary. Section 5 introduces dual Mathias forcing and results proved with it. Section 6 is devoted to the proof of the Key Lemma due to Carlson, who learned about Simpson's work in July 1982 when they met at the AMS Recursion Theory Institute at Cornell University and at an American Mathematical Society Meeting held in Toronto in August 1982, and found his proof shortly thereafter along with a stronger version which appears as Theorem 6.3.

Before we state the Key Lemma we need to introduce some additional definitions. For a finite alphabet A and some $\alpha \leq \omega$, an *A-partition of α* is a collection of pairwise disjoint, non-empty *blocks* whose union is all of $A \cup \alpha$ and such that each block has at most one element of A. The blocks with no element of A are called *free*. If $\alpha < \beta \leq \omega$ and X is an A-partition of β, let $X[\alpha] = \{x \cap (A \cup \alpha) : x \in X\} \setminus \{\emptyset\}$. Observe that $X[\alpha]$ is an A-partition of α. If X is an A-partition of ω, let X_A^* denote the collection of all $s \cup \{\{n\}\}$ such that $0 < n < \omega$, s is an A-partition of n with no free blocks and $s = Y[n]$ for some A-partition Y of ω which is a coarsening of X. With this notation in hand, we state Lemma 2.4 of [1984, 270]:

> *Carlson-Simpson Key Lemma:* Let A be a finite alphabet. If Y is an A-partition of ω and $Y_A^* = C_0^* \cup \cdots \cup C_{\ell-1}^*$, then there exists an A-partition Z which is a coarsening of Y such that $Z_A^* \subseteq C_i^*$ for some $i < \ell$.

The statement of the stronger Theorem 6.3 proved by Carlson in their [1984] requires the following notation. Let $\langle\omega\rangle_A^\omega$ denote the family of A-partitions for

Ramsey property for categories.

[381] The conjecture that became Lemma 2.4 is described as "an infinitary Hales-Jewett [13] type conjecture" and the reference [13] is to [Hales and Jewett, 1963].

which any two free blocks x and y satisfy $\max x < \min y$ or $x = y$ or $\max y < \min x$. Theorem 6.3 states that for a finite alphabet A, if $(\omega)^*_A = C_0 \cup \cdots \cup C_{\ell-1}$, then there is some $X \in \langle\omega\rangle^\omega_A$ such that $(X)^*_A \subseteq C_i$ for some i. In Section 6 the authors announce what they call *Carlson's Theorem:* A set $C \subseteq \langle\omega\rangle^\omega_A$ is Ramsey if and only if it has the property of Baire with respect to the Ellentuck topology on $\langle\omega\rangle^\omega_A$. They report that Carlson found the proof of this theorem in October 1982 along with a still strong result which gives a common generalization of the key lemma, Hindman's Finite Sums Theorem and the work of Milliken [1975a] and Taylor [1976], and that Carlson has derived results for this space analogous to the Dual Ramsey Theorem.[382] A note added in proof records the March 1983 verification by Carlson of a conjecture by Simpson on finite Borel colorings of k-dimensional affine subspaces of the infinite dimensional space F^ω over a finite field F which appears in [Carlson, 1987].

Carlson [1988] looked at the series of Ramsey results about partitions from the more abstract perspective of *Ramsey spaces*, which he describes as structures which satisfy an analog of Ellentuck's Theorem. A *Ramsey space* is a space which is a collection of infinite sequences with a topology such that every set which has the property of Baire is Ramsey and no open set is meager. In Section 2, he sets up a framework for this study; discusses the notion of *finitization* which is a sequence of approximations; and introduces a neighborhood base defined from the approximations, which allows him to prove critical topological closure results in his reduction of the question of whether a topological space is a Ramsey space to a more combinatorial question. He introduces axiom-like statements (see page 125 and statements A1-A3) and proves an abstract Ellentuck Theorem. His approach generalizes a large part of quantitative Ramsey theory. Hindman and McCutcheon [2002, 2559] single out Carlson's Lemma 5.9: "Experience suggests that Carlson's 'main lemma' ... implies those Ramsey-theoretic corollaries of his theorem in which a finite collection of finite objects is partitioned into finitely many classes."

During the 1982 Prague Symposium on Graph Theory, several people suggested collecting papers to showcase the new Ramsey theory of the time. Nešetřil and Rödl took up the challenge, wrote an extended introduction, solicited and selected appropriate papers and edited the volume *Mathematics of Ramsey Theory* [1990]. This warmly recommended survey includes Ramsey theory from a wide variety of perspectives including many not touched upon in this chapter. Of particular interest are the paper by Weiss titled *Partitioning Topological Spaces* and the paper [1990] in which Carlson and Simpson survey topological Ramsey theory, i.e. work related to their seminal paper [1984], the two Carlson papers discussed above [1987], [1988] and related work of Prömel and Voigt, including some of their canonical partition theorems [1983], [1985b]. Carlson and Simpson define a Ramsey space to be a partial ordering with approximations which satisfies an Ellentuck type theorem. There is a list of eight assumptions, A1–A8. Theorem 1 (due to Carlson) states that a partial ordering with approximation which satisfies

[382] Wolfsdorf [1988] in an article received by the editors January 28, 1983, independently proved some results of the kind found in Section 6 of the Carlson and Simpson paper [1984].

A1–A7 is a Ramsey space if and only if it satisfies A8. The definition of partial order with approximations and eight conditions are somewhat technical to state, so we have omitted them, but the authors assert on page 175 that "Generally, when proving a particular partial ordering with approximations is a Ramsey space assumptions A1–A7 are immediate." They describe Theorem 1 as an abstract Ellentuck Theorem. They discuss finitary consequences, finite dimensional analogs of Ellentuck type theorems, and canonical partitions as well.

Arnold Miller [1989] investigated Borel versions of infinite combinatorial theorems, and proved a parameterized version of the Galvin-Prikry Theorem.

Many more results were found in the 20th Century, but are beyond the scope of this chapter. For a detailed look at infinitary partition relations and Ramsey spaces from a modern perspective, see Todorcevic [2010]. To learn about their applications to Banach space theory, see [Argyros and Todorcevic, 2005] and [Gowers, 2003].

8.4 Partial order, not trees

Kunen and Miller (cf. [Fraïssé, 2000, 164]) showed that the class of denumerable partial orders includes a strictly decreasing ω_1-sequence.

Abraham [1987] revisited the Perles counter-example to Dilworth's chain decomposition theorem holding in the infinite and showed that if a partial order P has no infinite antichain and the length of the well-founded poset of antichains ordered by reverse inclusion has height $< {\omega_1}^2$, then P is the union of countably many chains.

In answer to a question of Galvin, Milner [1982] proved another infinitary analog of Dilworth's theorem by showing that every partially order set P in which every antichain has cardinality less than μ can be decomposed into fewer than $\mu^{<\mu}$ many directed sets.

Todorcevic [1988a] proved that every σ-chain complete poset with the finite cutset property is the union of countably many chains.[383] For context, he listed several earlier papers that studied posets with finite cutsets.

Rado [1981] conjectured that a family of intervals of a linearly ordered set is the union of countably many disjoint subfamilies if and only if every subfamily of size $\aleph_1$ has this property. Todorcevic [1983b] heard about Rado's conjecture, and investigated its non-trivial consistency strength.

8.5 Tree results

Shelah [1984a] proved what we will call his *Cohen Real to Suslin Tree Theorem*: Forcing to add a Cohen real also adds a Suslin tree. Shelah and Stanley [1982]

[383] A partial ordering $(P, <)$ is *σ-chain complete* if every bounded countable chain in P has both a supremum and an infimum. It has the *finite cutset property* if for every $p \in P$, there is a finite set S with $p \in S \subseteq P$ such that S intersects every maximal chain of P and every element of $S \setminus \{p\}$ is incomparable with p.

generalized the result to larger successor cardinals by proving that if $2^\kappa = \kappa^+$ and κ^{++} is inaccessible in L, then there is a κ^{++}-Suslin tree. Velleman proved that if κ is a regular limit cardinal, $2^{<\kappa} = \kappa$, and κ^+ is a regular limit cardinal in L, then forcing to add a Cohen subset of κ also adds a κ^+-Suslin tree. In his Theorem 1.11, Velleman proved that if there is a simplified $(\kappa, 1)$-morass with a complete amalgamation system, then there is a κ^+-Suslin tree. Velleman proved in [1984a] that a simplified $(\omega, 1)$-morass can be constructed in ZFC and in [1984b] that the existence of a $(\kappa, 1)$-morass for regular uncountable κ is equivalent to the existence of an ordinary $(\kappa, 1)$-morass. In Theorem 2.1 of [1984c], he proved that if κ is a regular cardinal and $2^{<\kappa} = \kappa$, then forcing over a model with a simplified $(\kappa, 1)$-morass to add a Cohen subset of κ also adds a complete amalgamation system to the simplified $(\kappa, 1)$-morass. With these results in hand, he derived as corollaries Shelah's Cohen Real to Suslin Tree Theorem and the Shelah-Stanley results quoted above. In the introduction he commented that his proof of the former is a "modified form of a proof by Mark Bickford,[384] and his proof of the later was based on Shelah and Stanley's original proof.

Todorcevic [1987] gave a different proof of Shelah's Cohen Real to Suslin Tree Theorem. Call a function $e : [\omega_1]^2 \to \omega$ a *coherent mapping* if for all $\alpha < \beta < \omega$, there are only finitely many ξ with $e(\xi, \alpha) \neq e(\xi, \beta)$. Note that in this case, the restriction of the function $e(\cdot, \beta)$ to α is almost equal to $e(\cdot, \alpha)$, in symbols, $e(\cdot, \alpha) =^* e(\cdot, \beta)\restriction\alpha$. It is not difficult to construct a coherent mapping by transfinite induction. Todorcevic observed that if e is a coherent mapping, then the following set is an Aronszajn tree under $\sqsubseteq$:

$$T(e) := \{e_\beta\restriction\alpha : \alpha \leq \beta < \omega_1\}.$$

Note that if e is a coherent mapping, then the *stretching* $\overline{e} : [\omega_1]^2 \to \omega$ of e which Todorcevic defined on page 271 of [1987] by setting

$$\overline{e}(\alpha, \beta) = 2^{e(\alpha,\beta)}(1 + 2|\{\xi < \alpha : e(\xi, \alpha) = e(\xi, \beta)\}|)$$

is coherent and each $\overline{e}_\beta$ is one-to-one. He used such a map to prove the following [1987, 292–293]:

> *Todorcevic's Cohen Real to Suslin Tree Construction:* If $r : \omega \to \omega$ is a Cohen real and $e : [\omega_1]^2 \to \omega$ is a coherent mapping such that each e_β is one-to-one, then $T(r \circ e)$ is a Suslin tree.

