1. Let \( f(x, y) = \sqrt{xy} \).

(i) Find the linear approximation of \( f(x, y) \) at the point \((1, 4)\). (3 points)

\[
\nabla f(x, y) = \left< \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right> \Rightarrow \nabla f(1, 4) = \left< 4, \frac{1}{4} \right>
\]

\[
f(1, 4) = \sqrt{4} = 2
\]

\[
L(x, y) = f(1, 4) + \nabla f(1, 4) \cdot (x-1, y-4)
\]

\[
\Leftrightarrow \quad L(x, y) = 2 + \left( x-1 + \frac{1}{4} (y-4) \right)
\]

(ii) Using the linearization from part (i), estimate \( f(0.99, 4.08) \). (1 point)

\[
f(0.99, 4.08) \approx L(0.99, 4.08)
\]

\[
= 2 + (0.99-1) + \frac{1}{4} (4.08-4) = 2 - 0.01 + 0.02 = 2.01
\]

2. Consider the function \( f(x, y) = \begin{cases} \frac{3x^2y}{x^6+2y^2} & : (x, y) \neq (0, 0) \\ 0 & : (x, y) = (0, 0) \end{cases} \).

(i) Show that \( f_x(0,0) \) and \( f_y(0,0) \) both exist. (2 points)

\[
f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0
\]

\[
f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.
\]

(ii) Determine if \( f(x, y) \) is differentiable at \((0,0)\). You must provide justification to receive full credit. (2 points)

Observe that

\[
\begin{align*}
x = 0: & \quad \lim_{y \to 0} \frac{3(0)^2y}{(0)^6+2y^2} = \lim_{y \to 0} \frac{0}{2y^2} = 0 \\
y = x^3: & \quad \lim_{x \to 0} \frac{3x^6}{x^6+2x^6} = \lim_{x \to 0} \frac{3x^6}{3x^6} = 1
\end{align*}
\]

So, \( \lim_{(x,y) \to (0,0)} f(x,y) \) DNE \( \Rightarrow f \) not continuous \( \Rightarrow f \) not differentiable at \((0,0)\).
Problem References:

1. MAC2313 L13 NYTI #2. Answer: (i) \( L(x, y) = 2 + (x - 1) + \frac{1}{2}(y - 4) \). (ii) 2.01.
2. MAC2313 L13 HW Assignment Problem #14. Answer: (i) \( f_x(0, 0) = f_y(0, 0) = 0 \). (ii) \( f \) not continuous at \((0, 0)\).