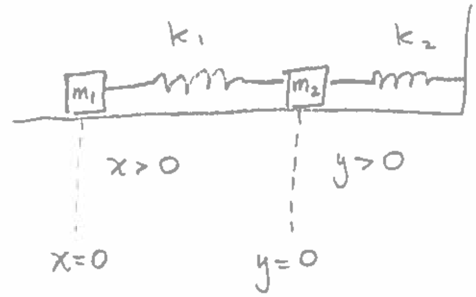
5.6

(1) let $x(t) :=$ displacement to right of m_1 block

$y(t) :=$ displacement to right of m_2 block



The block with mass m_1 has the force

$$F_1 = -k_1 x + k_1 y = k_1(y-x)$$

acting on it and the block with mass m_2 has the forces

$$F_2 = -k_2 y$$

$$F_3 = k_1 x - k_1 y = k_1(x-y)$$

So we have the system of differential equations

$$m_1 x''(t) = F_1 = k_1(y-x)$$

$$m_2 y''(t) = F_2 + F_3 = -k_2 y + k_1(x-y)$$

Substituting $m_1 = 1kg$, $m_2 = 2kg$, $k_1 = 4$ N/m, $k_2 = \frac{10}{3}$ N/m we get

$$D^2 x = 4y - 4x$$

$$2D^2 y = -\frac{10}{3}y + 4x - 4y = 4x - \frac{22}{3}y$$

or equivalently,

$$(D^2+4)x - 4y = 0 \quad (*)$$

$$-2x + \left(D^2 + \frac{11}{3}\right)y = 0 \quad (**)$$

Multiplying $(*)$ by $\left(D^2 + \frac{11}{3}\right)$ and $(**)$ by 4 then adding the resulting equations yields

$$(D^2+4)\left(D^2+\frac{11}{3}\right)x - 8x = 0$$

$$\Leftrightarrow \quad \left(D^4 + \frac{23}{3}D^2 + \frac{20}{3}\right)x = 0$$

$$\Leftrightarrow \quad (D^2+1)\left(D^2+\frac{20}{3}\right)x = 0 .$$

Hence, the roots of the associated auxiliary equation are

$$r = \pm i, \ \pm i\sqrt{\frac{20}{3}}.$$

So the solution for x is given by

$$\underline{x(t) = c_1 \cos t + c_2 \cos\left(t\sqrt{\frac{20}{3}}\right)}$$

(which is the sum of two linearly independent homogeneous solutions).

Now since

$$D^2 x(t) = D^2\left(c_1 \cos t + c_2 \cos\left(\sqrt{\tfrac{20}{3}}\, t\right)\right)$$

$$= -c_1 \cos t - \frac{20}{3} c_2 \cos\left(\sqrt{\tfrac{20}{3}}\, t\right)$$

we have that (by substituting into ⊛):

$$y(t) = \frac{1}{4}(D^2+4)\, x(t)$$

$$= \frac{1}{4} D^2 x(t) + x(t)$$

$$= -\frac{1}{4} c_1 \cos t - \frac{5}{3} c_2 \cos\left(\sqrt{\tfrac{20}{3}}\, t\right) + c_1 \cos t + c_2 \cos\left(\sqrt{\tfrac{20}{3}}\, t\right)$$

$$\Longleftrightarrow \quad y(t) = \frac{3}{4} c_1 \cos t - \frac{2}{3} c_2 \cos\left(\sqrt{\tfrac{20}{3}}\, t\right).$$

Now it simply remains to solve for c_1 & c_2 using the initial conditions:

$x(0) = -1 \qquad x'(0) = 0$

$y(0) = 0 \qquad y'(0) = 0$

Accordingly, we have

$$-1 = x(0) = c_1 + c_2 \quad (a)$$

$$0 = y(0) = \frac{3}{4}c_1 - \frac{2}{3}c_2 \quad (b)$$

From (b) we have that $c_1 = \frac{8}{9}c_2$ which, when substituted into (a), yields

$$-1 = \frac{8}{9}c_2 + c_2$$

$$\Leftrightarrow \quad -1 = \frac{17}{9}c_2$$

$$\Leftrightarrow \quad c_2 = -\frac{9}{17}.$$

Then $c_1 = \frac{8}{9}\left(-\frac{9}{17}\right) = -\frac{8}{17}$. Thus, our final solution to the system of differential equations is given by

$$\boxed{\begin{aligned} x(t) &= -\frac{8}{17}\cos t - \frac{9}{17}\cos\left(\sqrt{\tfrac{20}{3}}\,t\right) \\ y(t) &= -\frac{6}{17}\cos t + \frac{6}{17}\cos\left(\sqrt{\tfrac{20}{3}}\,t\right). \end{aligned}}$$