The most common constructions of κ-Aronszajn trees to that point had been subtrees of the collection of functions ${}^{<\kappa}L$ for some linear order L under end-extension. Todorcevic [1987] associated various *rho functions* or *characteristics* with the walks on ordinals discussed in the subsection on partition properties. He proved several "metric" properties of his rho functions, and used them to define trees whose properties could be described in terms of features of the rho functions

[384] Mark Bickford received his 1983 doctorate from the University of Wisconsin, Madison, where his advisor was Terrence Millar.

from which they came. Define the *right lexicographic ordering of* ${}^{<\kappa}L$ by $s <_0 t$ if and only if s properly end-extends t or for the least α on which s and t disagree, $s(\alpha) < t(\alpha)$. Define the *left lexicographic ordering of* ${}^{<\kappa}L$ similarly by $s <_1 t$ if and only if t properly end-extends s or for the least α on which s and t disagree, $s(\alpha) < t(\alpha)$. Call an arbitrary tree T of height ω_1 a *coherent tree* if it can be represented as a downwards closed subtree of some tree of the form $T(a)$, where $a : [\omega_1]^2 \to \omega$ is a coherent mapping.[385]

> *Todorcevic's Coherent Tree Theorem:* Suppose that $e : [\omega_1]^2 \to \omega$ is a coherent mapping with each e_β finite-to-one. Further suppose that $\vec{C} = \langle C_\alpha : \alpha < \omega \rangle$ is a c-sequence and ρ_0 and ρ_1 are defined from C. Then $T(e)$, $T(\rho_0)$ and $T(\rho_1)$ are all coherent Aronszajn trees, $T(\rho_0)$ is special, and $T(\rho_0, <_0)$ and $T(\rho_1, <_1)$ are Countryman lines.

Here $\rho_0 : [\omega_1]^2 \to {}^{<\omega}\omega$ is recursively defined by

$$\rho_0(\alpha, \beta) = \langle \mathrm{ot}(C_\beta \cap \alpha) \rangle ^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha),$$

and $\rho_1 : [\omega_1]^2 \to \omega$ is recursively defined by

$$\rho_1(\alpha, \beta) = \max\{\mathrm{ot}(C_\beta \cap \alpha), \rho_1(\alpha, \min(C_\beta \setminus \alpha))\}.$$

The definitions of ρ_0 and ρ_1 extend to larger successor cardinals. When working with walks on cardinals larger than ω_1, it is useful to have c-sequences with coherence properties like those of $\Box_\kappa$-sequences. Todorcevic used his new approach to revisit some results on large cardinals and about combinatorial properties of L. For example, he generalized the notion of special tree to limit cardinals as follows. Suppose $(T, <_T)$ is a tree of height κ. A function $f : T \to T$ is *regressive* if for all non-minimal $t \in T$, $f(t) <_T t$. T is *special* if there is a regressive function $f : T \to T$ such that for every $t \in T$, $f^{-1}(\{t\})$ is the union of fewer than κ many antichains. Todorcevic [1987, 266] proved that among inaccessible cardinals κ, there are no special κ-Aronszajn trees if and only if κ is Mahlo.

Laver [1987] proved that Shelah's Cohen Real to Suslin Tree Theorem does not extend to forcing with random reals. By adding many random reals to a universe satisfying $\mathrm{MA}_{\aleph_1}$, Laver obtained a model of "$2^{\aleph_0}$ is singular with cofinality ω_1" with no Suslin tree.

In chapter IX of [1982b], Shelah came up with some new proper forcing techniques for specializing an Aronszajn tree, including one that did not add reals, and some that specialized an Aronszajn tree on a stationary set but not on the rest of the tree. Thus he was able to build a model with no Suslin trees without making all Aronszajn trees special.

Several consistency results for the existence or non-existence of Aronszajn trees on $\aleph_n$'s were proved in the 1980's. Note that the non-existence of a κ-Aronszajn

[385] The definition of coherent tree given here is from Todorcevic [2007b].

tree for regular κ means that κ has the *tree property*, i.e. every κ-tree has a cofinal branch.

Abraham [1983] used a supercompact cardinal with a weakly compact cardinal above to show the consistency of having no $\aleph_2$-Aronszajn trees and no $\aleph_3$-Aronszajn trees. Included in his paper is a proof by Magidor that if there are no $\aleph_2$-Aronszajn trees and $\aleph_3$ is inaccessible in L (as would be the case if there were no $\aleph_3$-Aronszajn trees), then $0^\#$ exists, and that two weakly compact cardinal would be insufficient.[386]

Shelah [1982a] generalized bqo theory to uncountable cardinals and identified the least cardinal κ such that the class of unions of $\leq \lambda$ scattered orderings is κ-well-ordered (a generalization of wqo) and called these cardinals *beautiful cardinals.* He also gave a construction for rigid families of colored trees, and generalized the method to other structures.

Laver and Shelah [1981], starting from a weakly compact cardinal, used iterated forcing to construct a model of CH in which there are no $\aleph_2$-Suslin trees.

Baumgartner [1983b], as an application of Axiom A forcing,[387] proved that if the existence of an inaccessible cardinal is consistent, then so is MA + $2^{\aleph_0} = \aleph_2$ + every $\aleph_1$-tree has at most $\aleph_1$ uncountable branches.

Shelah and Stanley [1988] proved that a successor cardinal $\kappa = \lambda^+$ has a non-special Aronszajn tree when either $\square_\lambda$ holds or κ is not weakly compact in L.

Shai Ben-David and Shelah [1986] proved the consistency, relative to the existence of a supercompact cardinal, of the existence of λ^+-Suslin trees for every singular λ. Under the same assumption, they proved the consistency of the existence of an $\aleph_{\omega+1}$-Suslin tree together with the non-existence of any special $\aleph_{\omega+1}$-Aronszajn tree.

Chaz Schlindwein [1989], starting from an inaccessible cardinal, constructed a model of CH + SH (no Suslin trees) + KH (there is a Kurepa tree) in which not all Aronszajn subtrees are special.[388]

Masazumi Hanazawa [1982] defined a non-Suslin base for an Aronszajn tree $(T, <_T)$ to be a collection C of uncountable antichains such that for any uncountable $S \subseteq T$ there is an $X \in C$ such that for all $s \in X$ there is a $t \in S$ with $s \leq_T t$. Note that if T has a non-Suslin base, no subtree of T is Suslin. Hanazawa used Martin's Axiom to prove that no Aronszajn tree has a non-Suslin base of cardinality less than the continuum. He used $\diamondsuit^+$ to prove that there is a special Aronszajn tree which has no base of size $\aleph_1$, and asked whether $\diamondsuit^+$ was necessary.

Baumgartner [1985] defined *base for a tree* T of height ω_1 to be a collection $\mathcal{B}$ of subtrees of T with the property that every subtree of T contains one of the subtrees in $\mathcal{B}$. Note that the antichains used by Hanazawa can be turned into subtrees by downward closure, so the notion of Baumgartner generalizes the notion

[386]Recall Mitchell and Silver [Mitchell, 1972] showed that non-existence of $\aleph_2$-Aronszajn trees and $\aleph_4$-Aronszajn trees is equiconsistent with the existence of two weakly compact cardinals.

[387]Axiom A forcings are proper and were a precursor to proper forcings.

[388]In his Math Review MR1031772 (91i:03101) of this article, Di Prisco commented that models of ZFC exist satisfying every combination of the form $\pm$ CH $\pm$ SH $\pm$ KH $\pm$ EATS that are not ruled out by the fact that EATS (every Aronszajn tree is special) implies SH.

of Hanazawa. Baumgartner answered Hanazawa's question by showing that, in the model obtained from the Levy collapse of an inaccessible cardinal to ω_2, every Aronszajn tree has a base of size $\aleph_1$ and $\diamondsuit$ holds. He defined an anti-Suslin tree to be a tree with no Suslin subtree and called a Specker order (also called Aronszajn line) *anti-Suslin* if it has no Suslin suborder. Such orderings arise from anti-Suslin trees. From the above theorem and the equivalence of bases with non-Suslin bases for anti-Suslin trees, the following conclusion may be drawn: if it is consistent that there is an inaccessible cardinal, then it is consistent that for every anti-Suslin Specker order S there is a collection C of subsets of S such that C has cardinality $\leq \aleph_1$ and every uncountable subset of S contains an order-isomorphic copy of an element of C.

Moreover, Baumgartner showed that if there is a Kurepa tree with at least κ branches, then there is a special Aronszajn tree for which every non-Suslin base has cardinality at least κ. The proof he gave, due to Todorcevic and included with his permission, is the observation that the level-wise product of a Kurepa tree $(K, \leq_K)$ with κ branches and a special Aronszajn tree $(T, \leq_T)$, namely $KT = \{(s,t) : s \in K, t \in T, \text{ and } \ell(s) = \ell(t)\}$,[389] under the coordinate-wise ordering is the required special Aronszajn tree in which every base has cardinality $\geq \kappa$. Baumgartner asked whether it is consistent that both CH holds and no Aronszajn tree has a base of cardinality $\aleph_1$.

Abraham and Shelah [1985][390] investigated Aronszajn trees and embeddings of their subtrees, particularly those subtrees obtained by choosing all nodes from a closed unbounded (club) set of levels. They strengthened some of the notions of structure considered earlier. Recall that in [Shelah, 1976], two Aronszajn trees were said to be *near* if for some club $C \subseteq \omega_1$ they have Aronszajn subtrees which are isomorphic when restricted to C. Abraham and Shelah called S and T *club isomorphic* if for some club C, the subtrees $S{\restriction}C$ and $T{\restriction}C$ are isomorphic, and said S is *club embeddable* in T, if for some club C, $S{\restriction}C$ embeds in $T{\restriction}C$. Say that two Aronszajn trees are *really different* if for every club $C \subseteq \omega_1$ their restrictions to C are non-isomorphic. They deemed an Aronszajn tree to be *really rigid* if for any club C its restriction to C has no order automorphisms other than the identity. They proved a series of consistency results using forcing and cardinality considerations:

1. It is consistent to have GCH + there is no Suslin tree, every two Aronszajn trees are near and there is an Aronszajn tree U universal in the sense that every Aronszajn tree is club embeddable in U.

2. If $2^\omega < 2^{\omega_1}$, then there are 2^{ω_1} really rigid pairwise really different Aronszajn trees.

3. If $2^\omega < 2^{\omega_1}$, then there is no minimum Aronszajn tree P in the sense that P is club embeddable into every Aronszajn tree.

[389]Here $\ell(s)$ is the level of s.
[390]The paper was received July 1, 1983; an abstract for it dates back to 1979.

4. It is consistent to have GCH + "there are exactly ω_1 non-isomorphic Suslin trees and there is an Aronszajn tree U which is almost universal in the sense that every Aronszajn tree without Suslin subtrees is club embeddable in U."

