MHF 3202 (Larson) Subset relation is transitive

Theorem: Suppose A, B, and C are sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Assume $A \subseteq B$ and $B \subseteq C$. (o$\rightarrow$)

(We want to prove that $A \subseteq C$, i.e. $(\forall x)(x \in A \rightarrow x \in C.)$

Let x be arbitrary. (o$\forall$)

Assume $x \in A$. (o$\rightarrow$)

Since $A \subseteq B$, we know $x \in A \rightarrow x \in B$, by definition of subset.

From $x \in A$ and $x \in A \rightarrow x \in B$, we conclude that $x \in B$ by modus ponens (mp)

Since $B \subseteq C$, we know $x \in B \rightarrow x \in C$, by definition of subset.

From $x \in B$ and $x \in B \rightarrow x \in C$, we conclude that $x \in C$ by modus ponens (mp)

We assumed $x \in A$ and proved $x \in C$, so the implication $x \in A \rightarrow x \in C$ follows (c$\rightarrow$).

Since x was arbitrary, the implication is true for all x, i.e. $(\forall x)(x \in A \rightarrow x \in C)$ (c$\forall$).

Consequently $A \subseteq C$ by definition of subset.

We assumed $A \subseteq B$ and $B \subseteq C$ and proved $A \subseteq C$, so the implication and theorem follow. (victory statement)