5. MA + $\neg$CH is independent of "every two Aronszajn trees are isomorphic on a club".[391]

6. It is consistent to have MA + "2^ω is as large as you like and no two Aronszajn trees are really different."

They asked if the existence of a Suslin tree implied the existence of two really different Suslin trees [1985, 78], and they ask a basis problem [1985, 79]:

Problem. It is not clear whether we can get the consistency of:

(*) Any Specker order contains a suborder in K_s'.[392]
This is equivalent to the following consistency question.

(**) If T is an Aronszajn tree, $T = A_1 \cup A_0$, then there is an unbounded $B \subseteq T$ and $\ell \in \{0,1\}$ such that $x, y \in B \implies \text{g.l.b.}\{x,y\} \in A_\ell$.[393]

Ingrid Lindström,[394] in a paper based on her 1979 Stanford dissertation, investigated a pre-order $\leq$ defined on rooted, Hausdorff, well-pruned $\geq$ 2-full ω_1-trees by $S \leq T$ if and only if for some closed unbounded set $C \subseteq \omega_1$ there is an order-preserving and level-preserving mapping carrying the restriction of T to levels indexed by C, briefly $T{\restriction}C$ to $S{\restriction}C$. In the language of Abraham and Shelah, $S \leq T$ if T is club-embeddable in S. Intuitively, the relation $S \leq T$ holds for Suslin trees if forcing with T adjoins a branch through S. She points out that the equivalence classes (degrees) of Aronszajn trees under $\leq$ with the level-wise product and disjointed union with added root operations forms a distributive lattice, and uses $\diamondsuit$ to prove that all countable linear order types embed in it and that there is no maximal Aronszajn degree. Recall that Jech [1972] had used $\diamondsuit$ to proved the existence of $2^{\aleph_1}$ pairwise non-isomorphic Suslin trees. Lindström uses $\diamondsuit$ to proved the existence of $2^{\aleph_1}$ pairwise $\leq$-incomparable of several types of ω_1-trees: Suslin trees, special Aronszajn trees, non-special $\mathbb{R}$-embeddable Aronszajn trees.

[391] Shelah proved the consistency relative to Martin's Axiom for stable posets (see his paper with Abraham [1982] for the definition of stable). Then Abraham used generic reals to extend the result to the full Martin's Axiom. Shelah constructed a proper forcing with which to force two Aronszajn trees to be isomorphic on a club, using finite conditions. In §5 where the construction is given, the authors [1985, 100] remark "At the present state of knowledge, we do not know how to go with proper forcing beyond $2^{\aleph_0} = \aleph_2$".

[392] K_s' is the collection of the Countryman lines, so (*) may be rephrased as "every Aronszajn line embeds a Countryman line."

[393] The ordering of the indices $T = A_1 \cup A_0$ is the way it appears in [Abraham and Shelah, 1985]. The statement (**) is called the *Coloring Axiom for Trees* or CAT in [Todorcevic, 2007a]. In current usage one writes $x \wedge y$ for greatest lower bound of x and y.

[394] Ingrid Birgitta Lindström received her doctorate from Stanford University in 1979 under the direction of Ketonen. She is currently a senior lecturer at Uppsala University.

8.6 Linear orders

Baumgartner [1980] used iterated forcing to build a model of Martin's Axiom + $2^{\aleph_0} = \aleph_2$ in which (A) every uncountable subset of $\mathcal{P}(\omega)$ contains an uncountable chain or antichain and (C) any two $\aleph_1$-dense sets of reals are order-isomorphic. Then he proved that (A) + (C) implies that every uncountable Boolean algebra has an uncountable antichain.[395]

Abraham and Shelah [1981] answered a question of Baumgartner [1973] (see §7.8 for the statement of Baumgartner's question) when they proved that Martin's Axiom does not imply that all $\aleph_1$-dense sets of reals are isomorphic.

At the 1981 conference on ordered sets, Pouzet (cf. [Bonnet and Pouzet, 1982]) gave a series of lectures on what he called the most popular connection between partial orders and linear orders, namely that every partial order can be extended to a linear order (Szpilrajn Theorem [Marczewski, 1930]). One topic was the question of whether a partial order with a given property P can be extended to a linear order with the same property P or whether all extensions to a linear order will have this property.

At the same conference, Baumgartner [1982] surveyed and extended known results for uncountable linear orders. He reviewed Specker types, and proved that every Specker order is isomorphic to a lexicographically ordered Aronszajn tree and that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-dense Specker types. He highlighted the fact that the Countryman line constructed by Shelah with its converse are incompatible, no uncountable type is below both, and asserted that under the assumption of $\diamondsuit$, for any Aronszajn tree T there is an Aronszajn tree U such that no Aronszajn tree is embeddable in both T and U. He also proved under the stronger assumption of $\diamondsuit^+$, that there is a Specker type φ which is minimal, i.e. if ψ is embeddable in φ then φ is also embeddable in ψ.

Abraham, Matatyahu Rubin and Shelah [1985] presented a variety of consistency results, including the consistency of ZF + $2^{\aleph_0} < 2^{\aleph_1}$ + "for any two $\aleph_1$-dense subsets $A, B \subseteq \mathbb{R}$, there is an $\aleph_1$-dense $C \subseteq \mathbb{R}$ which is order-embeddable in both A and B." They also produced a ZFC example of two $\aleph_1$-dense sets for which the order-isomorphism cannot be taken to be C^1. Let $(K^H, \leq)$ be the set of order types of $\aleph_1$-dense homogeneous subsets of the reals the embeddability relation. They prove that for every finite model $(L, \leq)$, the statement $(K^H, \leq)$ is isomorphic to $(L, \leq)$ is consistent if and only if $(L, \leq)$ is a distributive lattice.

8.7 Other combinatorial results

In the paper just discussed above, Abraham, Rubin and Shelah introduced the *Semi-Open Coloring Axiom* (SOCA) and the *Open Coloring Axiom*, which we here will abbreviate (OCA$_{\mathrm{ARS}}$, along with a variety of other axioms. For a subset $Y \subseteq X$ of a topological space X, let $D(Y) = Y \times Y \setminus \{(y, y) : y \in Y\}$.

[395] Maria Losada and Todorcevic [2000] showed that $\mathrm{MA}_{\aleph_1}$ suffices.

Open Coloring Axiom (OCA_{ARS}): If X is a second countable space of power $\aleph_1$ and $\{U_0, \ldots, U_{n-1}\}$ is a cover of $D(X)$ consisting of symmetric open sets, then there is a partition of $X = \bigcup_{i<\omega} X_i$ such that for every $i < \omega$, there is an $\ell < n$ such that $D(X_i) \subseteq U_\ell$.

The trio of authors proved, relative to ZFC, the consistency of MA + SOCA and moreover the consistency of MA + SOCA + $\neg\mathrm{OCA}_{\mathrm{ARS}}$. They also prove that MA + $\mathrm{OCA}_{\mathrm{ARS}}$ implies $2^{\aleph_0} = \aleph_2$.

The Abraham-Rubin-Shelah paper [1985] was received by the journal March 3, 1982. That spring, Todorcevic [1989] formulated a different open coloring axiom as a proposition using PFA, and I have reformulated it below using coloring instead of partitions.

Open Coloring Axiom (OCA_T): For every coloring $f : [\mathbb{R}]^2 \to \{0, 1\}$ with $f^{-1}(\{0\})$ open, either there is an uncountable 0-homogeneous set Y, or $\mathbb{R}$ is the union of countably many 1-homogeneous sets.[396]

Todorcevic lists some straightforward consequences: (1) every uncountable subset of $\mathcal{P}(\omega)$ contains an uncountable chain or antichain, and (2) every function f from an uncountable set of reals into the reals is monotonic on some uncountable set.[397] In remarks at the end of chapter 8, Todorcevic credits Baumgartner [1973], [1980] with "the main advance towards the Open Coloring Axiom for sets of reals" and continues "the discovery of Baumgartner ... is that a CH diagonalization argument can be used in building [a] c.c.c. poset which forces homogeneous sets for certain open partitions on sets of reals" [Todorcevic, 1989, 86–87].

Moore [2002] proved that $\mathrm{OCA}_{\mathrm{ARS}}$ + $\mathrm{OCA}_{\mathrm{T}}$ implies $2^{\aleph_0} = \aleph_2$.

Aharoni, Nash-Williams and Shelah [1983] devised a general criterion for the existence of transversals. They used *critical families* $\mathcal{F}$, indexed families (functions) with a transversal that have the property that for every transversal φ, the range of φ is the union of the range of $\mathcal{F}$. They use critical families to identify a collection of *obstructions* and prove that a family has a transversal if and only if it does not have a subfamily in the collection of obstructions. In a follow-up, the authors [Aharoni *et al.*, 1984] gave a different criterion based on an idea of [Nash-Williams, 1978] that avoided use of critical families.

Shelah [1985] proved that if S is a stationary co-stationary subset of ω_1, then the Weak Diamond Principle for S does not follow from $2^{\aleph_0} < 2^{\aleph_1}$.

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[396] Moore [2002] used the following reformulation of $\mathrm{OCA}_{\mathrm{T}}$: if X is a separable metric space and $G \subseteq [X]^2$ is open, then either G is countably chromatic (there is a decomposition of X into countably many pieces X_i such that each $X_i \cap G = \emptyset$) or there is $H \subseteq X$ such that $[H]^2 \subseteq G$. That is, either X is the union of countably many independent sets or G includes an uncountable complete subgraph.

[397] The Axiom $\mathrm{OCA}_{\mathrm{T}}$ has applications in analysis, e.g. Ilijas Farah [] used it to establish the consistency of all automorphisms of the Calkin algebra being inner. His paper complements a 2006 construction of Phillips and Weaver [2007] of an outer automorphism using CH. Together these results solve a problem of Brown, Douglas and Fillmore [1977].

9 1990–2000: A SAMPLING

For the final decade of the 20th Century, I am presenting a miscellaneous collection of results I found interesting and make no claims that they are representative for this time. In particular, singular cardinal combinatorics became an increasingly important part of the subject, but it is a subject of which I know little. I warmly recommend the survey article by James Cummings [2005], a recent paper [2009] by Todd Eisworth and his chapter [2010] for the *Handbook of Set Theory*, especially Section 5 on square bracket partition relations,

Alan Dow [1992] provided a valuable update to the *Handbook of Set-Theoretic Topology* [Kunen and Vaughan, 1984] in a survey paper written based on a talk at the Symposium on Topology (Topsym) held in Prague.[398] Another valuable resource that came out of that meeting is the compilation *Open Problems in Topology* [van Mill and Reed, 1990].[399]

In 1998, Hindman and his frequent collaborator, Dona Strauss, made widely available their work on Ramsey theory, especially semi-group colorings, in their book [1998],[400] which includes interesting historical remarks.

Todorcevic [1998a] spoke on *Basis problems in combinatorial set theory* at the 1998 International Congress of Mathematicians in Berlin. Of particular note for this chapter was his discussion of c-sequences and their related distance functions as critical objects for the collection of uncountable subsets of ω_1. The variety of applications is astounding, and his recent book, *Walks on Ordinals and Their Characteristics* [2007b], is an important new resource. Todorcevic focused on five general areas: (1) uncountable subsets of ω_1, (2) binary relations on ω_1 under Tukey reducibility, (3) transitive relations both on ω_1 and on the reals, (4) uncountable linear orderings, and (5) topological spaces. In each case, he identified a list of critical objects and either indicated why it is known to be sufficiently rich that one can relate any object in the class to one of the critical objects or made a conjecture to that effect under the assumption of PFA. He referred readers interested in metamathematical aspects to the 1,000 page book by Woodin, saying that they "will find a satisfactory explanation in the recent monograph of Woodin" (see [Woodin, 1999]).

In his examination of uncountable subsets of ω_1, Todorcevic focused on the delightful fact that many critical objects in families of uncountable structures whose domain is a subset of ω_1 can be defined in terms of a c-sequence.[401] Todorcevic reviewed walks on ordinals, the trace function of places visited on a walk

[398] The Prague Topsym conference meets every five years.

[399] In his Math Review MR1078636 (92c:54001) of the problem book, John Walsh noted "The journal *Topology and its Applications* has agreed to provide a readily available source for determining the status of the various problems. Each issue of the journal will have space devoted to updating the status as progress is made on individual problems. A second book of problems [Pearl, 2007] has followed.

[400] This book was cited in the 2003 JAMS award citation honoring Hindman's work.

[401] For the general audience, Todorcevic omitted the term c-sequence in favor of describing one on ω_1 with minimal jargon as a sequence C which assigns to each ordinal α a subset c_α of smaller ordinals of minimal order type with $\alpha = \sup c_\alpha$.

from β to α, distance functions, ρ_0, and Todorcevic's square bracket function, $[\alpha\beta] = \min(\mathrm{Tr}(\xi, \beta) \setminus \alpha)$ where $\xi = \min\{\zeta : \rho_0(\zeta, \alpha) \neq \rho_0(\zeta, \beta)\}$. He connected an uncountable set $X \subseteq \omega_1$ with one which contains a closed unbounded subset using the square bracket operation (see [Todorcevic, 1987]), namely

$$\{[\alpha\beta] : \alpha \in X \wedge \beta \in X \wedge \alpha < \beta\}.$$

He pointed to the utility of this approach through citation of others' work: the construction by Saharon Shelah and Juris Steprāns [1987] of Ehrenfeucht-Faber groups, that is, extraspecial p-groups G in which each Abelian subgroup has cardinality less than $|G|$; the construction by Shelah and Steprāns [1988] of a Banach space on which every bounded linear operator is the sum of a multiple of the identity operator and an operator with a separable range; and the construction by Erdős, R. Daniel Mauldin, and Steve Jackson [1997] of a family H of ω_1 hyperplanes in $\mathbb{R}^n$, a family X of ω_1 points in $\mathbb{R}^n$ and a coloring $P : H \to \omega$ such that any n hyperplanes of distinct colors meet in at most one point, and there is no coloring $Q : X \to \omega$ such that each hyperplane $h \in H$ meets X in at most $n - 1$ points of its color, $P(h)$.

9.1 Partition calculus results

We split the discussion of partition calculus results into the same subsections as before, with the addition of one for polarized partitions so that this active area of research dating back to the 1950's has some representation in this chapter.

Cardinal resources

The earliest uncountable partition calculus result stated in the literature is the Dushnik-Miller Theorem

$$\kappa \to (\kappa, \omega)^2.$$

Implicit in the original proof and explicit in [Erdős and Rado, 1956] is the relation $\kappa \to (\kappa, \omega+1)^2$. Erdős and Hajnal [1971b, 22] asked in their problem paper arising from the 1967 UCLA set theory conference the following natural question about the limitations on the possible extension of the Dushnik-Miller Theorem:

> *Problem 8:* Can one prove without using the continuum hypothesis that $\omega_1 \nrightarrow (\omega_1, \omega + 2)^2$ holds?

Péter Komjáth [1998], [1999] examined what was known about this foundational theorem, derived some additional ZFC results, and clarified the boundary between what can be proved in ZFC with additional consistency results. Recall that Hajnal [1960] used CH to prove that $\omega_1 \nrightarrow (\omega_1, \omega+2)$ and the proof gave $\omega_1 \nrightarrow (\omega_1, (\omega{:}2))$ as he later pointed out in [Erdős and Hajnal, 1974]. Komjáth [1998] showed these two partition relations were different by forcing to produce a model in which the bipartite partition relation holds but the complete subgraph relation does not:

$\omega_1 \to (\omega_1, (\omega{:}2))^2$ and $\omega_1 \not\to (\omega_1, \omega+2)^2$. Recall that Baumgartner [1976a] proved the consistency of $\kappa \not\to (\kappa, (\kappa, (\omega{:}2))^2$ for every cardinal κ which is the successor of a regular cardinal. Laver [1975] proved that if κ is Mahlo, then one can force to make $\kappa \not\to (\kappa, (\omega{:}2))^2$ true and preserve the fact that κ is Mahlo. Komjáth [1999] proved that if λ is singular, then $\lambda \to (\lambda, (\mu{:}n))^2$ holds for $\mu < \lambda$ and $n < \omega$.

Recall that Todorcevic [1983a] proved that if λ is a cardinal of uncountable cofinality, there is a c.c.c. forcing for adding a witness to $\lambda \not\to (\lambda, \omega+2)^2$. Komjáth [1999] proved that under GCH, if κ is a regular cardinal and λ is a cardinal whose cofinality is greater than κ, then there is a κ-closed, κ^+-chain condition forcing which adds a witness to $\lambda \not\to (\lambda, \kappa+2)^2$. Recall that Erdős and Hajnal [1974] reported that Galvin had proved $\omega_2 \to (\omega_1, \omega+2)^2$ and that he had mentioned as open problems the questions about whether the following hold without the assumption of GCH: Does $\omega_2 \to (\omega_1, \omega+3)^2$? Does $\omega_2 \to (\omega_1, (\omega+2)_2)^2$? Komjáth pointed out that Galvin's proof generalizes to give $\omega_n \to (\omega_1, \omega+n)^2$ for positive $n < \omega$. Komjáth proved that for all $n < \omega$,

$$\omega_3 \to (\omega_1, \omega+n)^2.$$

He also showed that if the resource ω_3 shrinks to ω_2, then it is consistent that

$$\omega_2 \not\to (\omega_1, \omega+3)^2.$$

In a 1997 lecture series at Rutgers University, Shelah commented on a variety of problems he found interesting. A sample is Question 8.6 [2000a, 61] below:

> For which cardinals λ and which ordinals α with $2 \leq \alpha < \lambda$ does the partition relation $\lambda^{++} \to (\lambda^+ + \alpha)^2_{\aleph_0}$ hold? GCH may be assumed for simplicity.

Recall that Erdős and Hajnal [1971b, 23] asked the following natural question about the limitations on possible extensions of the Erdős-Rado Theorem:

> *Problem 10:* Assume GCH. Does then $\omega_{\rho+1} \to (\xi, \xi)^2$ hold for every $\xi < \omega_{\rho+1}$ and every ρ?

Baumgartner, Hajnal and Todorcevic [1993, 5], all of whom had given invited talks at the NATO Advanced Science Institute on Finite and Infinite Combinatorics in Sets and Logic which had been held in Banff in 1991, published their generalization of the Erdős-Rado theorem, now known as the Baumgartner-Hajnal-Todorcevic Theorem, in the conference volume using the notation $\log(\kappa)$ for the least cardinal μ with $2^\mu \geq \kappa$:

> *Balanced Baumgartner-Hajnal-Todorcevic Theorem:* If κ is a regular cardinal and $\lambda = 2^{<\kappa}$, then for $k < \omega$ and $\xi < \log(\kappa)$
>
> $$\lambda^+ \to (\kappa + \xi)^2_k.$$

Unbalanced Baumgartner-Hajnal-Todorcevic Theorem:
If κ is a regular cardinal and $\lambda = 2^{<\kappa}$, then for all $n < \omega$

$$\lambda^+ \to (\kappa^{\omega+2} + 1, \kappa + n)^2.$$

For $\kappa = \omega$, the results follow from the Baumgartner-Hajnal Theorem $\omega_1 \to (\alpha)^2_k$ for $k < \omega$ and $\alpha < \omega_1$, so the main interest is for κ uncountable. The authors note that in unpublished work, Hajnal proved for $\lambda = 2^{<\omega_1}$ that $\lambda^+ \to (\kappa + m)^2_2$ for $m < \omega$ "about thirty years ago" [Baumgartner *et al.*, 1993, 5], and Shelah proved the balanced version for $k = 2$ in his [1973]. The trio of authors used what has come to be called the *elementary submodel method* for their proof, which was different from the approach of Simpson [1970] in his model-theoretic proof of the Erdős-Rado Theorem. Later, Hajnal [2010] commented that this paper was the one which introduced non-reflecting ideals as a tool in the study of partition relations. Baumgartner, Hajnal and Todorcevic make use of elementary substructures of the form $(H(\mu), \in)$ where $H(\mu)$ is the set of all sets hereditarily of cardinality $< \mu$, and ideals on ordinals generated from such elementary substructures. They say "it is possible to recast our arguments in such a way as to fit them under the heading of ramification arguments but we choose not to do so, both because the proofs remain clearer this way, and because we believe that this may generally be a better approach to ramification arguments" [Baumgartner *et al.*, 1993, 2]. They also used the metamathematical approach taken in the proof of the Baumgartner-Hajnal Theorem of using forcing to get the desired result in a generic extension, and then using an absoluteness argument to prove the statement is outright true.

Milner [1994] emphasized the utility of using elementary substructures in an expository article designed to encourage more people to use this approach, in place of the ramification arguments of many earlier proofs of combinatorial results.

For the proceedings of the Hajnal conference of October 1999, Baumgartner wrote about the use of ideals in arguments involving elementary substructures, which were a feature of Hajnal's work. He started with a treatment of the Erdős-Rado Theorem using the non-stationary ideal. In this paper, he attributed to Hajnal the following unbalanced extension of the Erdős-Rado Theorem: for regular uncountable κ, all ξ with $2^\xi < \kappa$, and all $k < \omega$, the following partition relation holds:

$$(2^\kappa)^+ \to (\kappa + \xi)^2_k\,,$$

whose proof appeared in [Baumgartner *et al.*, 1993]. Baumgartner revisited the proof using many elements of the proof from that paper, and the perspective gained from nearly a decade.

Todorcevic [1998b] proved a common generalization of the Baumgartner-Hajnal Theorem for ω_1 [1973] and Prikry's Theorem [1972] when he showed that for all finite k and countable ordinals α the following partition relation holds:

$$\omega_1 \to ((\alpha; \omega_1), (\alpha)_k)^2.$$

That is, if $[\omega_1]^2$ is partitioned into $k+1$ pieces, then either the first piece includes $\{\{\xi,\eta\}_< : \xi \in A \wedge \eta \in A \cup B\}$ for some A of order-type α and some uncountable B or one of the other pieces includes $[A]^2$ for some A of order-type α.

At the end of the decade, Matthew Foreman[402] and Hajnal [2003] were writing up their results on partition properties of large successor cardinals, which gave further insight into possibilities for Problem 10. They showed that if the Continuum Hypothesis holds and there is an $\aleph_1$-dense ideal on ω_1, then, for $\alpha < \omega_2$,

$$\omega_2 \to ({\omega_1}^2 + 1, \alpha)^2.$$

At the International Congress of Mathematicians held in Berlin, Foreman [1998b] spoke on *Generic large cardinals: new axioms for mathematics?* In part of his talk, he discussed his work with Hajnal and pointed out that the assumption of the existence of an $\aleph_1$-dense ideal on $\aleph_2$ implies both hypotheses they used to prove their positive partition relation holds.[403] Note that Foreman [1998a] had shown the consistency of the existence of such an $\aleph_1$-dense ideal on ω_2 as well as the consistency of the existence of an $\aleph_1$-dense ideal on ω_1.

Now we shift attention to partitions of triples. Recall that Todorcevic [1983a] proved that under $\mathrm{MA}_{\aleph_1} + 2^{\aleph_0} = \aleph_2$, the following positive partition relation holds for all $\alpha < \omega_1$:

$$\omega_1 \to (\omega_1, \alpha)^2.$$

Using this result, Milner and Prikry [1991] proved that $\omega_1 \to (\omega \cdot 2 + 1, 4)^3$ also holds, using $\mathrm{MA}_{\aleph_1}$ and absoluteness as in the Baumgartner-Hajnal Theorem.

Albin Jones,[404] in his thesis [1999], [2000b], took full advantage of the model-theoretic approach to partition relations to prove the following partition relation for all $m, n < \omega$:

$$\omega_1 \to (\omega + m, n)^3.$$

> *Open Question:* Is it true that for all $\alpha < \omega_1$ and $n < \omega$, the following partition relation holds:
>
> $$\omega_1 \to (\alpha, n)^3?$$

Ordinal partition relations

Progress was made on the problem of determining which countable ordinals are partition ordinals. Galvin, in a paper with Larson [1975], had shown that no countable ordinal greater than ω^2 which is the ordinal product of two strictly smaller ordinals is a partition ordinal. Ordinals which do not have this property are

[402]Matthew Foreman received his doctorate in 1980 from the University of California, Berkeley, where his advisor was Solovay.

[403]In the mid-1990's, Woodin showed that if you have an $\aleph_1$-dense ideal on ω_2, then CH holds. See [Foreman, 2010] for a proof.

[404]Albin Jones received his doctorate from Dartmouth College, where Baumgartner was his advisor. His academic career, which included some time at the University of Kansas, was fairly short, but mathematics remains an avocation.

called *multiplicatively indecomposable*, and one can show that they have the form ω^α for some additively indecomposable ordinal α. Since additively indecomposable ordinals can be written in the form $\alpha = \omega^\beta$ and Chang [1972] had shown that ω^ω is a partition ordinal, ω^{ω^2} was the next test case.

Carl Darby[405] and Rene Schipperus independently came up with proofs that ω^{ω^2} is a partition ordinal in the 1990's. I remember hearing Darby's proof during a visit to Boise State University, possibly as early as 1991, the year of the first BEST meeting. Darby's work used Ramsey's Theorem and the *good sequences* developed by Galvin [1971] to represent the ordinal ω^{ω^2} and a careful analysis of the avoidable and unavoidable ways two such disjoint sequences could be interlaced to prove ω^{ω^2} is a partition ordinal, though his proof remains unpublished.

Rene Schipperus recalled enjoyable conversations with Baumgartner about partition problems and partition ordinals in the 1991 meeting in Banff. He worked for a long time on his own, spent some time speaking in a seminar at the University of Calgary with Claude Laflamme[406] and Milner[407] about this work. Eventually Schipperus wrote a thesis with Laver for which he won the Sacks prize for his proof that every denumerable ordinal of the form ω^{ω^β} where β is the sum of one or two indecomposable ordinals is a partition ordinal. In place of the lexicographically ordered good sequences used by Darby for his underlying set, Schipperus used finite labeled trees, carefully constructed to enable him to build his large subsets in a highly restricted way. The heart of his argument is a Ramsey dichotomy, proved using a game and the Nash-Williams Partition Theorem. Darby and Schipperus independently came up with new families of examples witnessing $\alpha \not\to (\alpha, k)$ for different values of k. All the known examples use oscillation type behavior where, for larger values, the blocks can be thought of as having a finite number of possible weights.

Darby [1999] developed a general method to show how to build finite sets witnessing a particular pattern of weighted blocks inside a large subset of an ordinal, depending on the order types of the original set and the large subset. This allowed him to reduce the question of counter-examples to one of finite combinatorics. Schipperus extended the notion of having a large "free" subset (see Lemma 7.2.2 of [Williams, 1977]) to enable him to construct the necessary finite sets.

These negative results are best expressed using the *additive normal form* for the exponent β for an ordinal of the form ω^{ω^β}, that is, its unique representation as a sum of non-decreasing additively indecomposable ordinals. Larson built on their work to improve one of the results obtained by both of them.

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[405]Carl Darby was a student of Laver at the University of Colorado, who earned his doctorate in 1990.

[406]In 1987, Claude Laflamme received his doctorate from the University of Michigan, Ann Arbor, where his advisor was Andreas Blass.

[407]Recall that Milner was the careful reader of Chang's proof who generalized it from 3 to all m; Milner died in 1997 and did not see the final version of the proof by Schipperus.

1. (Darby unpublished) If $\beta = \omega^{\alpha+1}$ and $m \to (4)^3_{2^{32}}$, then
$$\omega^{\omega^\beta} \not\to (\omega^{\omega^\beta}, m)^2.$$
2. (Darby [1999] for 6; Schipperus [1999] for 6; Larson for 5 [2000]) If $\beta \geq \gamma \geq 1$, then
$$\omega^{\omega^{\beta+\gamma}} \not\to (\omega^{\omega^{\beta+\gamma}}, 5)^2.$$
3. (Darby [1999]; Schipperus [1999]) If $\beta \geq \gamma \geq \delta \geq 1$, then
$$\omega^{\omega^{\beta+\gamma+\delta}} \not\to (\omega^{\omega^{\beta+\gamma+\delta}}, 4)^2.$$
4. (Schipperus [1999]) If $\beta \geq \gamma \geq \delta \geq \varepsilon \geq 1$, then
$$\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}} \not\to (\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}}, 3)^2.$$

The Ramsey game $R(\varphi, n, \psi)$ is played by two players, the Builder and the Spoiler, alternating choosing n-element subsets of a linear order L of order type φ, the resource. The Builder goes first and wins if and only if there is a set $L' \subseteq L$ of order type ψ, the goal, such that $[L']^n$ is a subset of the Builder's moves. Using Galvin's result from §7.3 that if φ is scattered and does not embed ω_1, then
$$\varphi \nrightarrow [\omega, \omega^2, \omega^3, \ldots]^1_\omega.$$
Darby and Laver [1998] showed that the Spoiler has a winning strategy in the Ramsey game $R(\omega^\omega, 3, \omega^\omega)$

We conclude this section with two open questions on ordinal partition relations, the first asked many times by Erdős and the second inspired by the proof by Hajnal [1960] of ${\omega_1}^2 \nrightarrow ({\omega_1}^2, 3)^2$ and explicitly asked by Baumgartner.

Open Questions:

1. Characterize the set of $\alpha < \omega_1$ for which $\alpha \to (\alpha, 3)^2$ holds.
2. Is it consistent that ${\omega_1}^2 \to ({\omega_1}^2, 3)^2$?

Erdős and Hajnal [1974] specifically asked about the consistency of the partition relation ${\omega_1}^2 \to ({\omega_1}^2, 3)^2$ with $\mathrm{MA}_{\aleph_1} + 2^{\aleph_0} = \aleph_2$.

Polarized partitions

Polarized partitions may be interpreted in different ways. One approach is to think of them as treating partitions of a product of two sets that can be thought of as a rectangle for which you seek a homogeneous set which is a product of subsets of the given factors. Another approach is to think of one of the factors parameterizing a family of partitions. For each of the examples below, we specify the partition relation.

Example 1: Erdős, Hajnal and Rado had used GCH to prove that for every infinite cardinal κ, the following partition relation holds (see Theorem 39 [1965, 172–176]):

$$\begin{pmatrix}\kappa^+ \\ \\ \kappa^+\end{pmatrix} \to \begin{pmatrix}\kappa^+ & & \kappa & \kappa \\ & \vee & & \\ \kappa & & \kappa^+ & \kappa\end{pmatrix}^{1,1}.$$

Here the partition relation asserts that if $f : \kappa^+ \times \kappa^+ \to \{0,1\}$ is a partition, then there are sets $A, B \subseteq \kappa^+$ so that one of the following happens:

1. $f \restriction A \times B$ is constantly 0 and $|A| = \kappa^+$ and $|B| = \kappa$;
2. $f \restriction A \times B$ is constantly 0 and $|A| = \kappa$ and $|B| = \kappa^+$;
3. $f \restriction A \times B$ is constantly 1 and $|A| = \kappa = |B|$.

Albin Jones [2000a] used elementary submodels to prove a ZFC version:

$$\begin{pmatrix}(2^{<\kappa})^+ \\ \\ (2^{<\kappa})^+\end{pmatrix} \longrightarrow \begin{pmatrix}(2^{<\kappa})^+ & & 2^{<\kappa} & \kappa \\ & \vee & & \\ 2^{<\kappa} & & (2^{<\kappa})^+ & \kappa\end{pmatrix}^{1,1}$$

Baumgartner and Hajnal [2001] added a plus to get a balanced partition relation in a paper received by the journal in 1999.

$$\begin{pmatrix}(2^{<\kappa})^{++} \\ \\ (2^{<\kappa})^+\end{pmatrix} \longrightarrow \begin{pmatrix}\kappa \\ \\ \kappa\end{pmatrix}^{1,1}$$

Example 2: Di Prisco and Todorcevic [1999] considered κ^*, the least cardinal κ satisfying the following partition relation:

$$\kappa \to \begin{pmatrix}2 \\ 2 \\ 2 \\ \vdots\end{pmatrix}^{<\omega}$$

This partition relation holds if for every $f \colon [\kappa]^{<\omega} \to 2$, there is a sequence $\langle H_n \in [\kappa]^2 : n < \omega\rangle$ such that for all $n < \omega$, the function f is constant on $\prod_{k<n} H_k)$. Di Prisco and Todorcevic [1999] prove that $\kappa^* = \aleph_\omega < 2^{\aleph_1}$ is equiconsistent with the existence of a measurable cardinal. The proof builds on the work of Peter Koepke [1984], [1983], who used core model arguments to prove that the free-subset property for ω_ω is equiconsistent with the existence of a measurable cardinal.[408]

[408] A subset X of a structure S is called *free* in S if no element x of X is in the structure generated inside S by $X \setminus \{x\}$. The *free-subset property for* ω_ω is the statement that for every structure S with $\omega_\omega \subseteq S$ which has at most countably many functions and relations there is an infinite subset $X \subseteq S$ free in S.

Velleman, in Theorem 2 of [1990], gave a short proof of the result of Todorcevic that $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ which avoided the use of walks, but which Velleman described as "essentially the same as the one given by Todorcevic."

Todorcevic [1994] examined square bracket relations for triples. He proved, for example, that $2^{\aleph_1} \nrightarrow [\aleph_1]^3_{10}$. This is a special case of a general phenomenon. He showed that Chang's Conjecture is equivalent to $\aleph_2 \to [\aleph_1]^3_{\aleph_1}$. He proved that the partition relation $\aleph_2 \nrightarrow [\aleph_1]^3_{\aleph_0}$ holds, and discussed the optimality of the result, showing the size of the tuples cannot be reduced to 2 under GCH, and the number of colors cannot be increased to $\aleph_1$ by the Chang's Conjecture result. His proof used the same basic ingredients as that for result $\aleph_1 \to [\aleph_1]^2_{\aleph_1}$ (see [1987]), but the new setting required significant reworking. He also proved that for every continuous coloring $f : [\mathbb{Q}]^2 \to 2$, there is a monochromatic homeomorphic copy of $\mathbb{Q}$. On the other hand, he proved that there is a continuous coloring $d : [\mathbb{Q}]^3 \to \mathbb{Q}$ which is *irreducible* in the sense that d realizes every color on the pairs of every subset $X \subseteq \mathbb{Q}$ homeomorphic to $\mathbb{Q}$. He used almost disjoint coding of the coloring $f : [\omega_1]^2 \to \omega$ from his seminal [1987] which witnesses $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_0}$ to construct a set $X \subseteq \mathbb{R}$ of cardinality $\aleph_1$ and a continuous coloring $g : [X]^3 \to X$ which is irreducible.

Todorcevic's Remark 4.3.(d) of [1994] on partitions of pairs is useful for applications of the oscillation map on pairs. A (continuous) order-preserving map from a partial order into the rationals composed with osc will take all possible values $< 2k - 1$ on chains of order type ω^k. Recall that in §7.7 we noted that the Baumgartner-Malitz-Reinhardt proof of the consistency of all Aronszajn trees being embeddable in the rationals could be modified to show that they all have strictly increasing continuous maps into the rationals and the composition of such a function applied to each element of a pair and followed with the oscillation map yields a partition into ω many pieces for which every chain of order type ω^ω has pairs of every color.

Justin Moore[409] [2000] modified the oscillation map of Todorcevic to produce a continuous coloring from the family $\mathbb{P}$ of strictly increasing functions into ω which is irreducible in the sense that every unbounded and well-ordered subset of $\mathbb{P}$ receives all the colors. He introduced an ideal $\mathcal{N}_*$ of measure zero sets and proved that there is a subset $X \subseteq {}^\omega\omega$ of size $\mathrm{add}(\mathcal{N}_*)$[410] and a coloring $c : [X]^2 \to \omega$ which is both continuous and irreducible on subsets of the same size. The proof uses lower bounds for finite Ramsey numbers.

[409] Justin Tatch Moore received his doctorate in 2000 at the University of Toronto under the direction of Todorcevic, held a post-doctoral position at the University of East Anglia where he worked with Džamonja, spent some years at Boise State University and is now at Cornell University.

[410] For any ideal $\mathcal{I}$ of subsets of a set X, the *additivity* of $\mathcal{I}$, denoted $\mathrm{add}(\mathcal{I})$, is the smallest number of sets in $\mathcal{I}$ whose union is not in $\mathcal{I}$. Additivity is one of the well-studied cardinal invariants of ideals. Steprāns, in his chapter, points out that Sierpiński asked about the additivity of the null and meager ideals in the first issue of *Fundamenta Mathematicae*.

Galvin, Hajnal and Komjáth [1995] introduced a strengthening of $\lambda \nrightarrow [\kappa]^2_\mu$: for cardinals $\lambda \geq \kappa$ and μ, $Q(\lambda, \kappa, \mu)$ holds if whenever $X \subseteq [\lambda]^2$ is a graph on λ which has no independent set of cardinality κ, then X can be written as the disjoint union of μ graphs, $X = \dot{\bigcup}\{Y_\alpha : \alpha < \mu\}$ none of which has an independent set of cardinality κ. They prove the following:

1. If $2^\kappa = \kappa^+$, then $Q(\kappa^+, \kappa^+, \kappa^+)$ holds.
2. If more than $\aleph_1$ Cohen reals are added to a model of ZFC, then $Q(\omega_1, \omega_1, \omega_1)$ holds in the resulting model.
3. It is consistent that $Q(\omega_1, \omega_1, 2)$ fails.
4. If $2^\omega < \omega_\omega$, then $Q(2^\omega, 2^\omega, \omega)$ holds.
5. Assuming $V = L$, if κ is a regular uncountable non-weakly compact cardinal, then $Q(\kappa, \kappa, \kappa)$.

Hajnal and Komjáth [1997] show that there is a forcing of size $\aleph_1$ which adds an order type θ of size $\aleph_1$ with the property that $\psi \not\to [\theta]^2_{\omega_1}$ for every order type ψ, regardless of its size. This is in sharp contrast to the consistency result of Shelah [1989] discussed in §8.2.

Structural results

Louveau, Shelah and Veličković [1993] generalized both the Galvin-Prikry Theorem and the Galvin and Blass Perfect Set Partition Theorems. They defined a notion of type of a perfect tree, and showed that for any type τ if the set of all subtrees of a given perfect tree T which have type τ is partitioned into two Borel classes, there there is a perfect subtree $S \subseteq T$ such that all subtrees of S of type τ belong to the same class.

Pierre Matet [1993] used games to characterize sets which are completely Ramsey with respect to a happy family $\mathcal{F}$.

Veličković and Woodin [1998] use a variation of the Todorcevic oscillation function for reals to define a (partial) continuous coloring of triples of real numbers (regarded as infinite non-decreasing sequences of integers) with infinite binary sequences, such that every infinite binary sequence is obtained on the set of triples from every superperfect set.[411] Specifically, given (x, y, z), a triple of distinct non-decreasing functions in ${}^\omega\omega$, if

$$O(x, y, z) := \{n \in \omega : z(n-1) \leq \min(x(n-1), y(n-1) \wedge \max(x(n), y(n)) < z(n)\}$$

is infinite and enumerated as $\langle n_k : k < \omega \rangle$, then $o(x, y, z)$ is the function $k \mapsto 0$ if and only if $x(n_k) \leq y(n_k)$.

[411] Recall a subtree of ${}^{<\omega}\omega$ is *superperfect* if every node has an infinitely branching successor (it need not be an immediate successor); and a subset of ${}^\omega\omega$ is superperfect if the subtree of ${}^{<\omega}\omega$ it generates is superperfect.

9.2 Linear and partial orders

Galvin [1994] gave a short, elegant proof of the Dilworth chain decomposition theorem. For more on the impact of this theorem see [Bogart *et al.*, 1990].

Gary Gruenhage and Joe Mashburn continued the work of Kurepa on decompositions of partially order sets into chains and antichains by analyzing countable width partial orders $(X, <)$ which are *order-separable*, i.e. there is a countable set D such that if $p < q$, then there is an $r \in D$ with $p < r < q$. They [1999] showed that such partial orders can be written as a union of countably many chains under the Open Coloring Axiom $\mathrm{OCA}_{\mathrm{T}}$, but not if there is a 2-entangled subset.[412]

Kunen and Tall [2000] examined the possibilities for the order types of the intersection of the reals with models of set theory, $\mathbb{R} \cap M$. Under CH they found there are exactly two possibilities: $\mathbb{Q}$ and $\mathbb{R}$. They also show the consistency of several other possibilities, including consistency of exactly three types. If the existence of a Ramsey cardinal is consistent, then so is the existence of $2^{2^{\aleph_0}}$ many non-order-isomorphic order types of cardinality continuum of the form $\mathbb{R} \cap M$. At the end of the paper, they recall that it is a long-standing open question whether it is consistent that $2^{\aleph_0} > \aleph_2$ and all $\aleph_2$-dense sets of reals are order-isomorphic. Then they pose [2000, 691]: "a potentially less demanding question: is it consistent that with $2^{\aleph_0} > \aleph_2$ that all the $\mathbb{R} \cap M$s of size $\aleph_2$ are order-isomorphic?"

Abraham and Bonnet [1999] generalize the Hausdorff characterization of the class of scattered linear orders to FAC partial orders, i.e. those in which every antichain is finite. Building on work of Bonnet and Pouzet [1969], they proved that the class of scattered FAC partial orderings may be characterized as the closure under lexicographic sums, inverses and augmentations (keep the domain the same but add new pairs to the partial order). They use the natural sum (Hessenberg sum) and introduce a Hessenberg based product and a Hessenberg based exponentiation which make their proofs go smoothly.[413]

Recall that Hausdorff used CH to prove the existence of a linear order universal in power $\aleph_1$[414] and Shelah [1980a] proved the consistent existence of a linear order universal in power $\aleph_1$ in a model of $2^{\aleph_0} = \aleph_2$. Kojman and Shelah [1992] proved the non-existence of universal orders and universal partial orders in power λ for a variety of cardinals λ. They noted that the problem of existence of a universal model for a first-order theory remains "unsettled in classical model theory only for cardinals $\lambda < 2^{<\lambda}$" and cite [Chang and Keisler, 1973] and VIII.4 of [Shelah, 1978a] on saturated, special and universal models. In Theorem 3.6 they used club guessing to prove that for every regular cardinal λ for which $\aleph_1 < \lambda < 2^{\aleph_0}$, there is no universal linear order in power λ. The first item of Conclusions 5.7 is that under the hypotheses on λ of Theorem 3.6 there are no universal partial

[412]Entangled subsets of partial orders were introduced by Shelah, shown to follow from CH and to be consistent with $\mathrm{MA}_{\aleph_1}$ in [Abraham *et al.*, 1985], and used by Roitman [1978] in her combinatorial reformulation of the questions (S) and (L).

[413]See [Džamonja and Thompson, 2006] for an extension to κ-scattered partial orders.

[414]Recall that a linear (partial) order $(P, <)$ is *universal in power* λ (alternatively *in cardinality* λ) if it has cardinality λ and is universal for the class of linear (partial) orders of cardinality λ.

orders in power λ. In Theorem 3.10, they generalized Theorem 3.6 by showing that if λ is a regular cardinal and there is some $\kappa^+ < \lambda < 2^\kappa$, then there is no universal linear order in power λ. They start Section 4 by noting that for a strong limit cardinal μ, any first-order theory T with $|T| < \mu$ has a universal model in power μ. They prove in Theorem 4.3 that if θ and $\kappa > \theta^+$ are regular cardinals, $\kappa < \mu$ and μ^* is the minimum size of a family $A \subseteq [\mu]^{\leq\kappa}$ which has the property that for all $X \in [\mu]^{\leq\kappa}$ there are fewer than κ members of A whose union covers A, and if there is a binary tree $T \subseteq {}^{<\theta}2$ of size $< \kappa$ with more than μ^* branches of length θ, then there is no linear order of size μ which is universal for linear orders of size κ. They show that if μ is a fixed point of the aleph function of the first order but not the second order[415] and $\sigma + \mathrm{cf}(\lambda) < \lambda$, then $\mu^* = \mu$ in the theorem stated above. They draw two corollaries, proving that there is no universal linear order of power μ for a singular cardinal if (1) μ is not a second order fixed point, and there is some $\sigma < \mu$ such that $2^{<\sigma} < \mu < 2^\sigma$, and if (2) $\aleph_\mu > \mu$ or μ is not a second order fixed point, $\mathrm{cf}(\mu) = \aleph_0$ or $2^{<\mathrm{cf}(\mu)} < \mu$, and $\mu \neq 2^{<\mu}$.

9.3 Trees

We discuss a variety of results, many of them consistency proofs, on existence or non-existence of trees. Later we look at the structure of families of trees under embeddings.

Existence results

Kojman and Shelah [1993] proved that if $\mu^{<\mu} = \mu$ and $2^\mu = \mu^+$, and the set $\{\alpha < \mu^+ : \mathrm{cf}(\alpha) < \mu\}$ contains a non-reflecting stationary set, then there is a Suslin tree on μ^+ in which every ascending chain of size $< \mu$ has an upper bound, i.e. is closed in the tree topology. They use a principle they describe as "closely related to club guessing", and recalled Gregory's result [1976] that GCH together with the existence non-reflecting stationary set of ω-cofinal elements of ω_2 implies the existence of an $\aleph_2$-Suslin tree (see §7.7).

Magidor and Shelah [1996] proved that if a singular cardinal κ is the limit of strongly compact cardinals, then there are no κ^+-Aronszajn trees. Recall that for regular cardinals, Jensen proved that in L, there is no λ-Aronszajn tree if and only if λ is weakly compact. Magidor and Shelah went on to show the consistency relative to large cardinals of there being no $\aleph_{\omega+1}$-Aronszajn tree, shedding light on the difficulty of determining whether or not successors of singular cardinals must support Aronszajn trees.

Cummings [1997] showed that, under certain cardinal and combinatorial assumptions, there are Suslin trees on successor cardinals larger than $\aleph_1$ which are immune to specialization in cardinal and cofinality preserving forcing extensions.

[415] A cardinal λ is a *fixed point of the aleph function of the first order* if $\lambda = \aleph_\lambda$. It is not a fixed point of *the second order* if $|\{\nu < \lambda : \nu = \aleph_\nu\} = \sigma < \lambda$.

Cummings and Foreman [1998] derived the consistency of no $\aleph_n$-Aronszajn trees for $2 \le n < \omega$ from the consistency of infinitely many supercompact cardinals, and also derived the consistency of the existence of a strong limit cardinal κ of countable cofinality for which there are no κ^{++}-Aronszajn trees from the consistency of a supercompact cardinal with a weakly compact cardinal above.

In a paper submitted in 1998, Foreman, Magidor and Ralf Schindler [2001] investigated the consistency strength of the above result of Cummings and Foreman [1998] and from it prove the existence of sharps for inner models with n Woodin cardinals. In their proof they use a method of induction due to Woodin.

In an unpublished manuscript from 1994, Miyamoto showed that $\text{cov}(\mathcal{M}) \ge \aleph_2$ and the Stick Principle[416] were sufficient to get a complete amalgamation system for every ω-morass which by work of Velleman in his [1990] is enough to build a Suslin tree. Brendle [2006] adapts Todorcevic's Cohen Real to Suslin Tree Construction and Miyamoto's proof to show that $\clubsuit$ together with the assumption that the meager idea $\mathcal{M}$ has a base of size $\aleph_1$ implies there is a Suslin tree.

Structure under embeddings

Taneli Huuskonen [1995] called a tree of height ω_1 *persistent* if it has a non-empty subtree in which each node has extensions of all countable heights. He introduced and identified the $\le$-smallest persistent tree T^0 whose nodes are sequences $(t_0, \ldots, t_n)$ of countable ordinals ordered by $(t_0, \ldots, t_n) \le (s_0, \ldots, s_m)$ if $n \le m$, $t_n \le s_n$, and $i < n$ implies $t_i = s_i$.

Todorcevic and Jouko Väänänen [1999] studied the quasi-order $(\mathbb{T}, \le)$ where $\mathbb{T}$ denotes the class of all trees of cardinality ω_1 with no uncountable branches, and $S < T$ if and only if there is a strictly increasing map from S into T. Note that there is no requirement that the maps be one-to-one, so they are not necessarily homomorphisms. They proved a variety of results about the structure of $(\mathbb{T}, \le)$, with one set of results for those trees below T^0 and another set for those at or above T^0 and $(\mathbb{T}/\equiv, \le)$. For example, they showed in Theorem 33 that if $2^{\aleph_0} < 2^{\aleph_1}$, then there are incomparable special Aronszajn trees under order-preserving maps. Recall that Abraham and Shelah [1985] had a similar result for homomorphic embeddings.

In the same paper, Todorcevic and Väänänen identified the tree $\sigma\mathbb{Q}$ as the universal element in the class of $\mathbb{R}$-embeddable Hausdorff trees. They showed that the class LO_1 of linear orders without uncountable well-ordered or conversely well-ordered sub-orderings has no universal element, i.e. there is no $K \in \text{LO}_1$ such that every element of LO_1 embeds in it. They also showed that there is no universal element for the class PO_1 of all those partial orders of cardinality at most continuum which have no increasing ω_1-sequence.[417]

[416] The *Stick Principle* asserts the existence of a family $\mathcal{A}$ of $\aleph_1$ many countable subsets of ω_1 such that every uncountable subset of ω_1 contains a member of $\mathcal{A}$.

[417] Džamonja and Katherine Thompson [2005] showed that the least size of a universal family for well-founded partial orders of power λ under order preserving embeddings is the least size of a universal family for well-orders of size λ, which is λ^+. They used club guessing arguments

Todorcevic [1998a] announced a Ramsey theoretic reformulation, under PFA, of the existence of a two-element basis for the class of Aronszajn lines.[418]

9.4 Combinatorial principles

In the early 1990's, Shelah [2000b] used his pcf theory to make a remarkable connection between instances of Jensen's Diamond Principle for successor cardinals and instances of the GCH:[419] for cardinals $\lambda \geq \beth_\omega$,[420] $2^\lambda = \lambda^+$ if and only if $\diamondsuit_{\lambda^+}$.[421]

In a paper received October 19, 1998, Cummings, Foreman, and Magidor [2001] considered, in the context of singular cardinals, implications between square and the scales of pcf theory, and proved consistency results between relatively strong forms of square and stationary set reflection.

Abraham and Todorcevic [1997] focused attention on P-ideals on ω_1.[422] They introduced a new property of ideals which they called $(*)$: for every P-ideal, either

1. there exists an uncountable subset $A \subseteq \omega_1$ such that $[A]^\omega \subseteq \mathcal{I}$; or

2. ω_1 may be decomposed into countably many subsets A_i such that $A_i \cap I$ is finite for all $i < \omega$ and $I \in \mathcal{I}$ (they say A_i is *orthogonal* to $\mathcal{I}$).

They observed that PFA implies $(*)$, but in this paper, showed that $(*)$ is consistent with CH. They proved that if $(*)$ holds, then there are no Suslin trees, and they drew many other consequences.

In a paper [Todorcevic, 2000] submitted in November 1999, Todorcevic simplifies the statement of $(*)$ to get an optimal P-ideal dichotomy, also called $(*)$: for every P-ideal $\mathcal{I}$ of countable subsets of some set S, either

1. there exists an uncountable subset $A \subseteq S$ such that $[A]^\omega \subseteq \mathcal{I}$; or

2. S can be decomposed into countably many sets orthogonal to $\mathcal{I}$.

He used this version to prove, among other results, that Jensen's $\square_\kappa$ fails for any regular $\kappa > \omega_1$. Todorcevic proved that this P-ideal dichotomy follows from PFA and that the consistency of the existence of a supercompact cardinal gives the consistency of $(*)$ + GCH.

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to prove that for regular λ strictly between $\aleph_1$ and $2^{\aleph_0}$, the least size of a universal family for well-founded partial orders of power λ under rank and order preserving embeddings is at least $2^{\aleph_0}$.

[418] Moore [2006a] showed that under the assumption of PFA, Aronszajn lines have a two-element basis and uncountable linear orders have a 5-element basis.

[419] The paper was received February 15, 1994, revised November 16, 1998, and appear in the December 2000 issue of the Israel Journal.

[420] Recall that $\beth_\omega$ is the least strong limit cardinal.

[421] Shelah [2010] improved his result to all uncountable λ.

[422] An ideal $\mathcal{I}$ on ω_1 is a *P-ideal* if all its members are countable and it is $\subseteq^*$-σ-directed.

10 POSTSCRIPT

In this final section we look back at some threads woven through this decade-by-decade account, and we hark back to the opening of the chapter with the quotes by Cantor and Motzkin. Inspired by Cantor, we look at *freedom* as shown in the generalization of notions of largeness, first examining those used in successive versions of the Regressive Function Theorem, then in generalizations of the Pigeonhole Principle, and in applications to partition relations and trees. We then examine the *winnowing* process mentioned by Cantor that shaped the notion of uncountable tree from the ramified sets, tables, and suites of Kurepa to the family of ω_1-trees classified as to whether they are special Aronszajn trees, non-special Aronszajn trees, Suslin trees, or Kurepa trees. The Motzkin quote inspired a look at structural results more generally, and we review the possibilities for describing the structure of uncountable linear orders that appeared in this century with a brief peek at one in the next century. We wrap up the postscript with a few remarks on interactions with the near neighbors of logic, topology and analysis.

First we look at the *freedom* to generalize notions of largeness for subsets of a given set. Our first lens is on the series of generalizations of the Regressive Function Theorem of Alexandroff and Urysohn [1929] that any regressive function on the set of infinite countable ordinals is constant on an uncountable set. Two aspects were generalized, the possibilities for the domain of the regressive function and the notion of largeness of the set on which the function was constant. Dushnik [1931] extended the family of possible domains to include the set of infinite ordinal numbers less than an uncountable successor cardinal κ, with cardinality κ the notion of largeness. Erdős [1950] then extended the domains to include the set of infinite ordinal numbers less than a singular cardinal κ of uncountable cofinality and used being cofinal in κ as the notion of largeness, but relaxed the constraint that the function be constant there to only require the values of the function to be bounded. Novák [1950] proved that a closed unbounded subset of the countable ordinals was a suitable domain and used being uncountable as the notion of largeness. Neumer [1951], apparently unaware of Novák's work, weaken it to stationary subsets of a regular cardinal κ, and used cardinality κ as the notion of largeness. Finally Fodor, after a rediscovery in [1955] of the work of Dushnik, strengthened Neumer's Theorem to show any function defined on a stationary set is constant on a stationary subset. Fodor was not alone in rediscovering work by others related to this theorem: Bloch [1953], Denjoy [1953], and Neumer [1958] did so as well. The history of this theorem exemplifies the sort of forgetting and remembering of theorems that happens over time, the independent discovery of generalizations and proofs with variations that evolve into an especially beautiful theorem.

The Regressive Function Theorem has been generalized to other contexts, but we will stop here to recall that the original theorem can be regarded as a kind of pigeonhole principle with $\aleph_1$ pigeons and $\aleph_1$ pigeonholes both labeled with infinite countable ordinals but a requirement that pigeons come to roost in pigeonholes with a smaller label than their own label and a conclusion that an uncountable number end up in the same pigeonhole. The generalizations may be similarly recast.

As we turn to look at various pigeonhole principles and how they are used to prove partition theorems, particularly for pairs, we keep in mind the slogan that is embedded in the Motzkin quote: *complete disorder is impossible.*

Recall the *Schubfachprinzip*, known in English as the *Pigeonhole Principle*, was identified as a mathematical principle by Dirichlet in the 1800s and its generalization to the transfinite has proved widely applicable. Basic cardinality arguments, with infinitely many objects deposited in finitely many places with at least one receiving infinitely many, or the extension with uncountable versus countable are now fundamental in mathematical practice.

The important concept of *cofinality* allowed more nuanced generalizations. This concept was brought into focus when a mistake in Bernstein's thesis lead to the (false) "proof" by Gyula König of the Continuum Hypothesis at the 1904 International Congress in Heidelberg. Hausdorff introduced the concept of cofinality [confinality] in his paper [1906], worked out the basic properties in his paper [1908], and included this material in his book [1914]. With the notion of cofinality, one can say more: if κ objects are distributed among fewer than cofinality of κ many cells, then one cell must have κ objects.

Now we turn to the use of pigeonhole principles in proofs of partition relations. Ramsey specified the *Axiom of Selections* as his pigeonhole principle, i.e. the Axiom of Choice. Recall that he constructed a sequence begin-homogeneous in the inner induction step of his proof.

Many proofs of Ramsey's Theorem exist. One example is an ultrafilter proof. Building an end-homogeneous set is another natural approach. In their rediscovery of Ramsey's Theorem, Erdős and Szekeres used an Ordered Pigeonhole Principle in which a suitably long sequence was required to have either a large monotone increasing subsequence or a large monotone decreasing subsequence.

Recall that Erdős, in his proof [1942] that if $b > a^a$, then $b \to (a^+)^2_a$, built a tree of subsets of the vertices of the complete graph on his underlying set, built a tree of subsets and a parallel tree of representatives with the entire set of vertices at the root of the first and a representative at the root of the second, and at each successive stage through the transfinite, select from each set on the highest level a representative for the tree of vertices, and split the remaining vertices in each subset according to the color class or graph to which the vertex and the representative belonged. The process is quite like the one used by Ramsey, but proceeds on all fronts simultaneously to create two trees: one of nested subsets and another of the representative vertices with the set of vertices along any branch forming a set which is end-homogeneous. One could say that he is using color classes determined by specific representatives as his pigeonholes.

In the proof of the Positive Stepping Up Lemma, Erdős and Rado [1956] essentially built a tree of sequences of ordinals with a largest element, where the recursive definition of each sequence started with its top element and was so constructed to be end-homogeneous for r-element subsets for $r \geq 2$. They used the functions $h : [\sigma]^r \to k$ for $\sigma < \ell$ as pigeonholes and determined that they had more sequences than such pigeonholes so one must be of length at least ℓ and they could use it to finish their proof.

Milner and Rado [1965] investigated the generalization of the Pigeonhole Principle to ordinals that is encompassed in the partition relation $\alpha \to (\alpha_0, \alpha_1, \dots)^1$, and discovered their paradoxical decomposition: for any uncountable cardinal κ and every $\beta < \kappa^+$, the negative relation $\beta \nrightarrow (\kappa^\omega)^1_\omega$ holds.

A variation of Laver's generalization of the Halpern-Läuchli Theorem was used by Milliken as his pigeonhole principle in the proof of Milliken's Ramsey Theorem for strongly embedded trees.

In Galvin's investigation of partition relations of the form $\varphi \to (\alpha)^2_2$ for all countable ordinals α, he (see [Erdős and Hajnal, 1974, 272] for a discussion of Galvin's "old theorems") focused attention on those uncountable linear orders φ that satisfy the pigeonhole principle represented by the relation $\varphi \to (\omega)^1_\omega$, where Erdős and Hajnal in their paper with Rado [1965] and in their problem paper [1971b] had focused attention on order types in which neither ω_1 nor $\omega_1{}^*$ is embeddable. The Baumgartner-Hajnal Theorem for order types validated Galvin's conjecture that this was the right approach. Galvin then tackled partial orders and conjectured the same condition would be critical (see [Erdős and Hajnal, 1974, 272]). He was able to show that the hypothesis $\varphi \to (\eta)^1_\omega$ sufficed in [1975]. Todorcevic [1985a] validated Galvin's conjecture with a proof that first showed the result was true for non-special Aronszajn trees and then used that result to derive his Poset Partition Theorem.

At the end of the century, co-ideals (the positive sets for their corresponding ideals) were used together with elementary submodels to prove partition relations for pairs starting with a cardinal resource, including the Balanced and Unbalanced Baumgartner-Hajnal-Todorcevic Theorems and the consistency result of Foreman and Hajnal.

Next we look at winnowing to enduring concepts, especially for trees. The first infinite trees were graph-theoretic trees used by Kőnig in his Infinity Lemma. The first systematic study of uncountable trees was by Kurepa who introduced ramified sets, ramified tables, and suites. His basic notion was more general than what we use today, allowing equivalent elements. His notion of homogeneous required a constant size across the entire ramified table for every set consisting of all elements with the same set of predecessors.

Eventually these special partial orders became known as trees, and the key feature that the set of predecessors be well-ordered became standard. Further winnowing followed. Now the most studied trees are those whose levels have smaller cardinality than their height, those for which nodes whose set of predecessors have limit order type are the unique elements which that set of predecessors, a feature which means that the tree topology is Hausdorff. These constraints allow trees of height ω_1 in which all levels are countable to be embedded in the collection of functions from countable ordinals into ω ordered by end-extension. This collection of trees includes Aronszajn trees, special Aronszajn trees, Suslin trees, and

Kurepa trees, with special examples such as those coming from Countryman lines and from the rho functions of Todorcevic. For those interested in learning more about Aronszajn trees and their structure, the survey by Moore [2008b] is warmly recommended.

As one way to look at the interaction of logic and infinite combinatorics, we revisit the basis question for uncountable linear orders. There we find consistency results via forcing, new axioms, and representation theorems.

Sierpiński [1946] showed that under the Continuum Hypothesis, there are $2^{\mathfrak{c}}$ many pairwise incomparable order types of sets of real numbers of power $\mathfrak{c} = 2^{\aleph_0}$, so consistently any basis for the uncountable linear orders is very large.

Moore [2006a] proved the consistent existence of a finite basis for uncountable linear orders. His accomplishment offers us the opportunity to review the steps leading up to his proof.

Recall that Baumgartner [1973] had given a forcing proof of the consistency that all $\aleph_1$-dense sets of reals are isomorphic, and Abraham and Shelah [1981] had shown that Martin's Axiom does not suffice. However, the Proper Forcing Axiom does suffice.

Baumgartner [1976b] had investigated basis questions, mostly due to Galvin, for the class of uncountable order types which are not the union of countably many well-orderings. Galvin had asked if the collection consisting of ${\omega_1}^*$ together with all real types and all Specker types, was a basis for the class of uncountable order types which are not a union of countably many well-orderings. Baumgartner had shown there were more, but they all embed ω_1. Baumgartner had shown in [1976b] that every Specker type embeds a lexicographically ordered Aronszajn tree, and in [1982] had showed that every Specker type is isomorphic to a lexicographically ordered Aronszajn tree.

Baumgartner singled out the Aronszajn lines (Specker types) for special attention with his Question 5(ii) [1976b, 221] which asked for the consistency of a finite basis for them, and asked with his fourth problem on page 275 [1982] on the consistency of the existence of a finite basis for uncountable linear orders. Under PFA, the questions are the same, since the union of a finite basis for the Aronszajn lines with $\{\omega_1, {\omega_1}^*, X\}$ where X is an $\aleph_1$-dense set of reals, is the desired basis.

As this chapter comes to a close, we note that just as finite combinatorics benefited by interaction with probability as indicated in the Motzkin quote, infinite combinatorics benefited by interaction with logic and other areas of mathematics.

From logic and other parts of set theory, the field benefited by research on forcing, large cardinals, ideals, model-theoretic methods, and an axiomatic approach that has led to a wide variety of additional axioms for consistency results.

From topology came the notion of compactness (Tychonoff's Theorem), closure operators, notions of largeness and smallness related to measure and category, interest in products, Suslin problem, Normal Moore Space Conjecture and S-space and L-space problems that stimulated work in set theory. Since trees are the foundation of many interesting topological spaces, set-theoretic topologists have motivations and intuitions that lead to results interesting to both areas.

Finite combinatorics has provided tools applied applied in many proofs and has been a source of inspiration, with many finite results, especially partition relations, generalized to the infinite and uncountable in a variety of ways.

Dynamical systems and ergodic theory have given new approaches to Ramsey theory and interactions between Ramsey theory and Banach spaces has benefited both areas.

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